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BALANCED MODEL REDUCTION OF LINEAR TIME-VARYING SYSTEMS

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Abstract: This paper treats model reduction of linear time-varying models in continuous time. The method proposed is based on time-varying Lyapunov inequalities and balancing of Gramians. An error bound for truncated models that generalizes the well-known 'twice-the-sum-of-the-tail'-formula for time-invariant balanced systems is obtained. Input-output stability of truncated balanced models is also proved.

Keywords: Model reduction, Linear systems, Time-varying systems, Error analysis, Stability criteria

1. INTRODUCTION

Model reduction is a vital step in the design process of a control system. Both the process model and the controller might need reduction during the design. The motivation is to reduce complexity of models and to make the implementational work safer and less complicated.

There are many ways to reduce systems, see surveys in for example (Andersson, 1999). However, many of the reduction techniques lack the possibility of giving an error bound on the reduced system. One method often used is balanced truncation. It is popular because it is easy to implement and gives a nice upper error bound, see (Enns, 1984; Glover, 1984). The method was originally developed for time-invariant models. It is natural to ask if a similar result also holds for the time-varying case. Balanced realizations for time-varying systems were studied in (Shokoohi *et al.*, 1983) and results on existence of such realizations were given. However, no error bound of truncated models was given. With methods developed for uncertain systems (Beck *et al.*, 1996; Andersson, 1999), error bounds are obtainable if the time-varying dynamics is described as uncertainty. Those bounds are conservative as the known time-variance is not used explicitly.

In (Lall *et al.*, 1998) an operator theoretic framework for balanced truncation of time-varying system is presented together with an error bound. Error bounds on truncated periodic balanced time-variable systems in discrete time can be found in (Longhi and Orlando, 1999; Varga, 2000).

In this paper reduction of time-variable systems in continuous time will be studied. An error bound based on the use of time-varying balanced Lyapunov inequalities is presented. The bound generalizes the 'twice-the-sum-of-the-tail'-formula used for time-invariant balanced systems. Stability properties of balanced truncated systems are also given. Finally an example is solved numerically using the described techniques.

2. PRELIMINARIES AND PROBLEM FORMULATION

The subject of the paper is linear time-varying systems. The weighted Euclidean norm will be used extensively in the proofs, $|\mathbf{x}(t)|^2_{\mathbf{P}} = \mathbf{x}^T(t)\mathbf{P}(t)\mathbf{x}(t)$ and $\mathbf{P}(t) \geq 0$. Matrix inequalities of the type $\mathbf{P}(t) \leq \mathbf{Q}(t)$ means that $\mathbf{P}(t) - \mathbf{Q}(t)$ is negative semidefinite for all t. $\mathbf{I}_n$ will denote the $n \times n$-dimensional identity matrix.

For n-dimensional signals $\mathbf{x}(t)$ in $\mathbf{L}_2^n[0,T]$ the norm

$$\|\mathbf{x}\|_2 = \left(\int_0^T |\mathbf{x}(t)|^2 dt\right)^{1/2}$$

is defined, and for linear operators $\mathbf{G} : \mathbf{L}_2^m[0,T] \to \mathbf{L}_2^p[0,T]$ the induced norm

$$\|\mathbf{G}\| = \sup_{\|\mathbf{u}\|_2=1} \|\mathbf{G}\mathbf{u}\|_2 .$$

The original linear nth order system will be denoted by $\mathbf{G}$, and its state-space realization is given by

$$\mathbf{G} : \begin{cases} \dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \mathbf{B}(t)\mathbf{u}, \quad \mathbf{x}(0) = 0 \\ \mathbf{y} = \mathbf{C}(t)\mathbf{x} + \mathbf{D}(t)\mathbf{u} \end{cases} \tag{1}$$

with the signals $\mathbf{x} \in \mathbf{L}_2^n[0,T], \mathbf{u} \in \mathbf{L}_2^m[0,T]$, and $\mathbf{y} \in \mathbf{L}_2^p[0,T]$. The matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ are time-varying, continuous in t, and with dimensions so that the multiplications and additions are defined. With these conditions existence and uniqueness of solutions to (1) is guaranteed, see for example (Rugh, 1996). When the infinite time-horizon case is studied the system is assumed to be asymptotically stable.

