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SINGULAR LQ SIGNAL TRACKER - A DATA-BASED SYNTHESIS APPROACH

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Abstract: An LQ optimal tracker will achieve zero tracking error when the input penalty is removed. However, the problem becomes singular, and a solution hides behind several coordination transformations. We will highlight a discrete time version of this singular LQ optimization. We will also present data-based formulas to design this optimal tracker at the absence of a parametric plant model. We will show that numerical formulas which use input and output data from open-loop system tests are constructed in the way as to eliminate the need for explicit knowledge of plant model parameters. *Copyright © 2002 IFAC*

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1. INTRODUCTION

In many control system synthesis, it is often desirable to have the system output follows certain reference signal. Normally, this synthesis will involve change of system poles through feedback designs and also change in system zeros through feedforward compensations. Moreover, explicit knowledge of a parametric plant model must first be estimated, often from input and output data through system identification processes (Åström and Wittenmark, 1989; Ljung, 1987). However, by performing the model estimation and the controller design in two separate steps, we have complicated the whole task considerably. It is also true that "identification and control are often performed with independent, and possibly noncoherent, working criteria, hence necessitates a complicate joint design process to reach an acceptable final design" (Gevers, 1983).

Recently, Chan (1995a, 1995b, 1994) has achieved data-based synthesis of signal trackers that totally obviate the need for explicit knowledge of plant model parameters. However, these methods requires that the system is open-loop stable, and the designs are not optimal.

On the other hand, data-based formulas that enables linear LQ regulator design with unknown plant model has also been developed (Chan, 1996; Chan, 2000). In this work, we extend the previous data-based LQR formulations for an optimal signal tracker design.

The proposed LQ tracker design different from the data-based LQR synthesis in three folds. First, the output penalty of the LQR synthesis is replaced with the tracking error penalty. Also, because an exact signal tracking is sought, penalty in the control input is set to zero. However, setting the control input penalty to zero causes the optimization problem to become singular. In order to resolve the difficulties, several mathematical transformations are proposed for continuous time systems (Kelly, 1964; Goh, 1967; Speyer, 1971; Jacobson, 1971; Moore, 1971), and later for discrete time systems as well (Chan, 1986). The procedure leading to the minimization of this new functional is termed singular linear quadratic (SLQ) optimization. In this paper, a discrete-time SLQ optimization is highlighted. Secondly, omitting the input penalty also generates numerical difficulty for the data-based LQ synthesis, when the plant is inverse unstable. Resolving this numerical problem will require modifications to the previous data-based LQ formulas. Thirdly, construction of a data-based signal tracking pre-filter will be added. This data-based pre-filter will achieve perfect signal tracking in finite time.

2. LQ TRACKER FOR SISO DISCRETE SYSTEMS

2.1 The singular LQ formulation.

Consider the following SISO discrete system:

$$\mathbf{x}(k+1) = A\mathbf{x}(k) + Bu(k), \quad y(k) = C\mathbf{x}(k) \quad (1)$$

where y is the output, u is the input, $\mathbf{x}$ is a $n \times 1$ state vector and A, B and C are constant matrices. We seek a signal tracker design by minimizing, with respect to the control input u and subject to the constraint equations (1), a cost functional

$$J = \frac{1}{2}\sum_{k=0}^{p}(y_f(k) - y(k))^2 \quad (2)$$

where $y_f(k)$ is the signal to be tracked and p is the final time. Note that we have set the tracking error penalty to unity and omitted the control penalty from the standard LQ formulation, in order to fully exploit the control input resources available to perfect the signal tracking performance.

However, the Euler-Lagrange equations for this functional becomes linear in $u(k)$, and will yield no information on the optimal $u(k)$. In other words, the problem is singular. The difficulty can be resolved through mathematical transformations on the defined problem, a highlight of this technique is quoted from Chan (1986) as follows.