With inspiration from Linear Fractional Transformations (1) can be interpreted as a feedback system as depicted in Figure 1 with a static time-varying matrix $\mathbf{M}(t)$

$$\mathbf{M}(t) = \begin{bmatrix} \mathbf{A}(t) & \mathbf{B}(t) \\ \mathbf{C}(t) & \mathbf{D}(t) \end{bmatrix}.$$

With the additional signal $\mathbf{z} \in \mathbf{L}_2^n[0,T]$ the system can be written as

$$\begin{bmatrix} \mathbf{z} \\ \mathbf{y} \end{bmatrix} = \mathbf{M}(t) \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix}, \quad \mathbf{x} = \frac{1}{s}\mathbf{I}_n \cdot \mathbf{z}.$$

$1/s$ here denotes integration in time. It is often suitable to change the coordinate representation in the realization of $\mathbf{G}$. Here Lyapunov coordinate transformations $\mathbf{T}(t)$, $\mathbf{x}(t) = \mathbf{T}(t)\tilde{\mathbf{x}}(t)$, will be considered. $\mathbf{M}(t)$ then transforms according to

$$\mathbf{M}(t) = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \xrightarrow{\mathbf{T}} \tag{2}$$
$$\tilde{\mathbf{M}}(t) = \begin{bmatrix} \tilde{\mathbf{A}} & \tilde{\mathbf{B}} \\ \tilde{\mathbf{C}} & \tilde{\mathbf{D}} \end{bmatrix} = \begin{bmatrix} \mathbf{T}^{-1}(\mathbf{A}\mathbf{T} - \dot{\mathbf{T}}) & \mathbf{T}^{-1}\mathbf{B} \\ \mathbf{C}\mathbf{T} & \mathbf{D} \end{bmatrix}.$$

so that the input-output map is invariant.

A truncated realization of $\mathbf{G}$ will be denoted by $\hat{\mathbf{G}}$. This is obtained by first specifying the following structure on $\mathbf{M}(t)$

$$\mathbf{M}(t) = \begin{bmatrix} \mathbf{A}_{11}(t) & \mathbf{A}_{12}(t) & \mathbf{B}_1(t) \\ \mathbf{A}_{21}(t) & \mathbf{A}_{22}(t) & \mathbf{B}_2(t) \\ \mathbf{C}_1(t) & \mathbf{C}_2(t) & \mathbf{D}(t) \end{bmatrix}, \tag{3}$$

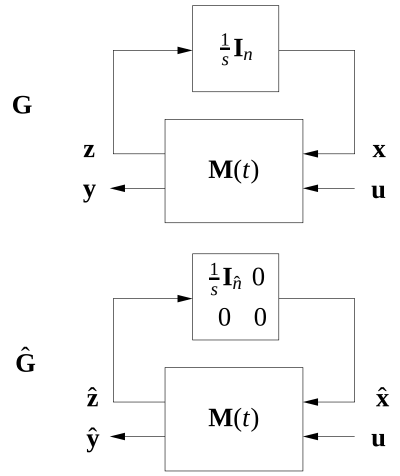

Fig. 1 The systems $\mathbf{G}$ and $\hat{\mathbf{G}}$ viewed as feedback systems: a time-varying matrix is interconnected with a dynamical block containing integrators. It is desired to measure the difference between $\mathbf{y}$ and $\hat{\mathbf{y}}$ when both systems are driven by the same signal $\mathbf{u}$.

with $\dim \mathbf{A}_{11} = \hat{n} \times \hat{n}$. Then use the interconnection

$$\begin{bmatrix} \hat{\mathbf{z}}_1 \\ \hat{\mathbf{z}}_2 \\ \hat{\mathbf{y}} \end{bmatrix} = \mathbf{M}(t) \begin{bmatrix} \hat{\mathbf{x}}_1 \\ \hat{\mathbf{x}}_2 \\ \mathbf{u} \end{bmatrix}, \begin{bmatrix} \hat{\mathbf{x}}_1 \\ \hat{\mathbf{x}}_2 \end{bmatrix} = \frac{1}{s} \begin{bmatrix} \mathbf{I}_{\hat{n}} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{z}}_1 \\ \hat{\mathbf{z}}_2 \end{bmatrix}.$$

Notice that $\hat{\mathbf{x}}_2 = 0$, but $\hat{\mathbf{z}}_2 \neq 0$ in general. A realization for $\hat{\mathbf{G}}$ is the following $\hat{n}$th order system

$$\hat{\mathbf{G}} : \begin{cases} \dot{\hat{\mathbf{x}}}_1 = \mathbf{A}_{11}(t)\hat{\mathbf{x}}_1 + \mathbf{B}_1(t)\mathbf{u}, \quad \hat{\mathbf{x}}_1(0) = 0. \\ \hat{\mathbf{y}} = \mathbf{C}_1(t)\hat{\mathbf{x}}_1 + \mathbf{D}(t)\mathbf{u}. \end{cases} \tag{4}$$

The objective of this paper is to find a low order system $\hat{\mathbf{G}}$ *such that the error* $\|\mathbf{G} - \hat{\mathbf{G}}\|$ *is small, or at least bounded by a computable number.*

3. A TECHNICAL LEMMA

When using balanced truncation on an asymptotically stable time-invariant system, it is assumed that there are diagonal solutions to the controllability and observability Lyapunov equations. In order for the error bound to hold it is enough to have solutions to Lyapunov inequalities, which is proven in for example (Andersson, 1999). When looking for corresponding results for time-variable systems, time-varying Lyapunov inequalities are a natural start:

$$\mathbf{A}(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{A}^T(t) - \dot{\mathbf{P}}(t) + \mathbf{B}(t)\mathbf{B}^T(t) \leq 0 \tag{5}$$
$$\mathbf{Q}(t)\mathbf{A}(t) + \mathbf{A}^T(t)\mathbf{Q}(t) + \dot{\mathbf{Q}}(t) + \mathbf{C}^T(t)\mathbf{C}(t) \leq 0 \tag{6}$$

for all t. If these are solved with equality, $\mathbf{P}$ is the controllability Gramian and $\mathbf{Q}$ the observability Gramian with suitably chosen initial conditions. The following lemma is the first step in establishing an error bound:

LEMMA 1
If there exist solutions

$$\mathbf{P}(t) = \begin{bmatrix} \mathbf{P}_1(t) & 0 \\ 0 & p(t)\cdot \mathbf{I}_{n-\hat{n}} \end{bmatrix}, \tag{7}$$

$$\mathbf{Q}(t) = \begin{bmatrix} \mathbf{Q}_1(t) & 0 \\ 0 & q(t)\cdot \mathbf{I}_{n-\hat{n}} \end{bmatrix} \tag{8}$$

to the Lyapunov inequalities (5) and (6) with $\dim \mathbf{P}_1 = \dim \mathbf{Q}_1 = \dim \mathbf{A}_{11} = \hat{n}\times\hat{n}$, $\mathbf{P}(t) > 0$ and $\mathbf{Q}(t) \geq 0$ for all t, and $\hat{n} \leq n$ the solutions to (1) and (4) satisfy

$$\int_0^T a(t)|\mathbf{y}-\hat{\mathbf{y}}|^2 dt \leq \int_0^T 4b(t)|\mathbf{u}|^2 dt.$$

where $a(t)$ and $b(t)$ are positive and non-increasing scalar functions that satisfy

$$b(t)p^{-1}(t) = a(t)q(t). \tag{9}$$

Proof. The proof is separated into three steps: first the two Lyapunov inequalities are transformed into two scalar inequalities. In the final step they are added and a cross-coupling term is cancelled. Notice that $\mathbf{x}^T = [\mathbf{x}_1^T \quad \mathbf{x}_2^T]^T$ and $\hat{\mathbf{x}}^T = [\hat{\mathbf{x}}_1^T \quad \hat{\mathbf{x}}_2^T]^T$ to conform with $\mathbf{M}(t)$ in (3).

1. (5) may be written as

$$\mathbf{P}^{-1}\mathbf{A} + \mathbf{A}^T\mathbf{P}^{-1} + \frac{d}{dt}(\mathbf{P}^{-1}) + \mathbf{P}^{-1}\mathbf{B}\mathbf{B}^T\mathbf{P}^{-1} \leq 0.$$

Using Schur complements this is equivalent to

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}^T \begin{bmatrix} \mathbf{0} & \mathbf{P}^{-1} \\ \mathbf{P}^{-1} & \frac{d}{dt}\mathbf{P}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \leq \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}. \tag{10}$$