2.2 Highlights of a discrete SLQ optimization.

Introduce the following change of variable transformation for $q = 1, 2, \cdots$ (Chan, 1986):

$$u_q(k) = u_{q-1}(k-1),\ B_q = A^qB,\ E_q = C^TCB_{q-1},$$
$$R_q = (CB_{q-1})^2,\ \mathbf{x}_q(k) = \mathbf{x}_{q-1}(k) - B_{q-1}u_q(k)$$

where $\mathbf{x}_0(k) = \mathbf{x}(k)$, $u_0(k) = u(k)$ and $B_0 = B$. Then, the plant dynamics will be transformed into

$$\mathbf{x}_q(k+1) = A_*\mathbf{x}_q(k) + B_qu_*(k), \quad (3)$$

and the transformed cost functional becomes

$$J = \frac{1}{2}\sum_{k=0}^{p}\left(\mathbf{x}_q(k)^TQ_*\mathbf{x}_q(k) + u_*(k)^2R_q\right) \quad (4)$$

where

$$A_* = A - B_qR_q^{-1}E_q^T, \quad Q_* = C^TC - E_qR_q^{-1}E_q^T$$

and

$$u_*(k) = u_q(k) + R_q^{-1}E_q^T\mathbf{x}_q(k).$$

The transformed J will be quadratic in u_*, and the problem is reduced to a standard LQ formulation, when $CB_{q-1} \neq 0$ for some $q > 0$. This value of q is then called the *order of singularity* of the formulation. For single variable systems, this q equals the relative order of the z-domain transfer function $C(zI - A)^{-1}B$ (Chan, 1986).

However, the transformed system is not controllable, and the existence of a steady state solution is not ensured. To resolve the difficulties, an order reduction transformation is introduced as follows:

$$\mathbf{x}_q(k) = \Omega\begin{bmatrix}\mathbf{x}_r(k)\\ \mathbf{x}_d(k)\end{bmatrix},\ \Omega = [\,\Phi_1 \quad \Phi_2\,] \quad (5)$$

where $\Phi_2 = [\,B_{q-1} \quad \cdots \quad B_1 \quad B\,]$ and Φ_1 is a $n \times (n-q)$ matrix which makes Ω full rank. Also, the dimension of $\mathbf{x}_r$ is $n - q$ and that of $\mathbf{x}_d$ is q.

It can shown that $\mathbf{x}_d$ corresponds to the q uncontrollable null poles of $\mathbf{x}_q$. Moreover, the following independent dynamics of $\mathbf{x}_r$ and the associating reduced order LQ problem are defined:

$$\mathbf{x}_r(k+1) = A_r\mathbf{x}_r(k) + B_ru_r(k) \quad (6)$$

and

$$J_r = \frac{1}{2}\sum_{k=0}^{p}\left(\mathbf{x}_r(k)^TQ_r\mathbf{x}_r(k) + u_r(k)^2R_q\right) \quad (7)$$

where

$$u_r(k) = u_q(k) + R_q^{-1}E_q^T\Phi_1\mathbf{x}_r(k) = u_*(k),$$
$$A_r = \Psi A_*\Phi_1, \quad Q_r = \Phi_1^TQ_*\Phi, \quad B_r = \Psi_1B_q$$

and Ψ is the upper $n - q$ rows of Ω^{-1}. Then, the reduced matrix pair, (A_r, B_r), is controllable, and a steady state LQ solution is ensured (Chan, 1986).

2.3 The controller structure and the closed-loop system of a SLQ optimization

By solving the LQ problem of $\mathbf{x}_r$ and transform the solution back to space of $\mathbf{x}$, we obtain the following steady state SLQ control law:

$$u(k) = -K\mathbf{x}(k) + \mathcal{F}\{y_f(k)\} \quad (8)$$

where K is an $1 \times n$ constant feedback gain and $\mathcal{F}\{\,\}$ is a dynamic filtering on $y_f(k)$.