Now multiply (10) from the right with $[\mathbf{x}^T + \hat{\mathbf{x}}^T,\ 2\mathbf{u}^T]^T$, and from the left with the transpose. After some simplifications and using the facts $\hat{\mathbf{x}}_2 = 0$ and $\hat{\mathbf{z}}_2 = \mathbf{A}_{21}\hat{\mathbf{x}}_1 + \mathbf{A}_{22}\hat{\mathbf{x}}_2 + \mathbf{B}_2\mathbf{u}$ this can be written

$$\begin{bmatrix} \dot{\mathbf{x}}_1 + \dot{\hat{\mathbf{x}}}_1 \\ \dot{\mathbf{x}}_2 + \hat{\mathbf{z}}_2 \\ \mathbf{x}_1 + \hat{\mathbf{x}}_1 \\ \mathbf{x}_2 \end{bmatrix}^T \begin{bmatrix} 0 & \mathbf{P}^{-1} \\ \mathbf{P}^{-1} & \frac{d}{dt}\mathbf{P}^{-1} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_1 + \dot{\hat{\mathbf{x}}}_1 \\ \dot{\mathbf{x}}_2 + \hat{\mathbf{z}}_2 \\ \mathbf{x}_1 + \hat{\mathbf{x}}_1 \\ \mathbf{x}_2 \end{bmatrix} \leq 4|\mathbf{u}|^2.$$

Simplifications using the structure of $\mathbf{P}$ and differentiation rules give

$$\frac{d}{dt}|\mathbf{x}_1 + \hat{\mathbf{x}}_1|^2_{\mathbf{P}_1^{-1}} + \frac{d}{dt}|\mathbf{x}_2|^2_{p^{-1}} + 2p^{-1}\hat{\mathbf{z}}_2^T\mathbf{x}_2 \leq 4|\mathbf{u}|^2. \tag{11}$$

2. (6) may be written as

$$\begin{bmatrix} \mathbf{A} \\ \mathbf{I} \end{bmatrix}^T \begin{bmatrix} \mathbf{0} & \mathbf{Q} \\ \mathbf{Q} & \dot{\mathbf{Q}} \end{bmatrix} \begin{bmatrix} \mathbf{A} \\ \mathbf{I} \end{bmatrix} + \mathbf{C}^T\mathbf{C} \leq 0. \tag{12}$$

Multiply (12) from the right with $[\mathbf{x} - \hat{\mathbf{x}}]$ and from the left with its transpose. This can be simplified to

$$\begin{bmatrix} \dot{\mathbf{x}}_1 - \dot{\hat{\mathbf{x}}}_1 \\ \dot{\mathbf{x}}_2 - \hat{\mathbf{z}}_2 \\ \mathbf{x}_1 - \hat{\mathbf{x}}_1 \\ \mathbf{x}_2 \end{bmatrix}^T \begin{bmatrix} \mathbf{0} & \mathbf{Q} \\ \mathbf{Q} & \dot{\mathbf{Q}} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_1 - \dot{\hat{\mathbf{x}}}_1 \\ \dot{\mathbf{x}}_2 - \hat{\mathbf{z}}_2 \\ \mathbf{x}_1 - \hat{\mathbf{x}}_1 \\ \mathbf{x}_2 \end{bmatrix} + |\mathbf{y} - \hat{\mathbf{y}}|^2 \leq 0$$

Using the structure of $\mathbf{Q}$ and differentiation rules we obtain

$$\frac{d}{dt}|\mathbf{x}_1 - \hat{\mathbf{x}}_1|^2_{\mathbf{Q}_1} + \frac{d}{dt}|\mathbf{x}_2|^2_q - 2q\hat{\mathbf{z}}_2^T\mathbf{x}_2 + |\mathbf{y} - \hat{\mathbf{y}}|^2 \leq 0. \tag{13}$$

3. In the expressions (11) and (13) the terms that include $\hat{\mathbf{z}}_2^T\mathbf{x}_2$ are troublesome because they contain coupling between $\mathbf{G}$ and $\hat{\mathbf{G}}$ and are sign indefinite. Notice however that by multiplying (11) by $b(t)$ and (13) by $a(t)$ and adding the inequalities these terms are cancelled with (9) in mind.

If the sum of the inequalities are integrated over $[0, T]$ and the additional assumption of non-increasing a and b the result follows as all terms are positive,

$$\int_0^T a(t)\frac{d}{dt}|\mathbf{x}_1 - \hat{\mathbf{x}}_1|^2_{\mathbf{Q}_1} dt = \left[a(t)|\mathbf{x}_1 - \hat{\mathbf{x}}_1|^2_{\mathbf{Q}_1}\right]_0^T - \int_0^T \dot{a}(t)|\mathbf{x}_1 - \hat{\mathbf{x}}_1|^2_{\mathbf{Q}_1} dt \geq 0,$$

for example. □

4. TIME-VARYING BALANCED REALIZATIONS AND TRUNCATION

In (Shokoohi *et al.*, 1983) balancing of time-variable systems in continuous time is studied. Essentially this means that there is a positive definite matrix $\Sigma(t) = \text{diag}\{\sigma_1(t)\mathbf{I}_{s_1}, \ldots, \sigma_N(t)\mathbf{I}_{s_N}\}$ with $s_1 + s_2 + \cdots + s_N = n$, that fulfills the Lyapunov inequalities

$$\begin{aligned} \tilde{\mathbf{A}}(t)\Sigma(t) + \Sigma(t)\tilde{\mathbf{A}}^T(t) - \dot{\Sigma}(t) + \tilde{\mathbf{B}}(t)\tilde{\mathbf{B}}^T(t) &\leq 0 \\ \Sigma(t)\tilde{\mathbf{A}}(t) + \tilde{\mathbf{A}}^T(t)\Sigma(t) + \dot{\Sigma}(t) + \tilde{\mathbf{C}}^T(t)\tilde{\mathbf{C}}(t) &\leq 0. \end{aligned} \tag{14}$$

To find such Gramians a coordinate transformation is often needed. In (Shokoohi *et al.*, 1983) sufficient conditions for the existence of a Lyapunov transformation, (2), that makes the realization balanced is presented. If there are solutions to (5)-(6) those are related to $\Sigma(t)$ by

$$\begin{aligned} \Sigma(t) &= \mathbf{T}^{-1}(t)\mathbf{P}(t)\mathbf{T}^{-T}(t) \\ \Sigma(t) &= \mathbf{T}^T(t)\mathbf{Q}(t)\mathbf{T}(t). \end{aligned} \tag{15}$$

Thus at each time-instant t, $\mathbf{T}(t)$ can be calculated. If there is a balanced representation of $\mathbf{G}$ Lemma 1 is applicable:

THEOREM 1
Suppose $\mathbf{G}$ has a balanced realization (14) on the interval [0,T] with $\Sigma(t) = \text{diag}\{\Sigma_1(t), \Sigma_2(t)\}$

$$\begin{aligned} \Sigma_1(t) &= \text{diag}\{\sigma_1(t)\mathbf{I}_{s_1}, \ldots, \sigma_r(t)\mathbf{I}_{s_r}\} \\ \Sigma_2(t) &= \text{diag}\{\sigma_{r+1}(t)\mathbf{I}_{s_{r+1}}, \ldots, \sigma_N(t)\mathbf{I}_{s_N}\} \end{aligned}$$

where each diagonal element $\sigma_i(t), \quad i \in [r+1, N]$ is either non-increasing or non-decreasing for all $t \in$

$[0, T]$. The truncated $(s_1 + \cdots + s_r)$-order system $\hat{\mathbf{G}}$ is then balanced by $\Sigma_1(t)$ and

$$\left\|\mathbf{G} - \hat{\mathbf{G}}\right\| \leq 2 \sum_{k=r+1}^{N} \sup_{t\in[0,T]} \sigma_k(t). \quad (16)$$

If the monotonicity condition is violated by some $\sigma_k(t)$ the corresponding term in the sum (16) is replaced by

$$\sqrt{\sigma_k(0)\sigma_k(T)\exp\left(\int_0^T \left|\frac{d}{d\tau}\log\sigma_k(\tau)\right| d\tau\right)}. \quad (17)$$

Proof. First assume the monotonicity condition is valid. Use Lemma 1 with $\mathbf{P}(t) = \mathbf{Q}(t) = \Sigma(t)$. Start by removing the states related to $\sigma_N(t)$, by selecting $p = q = \sigma_N$. By assumption there are two possibilities: $\dot{\sigma}_N(t) \geq 0$ or $\dot{\sigma}_N(t) \leq 0$ for all t. First consider $\dot{\sigma}_N(t) \leq 0$. Then choose $b(t) = \sigma_N^2(t)$ and $a(t) = 1$ in Lemma 1. In the other case choose $a(t) = \sigma_N^{-2}(t)$ and $b(t) = 1$. In both cases it follows that

$$\begin{aligned} \|\mathbf{G} - \mathbf{G}_{N-1}\| &= \sup_{u\neq 0}\left(\frac{\int_0^T |\mathbf{y} - \mathbf{y}_{N-1}|^2 dt}{\int_0^T |\mathbf{u}|^2 dt}\right)^{1/2} \\ &\leq 2 \sup_{t\in[0,T]} \sigma_N(t). \end{aligned}$$