In general, the closed-loop system of (8) will be

$$y(z) = \bar{h}_{cl}(z)\bar{r}(z)$$

where $\bar{r}(z)$ is the Z-transform of $\mathcal{F}\{y_f(k)\}$ and

$$\bar{h}_{cl}(z) = C(zI - A + BK)^{-1}B.$$

For SISO systems, we also have $Q_r = 0$ and

$$|zI - A_r| \times b_q = \mathbf{b}(z) = \begin{vmatrix} zI - A & -B \\ C & 0 \end{vmatrix},\ b_q \in \mathcal{R}.$$

Consequently, if we decompose $\mathbf{b}(z)$ into a real gain b_q, a monic and Hurwitz portion $\mathbf{b}_*(z)$ and another monic portion, $\mathbf{b}^{\circ}(z)$, which contains only unstable roots, then the following property of $\bar{h}_{cl}(z)$ is obtained from a symmetrical root locus analysis (Vaughan, 1975)

$$\bar{h}_{cl}(z) = \frac{\mathbf{b}(z)}{z^q \mathbf{b}_{\circ}(z)\mathbf{b}_{*}(z)} = \frac{b_q \mathbf{b}^{\circ}(z)}{z^q \mathbf{b}_{\circ}(z)} \qquad (9)$$

where $\mathbf{b}_{\circ}(z)$ is a Hurwitz polynomial derived from $\mathbf{b}^{\circ}(z)$ by inverting all of its roots. In addition, the factor, z^q, in the denominator corresponds to the q null poles of $\mathbf{x}_d$. The presence of this factor also indicates that this is a true SLQ result. However, unlike a continuous SLQ optimization, in which a true SLQ result would resort to bang-bang type control laws, a finite feedback gain is maintained in a discrete SLQ solution (Chan, 1986).

The following dynamics of the feedforward filter also result from the SLQ optimization:

$$\bar{r}(z) = \frac{\bar{y}_f(z)}{z^q \bar{h}_{cl}(z)} = \frac{\mathbf{b}_{\circ}(z)}{b_q \mathbf{b}^{\circ}(z)} \bar{y}_f(z), \qquad (10)$$

or

$$\mathcal{F}\{y_f(k)\} = \mathcal{Z}^{-1}\Big\{ \frac{\mathbf{b}_{\circ}(z)}{b_q \mathbf{b}^{\circ}(z)} \bar{y}_f(z) \Big\} \qquad (11)$$

where $\mathcal{Z}^{-1}\{\ \}$ denotes the inverse z-transform. As a result, a perfect signal tracking is obtained.

When the plant is of minimum phase, we will have $\mathbf{b}_{\circ}(z) = \mathbf{b}^{\circ}(z) = 1$; hence, $\mathcal{F}\{y_f(k)\} = b_q^{-1} y_f(k)$ for all k. In this case, signal tracking will be achieved in q steps. For nonminimum phase plants, (11) defines an unstable filtering of $y_f(k)$. In order to implement this filter, a finite-horizon reversed-time filtering technique (Chan, 1986) can be used. In this case, future values of $y_f(k)$ are needed and the signal tracking will be achieved asymptotically.

3 DATA-BASED SYNTHESIS OF THE SLQ OPTIMAL DESIGN

3.1 Foreword.

We can see from (8) that a SLQ design consists of two parts: feedback regulator, $-K\mathbf{x}(k)$, and feedforward tracker, $\mathcal{F}\{y_f(k)\}$. The state feedback portion can be realized with the following output feedback design (Chan, 1996):

$$\bar{u}(z) = -\frac{\mathbf{f}(z)}{\mathbf{g}(z)}\bar{y}(z) + \frac{z^m}{\mathbf{g}(z)}\bar{r}_f(z), \qquad (12)$$

$$\mathbf{f}(z) = \sum_{i=0}^{m} f_i z^{m-i}, \quad \mathbf{g}(z) = z^m + \sum_{i=1}^{m} g_i z^{m-i}$$

where $\bar{u}(z)$, $\bar{y}(z)$ and $\bar{r}_f(z)$ are the z-transforms of $\{u(k)\}$, $\{y(k)\}$ and $\mathcal{F}\{y_f(k)\}$, and f_i and g_i are real constants to be computed as follows:

$$\begin{aligned}&[\, g_1 \ \cdots \ g_m \ \ f_0 \ \cdots \ f_m \,] \\ &= -[\, u_{slq}(1) \ \cdots \ u_{slq}(\mathbf{k}) \,]\, W \ [W^T W]^{-1}\end{aligned} \qquad (13)$$

where

$$W = \begin{bmatrix} u_{slq}(0) & \cdots & u_{slq}(-m) & y_{slq}(1) & \cdots & y_{slq}(1-m) \\ & & \vdots & & & \vdots \\ u_{slq}(\mathbf{k}-1) & \cdots & u_{slq}(\mathbf{k}-m) & y_{slq}(\mathbf{k}) & \cdots & y_{slq}(\mathbf{k}-m) \end{bmatrix}.$$

Note that $\mathbf{k}$ is some integer larger than $2m+1$, and $u_{slq}(k)$ and $y_{slq}(k)$ are the output response and the input response of the SLQ closed-loop system to an unit pulse command. Moreover, the closed-loop stability of this design is ensured, if the following criterion is satisfied (Chan, 1996):

$$\xi = \max_{k=1}^{\mathbf{k}} \big\| \varepsilon(k) \big\| < 1 \qquad (14)$$

where $\{\varepsilon(k)\}$ is the discrete Fourier transform of the sequence $\{e(k)\}$, defined as follows:

$$\begin{aligned}&[\, e(1) \ \cdots \ e(\mathbf{k}) \,] \\ &= [\, u_{slq}(1) \ \cdots \ u_{slq}(\mathbf{k}) \,] \\ &\quad \times \big[I + W \ [W^T W]^{-1} \, W^T \,\big].\end{aligned} \qquad (15)$$

This criterion is used in designs using trail values of m, a procedure that is necessary due to the absence of a parametric plant model.

We can see that neither (13) nor (14) involve explicit knowledge of the plant model. However, implementation of (13) and (14) requires data of $u_{slq}(k)$ and $y_{slq}(k)$. In the absence of a system model, these data are prepared from open-loop plant test response through data-based computations. For LQR designs, formulas for this computation has been developed (Chan, 1996). For the SLQ synthesis, modifications to data-based LQR formulas are essential to cope for the missing of input penalty in SLQ optimization. The modified formulas for a data-based SLQ computation is shown in Section 3.2. The other unique feature of this work, a data-based version of the SLQ tracker design then follows, in Section 3.3.

3.2 Data-based SLQ computation.

Let the input and the output data of an plant test be denoted as $\{r_c(k)\}$ and $\{y_c(k)\}$, respectively. Then, data of $\{u_{slq}(k)\}$ and $\{y_{slq}(k)\}$ can be computed from $\{r_c(k)\}$ and $\{y_c(k)\}$ as follows:

$$\begin{bmatrix} u_{slq}(1) \\ \vdots \\ u_{slq}(\mathbf{k}) \end{bmatrix} = -\Big(H^T H\Big)^{-1} H^T \begin{bmatrix} h(2) \\ \vdots \\ h(p) \end{bmatrix} \qquad (16a)$$

and

$$\begin{bmatrix} y_{slq}(1) \\ \vdots \\ y_{slq}(\mathbf{k}) \end{bmatrix} = P_{\mathbf{k}} \begin{bmatrix} u_{slq}(0) \\ \vdots \\ u_{slq}(\mathbf{k}-1) \end{bmatrix} \qquad (16b)$$

where $\mathbf{k} < p-1$ for both equations and $u_{slq}(0) = 1$ for (16b). In addition,

$$H = \begin{bmatrix} h(1) & & h(2-\mathbf{k}) \\ \vdots & & \vdots \\ h(p-1) & \cdots & h(p-\mathbf{k}) \end{bmatrix},$$

$$\begin{bmatrix} h(1) \\ \vdots \\ h(p) \end{bmatrix} = \begin{bmatrix} r_c(0) & & \\ \vdots & \ddots & \\ r_c(p-1) & \cdots & r_c(0) \end{bmatrix}^{-1} \begin{bmatrix} y_c(1) \\ \vdots \\ y_c(p) \end{bmatrix}$$

and $P_{\mathbf{k}}$ is a subscript dependent square convolution matrix of $\{h(k)\}$ defined as follows:

$$P_l = \begin{bmatrix} h(1) & & \\ \vdots & \ddots & \\ h(l) & \cdots & h(1) \end{bmatrix},\ l = 1, 2, 3 \cdots.$$