Next notice that $\mathbf{G}_{N-1}$ is still balanced with the rest of $\Sigma(t)$. Thus remove $\sigma_{N-1}(t)$ from $\mathbf{G}_{N-1}$, and then repeat the scheme until the system $\hat{G}$ is reached. Finally notice that

$$\begin{aligned} &\left\|\mathbf{G} - \hat{\mathbf{G}}\right\| = \|\mathbf{G} - \mathbf{G}_r\| = \\ &\|\mathbf{G} - \mathbf{G}_{N-1} + \mathbf{G}_{N-1} + \cdots + \mathbf{G}_{r+1} - \mathbf{G}_r\| \\ &\leq 2 \sum_{k=r+1}^{N} \sup_t \sigma_k(t) \end{aligned}$$

by the triangular inequality.

If $\sigma_N(t)$ is not monotonic an error bound is still obtainable, even though it is large in general. In Lemma 1 choose $\frac{b(t)}{a(t)} = \sigma_N^2(t)$ with $a(t)$ and $b(t)$ decreasing. Such a choice is

$$\begin{aligned} \log a(t) = &-\log\sigma_N(T) + \\ &\int_t^T \frac{d}{d\tau}\log\sigma_N(\tau) + |\frac{d}{d\tau}\log\sigma_N(\tau)| d\tau \end{aligned}$$

and

$$\begin{aligned} \log b(t) = &\log\sigma_N(T) + \\ &\int_t^T -\frac{d}{d\tau}\log\sigma_N(\tau) + |\frac{d}{d\tau}\log\sigma_N(\tau)| d\tau. \end{aligned}$$

This gives the bound

$$\begin{aligned} &\|\mathbf{G} - \mathbf{G}_{N-1}\|^2 \leq \\ &4\sigma_N(0)\sigma_N(T)\exp\left(\int_0^T |\frac{d}{d\tau}\log\sigma_N| d\tau\right). \end{aligned}$$

which leads to (17). □

This is a generalization of the well known balanced truncation formula first derived in (Enns, 1984; Glover, 1984). Here, however, there is no need to worry about stability and the distinctness of σ_i as will be discussed in section 5. The monotonicity conditions on $\Sigma_2(t)$ seem to be hard constraints. In section 6 it will be discussed how the free choice of the boundary conditions can be utilized to fulfill them. If $\Sigma_2(t)$ is restricted to be constant basically the error bound in (Lall *et al.*, 1998) is obtained.

5. INPUT-OUTPUT STABILITY OF TRUNCATED REALIZATIONS

For asymptotically stable *time-invariant* systems $\mathbf{G}$, it is well known that there exists a coordinate system with a balanced realization, i.e. there is a constant diagonal Gramian that fulfills

$$\begin{aligned} \mathbf{A}\Sigma + \Sigma\mathbf{A}^T + \mathbf{B}\mathbf{B}^T &= 0 \\ \Sigma\mathbf{A} + \mathbf{A}^T\Sigma + \mathbf{C}^T\mathbf{C} &= 0. \end{aligned}$$

In Theorem 1 stability is not mentioned explicitly, still it gives predictions of input-output stability if $T \to \infty$. This might seem to be a contradiction to known time-invariant results. There are examples of truncated time-invariant balanced realizations that loses asymptotic stability when $\sigma_r = \sigma_{r+1}$, and have poles on the imaginary axis, see for example (Zhou and Doyle, 1998). However, with Lemma 1 it is seen that those poles are unobservable or uncontrollable:

THEOREM 2
Assume the realization (1) is balanced and asymptotically stable. Then every truncated realization (4) constructed from removing a *monotonic* Σ_2 is input-output stable, and $\hat{\mathbf{x}}_1(t)$ is bounded.

Proof. The result will here be proven for constant Σ_2. The proof for time-varying monotonic Σ_2 is essentially the same but is a bit more technical.

For asymptotically stable and completely controllable and observable systems (1), $\Sigma(t)$ is positive definite and bounded for all t. In Lemma 1 choose

$$\begin{aligned} &\mathbf{P}_1 = \mathbf{Q}_1 = \Sigma_1 = \text{diag}\{\sigma_1(t)\mathbf{I}_{s_1}, \ldots, \sigma_{N-1}(t)\mathbf{I}_{s_{N-1}}\}, \\ &p = q = \sigma_N = 1 \end{aligned}$$

without loss of generality. By following the steps in the proof of Lemma 1

$$\begin{aligned} \frac{d}{dt}|\mathbf{x}_1 + \hat{\mathbf{x}}_1|^2_{\Sigma_1^{-1}} + \frac{d}{dt}|\mathbf{x}_1 - \hat{\mathbf{x}}_1|^2_{\Sigma_1} + 2\frac{d}{dt}|\mathbf{x}_2|^2 + \\ |\mathbf{y} - \hat{\mathbf{y}}|^2 \leq 4|\mathbf{u}|^2 \end{aligned} \quad (18)$$

is obtained. Next integrate the inequality and use the fact the system is at rest at $t = 0$. Also use the inequality $|a+b|^2 \geq \frac{1}{2}|a|^2 - |b|^2$:

$$\frac{1}{2}|\hat{\mathbf{x}}_1(T)|^2_{\Sigma_1^{-1}+\Sigma_1} + \int_0^T |\mathbf{y}-\hat{\mathbf{y}}|^2 dt \leq$$

$$\int_0^T 4|\mathbf{u}|^2 dt + 2|\mathbf{x}_1(T)|_{\Sigma_1^{-1}+\Sigma_2} - 2|\mathbf{x}_2(T)|^2.$$

If input signals of finite length in $[0,\tau]$ and finite energy are applied, then $\int_0^T |\mathbf{u}|^2 dt \leq K$ for all $T \geq \tau$. From the asymptotic stability of (1): $\mathbf{x}_1(T) \to 0$ and $\mathbf{x}_2(T) \to 0$ as $T \to \infty$. Thus the left hand side of the inequality is bounded by $4K$. Notice that $\int_0^T |\mathbf{y}-\hat{\mathbf{y}}|^2 dt$ is increasing with T. Then it can be concluded that $\hat{\mathbf{x}}_1(T)$ is bounded for all T and $\hat{\mathbf{y}}(t) \to \mathbf{y}(t) \to 0$. Thus (4) is input-output stable. □

6. NUMERICAL EXAMPLE

Here Theorem 1 will be applied on a simple time-varying system.

EXAMPLE 1—TIME-VARYING BALANCING AND REDUCTION
The original second-order system $\mathbf{G}$ has the following realization

$$\begin{aligned} \mathbf{A}(t) &= \begin{pmatrix} e^t & 1 \\ 1 & 2-e^t \end{pmatrix}, \quad \mathbf{B}(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \mathbf{C}(t) &= \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad D(t) = 0, \end{aligned} \tag{19}$$

on the interval $t \in [0,1]$. The first step of the reduction is to find a suitable coordinate system. (5)-(6) are integrated with equality, as linear matrix inequalities are more expensive to solve. Then $\Sigma(t) = \text{diag}\{\sigma_1(t), \sigma_2(t)\}$ is computed, as

$$\mathbf{P}(t)\mathbf{Q}(t) = \mathbf{T}(t)\Sigma^2(t)\mathbf{T}^{-1}(t), \tag{20}$$

i.e. the eigenvalues of $\mathbf{P}(t)\mathbf{Q}(t)$. It is desired that at least $\sigma_2(t)$ is monotone. Here it will be made decreasing. An ad hoc procedure to obtain this can be described as follows: First choose a small $\mathbf{Q}(1)$,

$$\mathbf{Q}(1) = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix},$$

for example. Then integrate (6) backwards in time. When $\mathbf{Q}(0)$ is known any $\Sigma(0)$ can be assigned, and a corresponding $\mathbf{P}(0)$ can be found from (20). Then integrate (5) forward in time. After this is done, check if $\sigma_2(t)$ is decreasing. If not, $\Sigma(0)$ can be increased and the procedure repeated. Here

$$\mathbf{P}(0) = \begin{pmatrix} 1.63 & 0.65 \\ 0.65 & 0.87 \end{pmatrix}.$$

solves the problem. There is no proof that this will work in general or even generate good approximations, and there are many variations on this theme. Better understanding and methods to obtain solutions is a topic for future research.