Equation (16a) is derived from modification of a data-based LQR formula (Chan, 1996) in which $\left(H^T H\right)^{-1}$ takes on a general form $\left(\mathbf{q}P_p^T P_p + \mathbf{r}I\right)^{-1}$ where $\mathbf{r} > 0$ and $\mathbf{q} \geq 0$ are scalars, I is an identity matrix and P_p is the P_l at $l = p$. Besides setting $\mathbf{r} = 0$ and $\mathbf{q} = 1$, the key point in modifying the LQR formulation into (16a) is the omission of the last $p - 1 - \mathbf{k}$ columns from P_p. This arrangement is equivalent to forcing $u_{slq}(k)$ to vanish for all $k > \mathbf{k}$, an arrangement that is necessary to make (16a) valid for nonminimum phase systems where an unrestricted SLQ solution, the inverse of a square P_p, will grow indefinitely with p. Moreover, the existence of $\left(H^T H\right)^{-1}$ is ensured even if the first few $h(k)$ are zero. As a result, the formula is applicable for systems with an arbitrary (and unknown) relative order.

3.3 Data-based SLQ tracker design

A data-based SLQ feedforward tracker is synthesized here. This synthesis will use the SLQ response data computed in Section 3.2.

From (14), we have

$$\mathcal{Z}\Big\{\mathcal{F}\{y_f(k)\}\Big\} = \frac{1}{z^q\,\bar{h}_{cl}(z)}\bar{y}_f(z). \qquad (17)$$

Denote $\mathbf{b}^\circ(z) = (z-\gamma_1^\circ)\cdots(z-\gamma_\ell^\circ)$ for some $\ell \leq q$, then we also infer from (13) that

$$\frac{1}{z^{-q}\,\bar{h}_{cl}(z^{-1})} = \frac{\prod_{i=1}^{\ell} \frac{1}{z} - \frac{1}{\gamma_i^\circ}}{b_q \prod_{i=1}^{\ell} \frac{1}{z} - \gamma_i^\circ} = \frac{1}{d^2} z^q\,\bar{h}_{cl}(z) \quad (18)$$

where

$$d = b_q \times \gamma_1^\circ \times \cdots \times \gamma_\ell^\circ.$$

These equations indicate that the reversed-time transfer function of $\mathcal{F}\{y_f(k)\}$ is proportional to $\bar{h}_{cl}(z)$. In addition, $\{y_{slq}(k)\}$ is in fact a time sequence of $\bar{h}_{cl}(z)$. As a result, if we denote $r_f(k)$ as the filtered sequence of $\mathcal{F}\{y_f(k)\}$ and $y_{slq}(q)$ the first nonzero data of $\{y_{slq}(k)\}$, then the following reversed-time convolution formula is inferred:

$$r_f(k) = \frac{1}{d^2} \sum\nolimits_{i=0}^{\infty} y_{slq}(q+i) y_f(k+i). \qquad (19)$$

Also, the constant d can be computed as follows:

$$d = \|\,\bar{h}_{cl}(z)\,\|_{z=1} = \sum\nolimits_{k=0}^{\infty} y_{slq}(k). \qquad (20)$$

In practice, $y_{slq}(k)$ will become negligible for all $k > \mathbf{k}$. Hence, a finite-horizon implementation of (19) may be formulated as follows:

$$r_f(k) = \frac{1}{d^2} \sum\nolimits_{i=0}^{\mathbf{k}} y_{slq}(q+i) y_f(k+i). \qquad (21)$$

Note that (21) is applicable also for minimum phase systems. In that case, we have $y_{slq}(k) = 0$ for all $k \neq q$; hence, $r_f(k) = y_f(k)/y_{slq}(q)$.