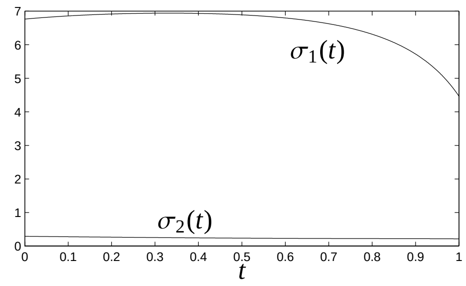

Fig. 2 The elements of $\Sigma(t)$ in Example 1. $\sigma_2(t)$ is decreasing.

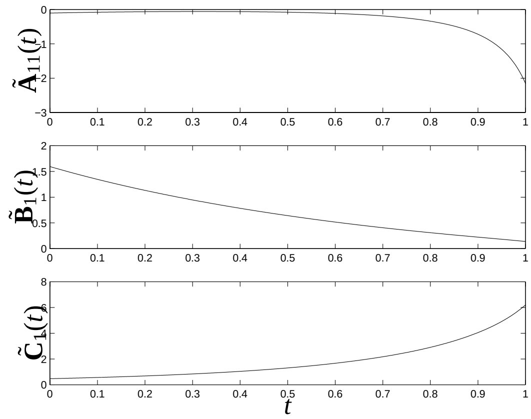

Fig. 3 The realization of the balanced truncated first order system $\hat{\mathbf{G}}$.

The resulting elements of $\Sigma(t)$ are seen in Figure 2. After a suitable solution $\Sigma(t)$ has been found, a balancing $\mathbf{T}(t)$ is calculated at time instants t, in the same way as for time-invariant balancing, see (Zhou and Doyle, 1998). Now $(\mathbf{A},\mathbf{B},\mathbf{C}) \xrightarrow{\mathbf{T}} (\tilde{\mathbf{A}},\tilde{\mathbf{B}},\tilde{\mathbf{C}})$. The realization $(\tilde{\mathbf{A}}_{11},\tilde{\mathbf{B}}_1,\tilde{\mathbf{C}}_1)$ is shown in Figure 3. From Theorem 1 it is known that

$$\left\|\mathbf{G}-\hat{\mathbf{G}}\right\| \leq 2 \sup_{t\in[0,1]} \sigma_2(t) = 0.58.$$

To see how conservative this estimate is, the real value of $||\mathbf{G}-\hat{\mathbf{G}}||$ can be calculated by a bisection algorithm that involves repeated solving of time-varying Riccati equations, see for instance (Tadmor, 1990). In this case it is concluded that

$$0.057 < \left\|\mathbf{G}-\hat{\mathbf{G}}\right\| < 0.058.$$

Thus the estimate is a factor 10 too large. To see how well $\hat{\mathbf{G}}$ works in practice a step response test is performed, which is seen in Figure 4. As seen the reduced model is very close to the original system. Furthermore,

$$||\mathbf{u}||_2 = 1, \quad ||\mathbf{y}-\hat{\mathbf{y}}||_2 = 0.054$$

which is close to the worst possible case. It is concluded that the error was overestimated, but the procedure generated a fine approximation. □

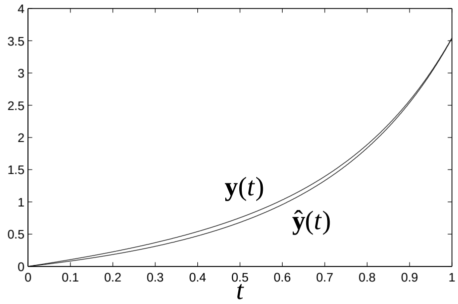

Fig. 4 Step output response of original system **G** and reduced system $\hat{\mathbf{G}}$.

7. CONCLUSIONS

Balanced truncation of time-varying systems has been studied and an error bound for truncated realizations was obtained. The proof was relatively direct and also gave insight in the stability issue of truncated models. The distinctness of the elements in Σ is not necessary to obtain input-output stability of reduced models. Thus Σ_1 and Σ_2 may have entries in common. The bound also introduces a monotonicity condition on $\Sigma_2(t)$, which gives more freedom in the search of solutions than the bound presented in (Lall *et al.*, 1998) does.

How to obtain suitable solutions to the Lyapunov inequalities in general should be further studied. To further understand the monotonicity condition is also of interest.

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