It is also possible to rewrite (21) into as follows:

$$r_f(k) = \frac{1}{d^2} \sum\nolimits_{i=0}^{\mathbf{k}} y_{slq}(q+i) y_f(k+i+q). \quad (22)$$

This is equivalent to multiply (10) with z^q, making $\mathcal{F}\{y_f(k)\}$ a noncausal filter. In (22), future values of $y_f(k)$ are required even if the system is of minimum phase. However, the design will be benefited in that a steady-state signal-tracking will be achieved without time delay. In addition, an estimate of q is obtained from the first nonzero $h(k)$.

4. A DESIGN EXAMPLE

The proposed method is tested on an example plant defined by the following open-loop transfer function:

$$\bar{h}_{ol}(z) = \frac{z^2 + 1.1z + 1.25}{z^4 + 0.9z^3 + 0.8z^2 + 0.7z + 0.6}.$$

We assume that $\bar{h}_{ol}(z)$ is not explicitly known.

The plant response to $r_c(k) \equiv 1$ is generated. A set of $\{y_c(k)\}$ up to $p = 200$ is collected. A set of $\{u_{slq}(k)\}$ and a set of $\{y_{slq}(k)\}$ are then computed from (16a) and (16b), with $\mathbf{k} = 120$.

A trial process for the computation of f_i and g_i has the following results:

$m =$	1	2	3
$\xi =$	2.95	1.27	0

Obviously, a third order controller is the solution; its parameters are as follows:

$$\mathbf{f}(z) = -0.1159z^3 + 0.0954z^2 + 0.04z + 0.3708$$

and

$$\mathbf{g}(z) = z^3 - 0.02z^2 + 0.1339z - 0.7724.$$

This set of $\{u_{slq}(k)\}$ and $\mathbf{f}(z)$ and $\mathbf{g}(z)$ match a model-based design using MATLAB.

A test with noisy data is also included by injecting into the data of $\{y_c(k)\}$ an unbiased and pseudo-random noise; the root-mean-square noise-to-signal ratio of the data is 1.2%. Following the same procedure of the noise free test, the following results are obtained from a trial process for the computation of f_i and g_i:

$m =$	1	2	3
$\xi =$	1.67	116	0.04

Note that the presence of data noise causes $\xi \neq 0$ even at $m = 3$. However, an abrupt drop in ξ is still evident at $m = 3$. As a result, the following 3rd order controller is chosen:

$$\mathbf{f}(z) = -0.1202z^3 + 0.0973z^2 + 0.0429 + 0.368$$

and

$$\mathbf{g}(z) = z^3 - 0.0223z^2 + 0.1364z - 0.7709.$$

It is also observed that data of $\{y_{slq}(k)\}$ decays down to the noise level for $k > 60$. As a result, the following signal-tracking SLQ controller is formed:

$$\begin{aligned} u(k) &= 0.0223u(k-1) - 0.1364u(k-2) \\ &+ 0.7709u(k-3) - 0.1202y(k) \\ &+ 0.0973y(k-1) + 0.0429y(k-2) \\ &+ 0.368y(k-3) + \frac{1}{d^2}\sum_{i=0}^{60} y_{slq}(i+1)y_f(k+i) \end{aligned}$$

and

$$d = \sum\nolimits_{l=1}^{60} h_{cl}(l).$$

The result of the noisy data test is verified through a closed-loop simulation with $y_f(k) = \sin(0.1k)$. The result shows a maximum tracking error of less than 2% and a root-mean-square tracking error of less than 0.35%. These results are quite encouraging, in view of the fact that the row data used in the controller design has a noise-to-signal ratio of 1.2%. A finite and bounded control input is also recorded, with $\max_k \| u(k) \| < 1.2$.

5. CONCLUDING REMARKS

A method for computing a discrete time SLQ output tracking controller for single variable systems has been presented. In this method, the controller is computed directly from input and output data from an open-loop system test in such a way that explicit identification of plant model parameters becomes unnecessary. As a result, SLQ tracker design and open-loop system testing can be combined into one procedure; thereby, greatly reducing the effort and the time spent for the task.

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