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A ROBUST OUTPUT FEEDBACK CONTROL LAW FOR MIMO PLANTS

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Abstract: In this paper a new robust control strategy for MIMO LTI plants is presented. The proposed approach results in an output feedback and the resulting control system exhibits strong robustness properties. The derivation of the control policy is based on the mathematical machinery of the Singular Perturbation Theory. More specifically, a high order sliding motion is imposed, and a suitable definition of a time-varying sliding manifold allows the controlled plant output to robustly track a prescribed reference trajectory. Simulations are presented considering a classic IFAC benchmark problem, namely a three-input-three-output binary distillation column with an unmatched disturbance and bounded control variables. The results of the simulation show the effectiveness of the proposed approach.

Keywords: Robust control, Sliding manifold, Output feedback

1. INTRODUCTION

Robustness is a key issue in automatic control applications. Generally, practical control system design suffers from model errors, parameter uncertainties, unmodeled dynamics and unknown disturbances that must be compensated for by a suitable feedback strategy. This is why almost all of the closed-loop strategies must face the problem of robustness. In the last three decades this problem has been attacked from many directions. For instance, robustness against structured (Ackermann, 1997) and unstructured (Dorato, 1992) uncertainties has been considered, leading to powerful conditions for the design of robustly stable feedback controllers (i.e. H_2 and H_∞ control design strategy). At the end of the 1970s, the LTR procedure showed how it is possible to recover the excellent LQ robustness properties when using an output feedback based on a state observer. Sliding manifold control strategies (Utkin, 1978) are also devoted to confer strong robustness properties. The main advantage of these strategies is their ability to deal with MIMO and nonlinear plants subject to severe uncertainties. The basic idea is to split the design of the control action into two subproblems: first the definition of the desired behavior of the closed-loop system by assigning a suitable sliding surface (Zinober, 1990) onto which the system state evolution has to be confined, next the design of a control action that steers the system state towards the manifold. As a consequence, when the system slides on the manifold its behaviour is the nominal one, and robustness is automatically guaranteed. In order to force the state evolution onto the manifold different approaches ar possible, e.g. VSC (Utkin, 1992) or high-gain strategies (Young *et al.*, 1977), with both state and output feedback.

There is an unavoidable drawback of output feedback: asymptotic tracking requires minimum phase plants (Grizzle *et al.*, 1994). The use of a static output feedback has been suggested (Zak and Hui, 1993), however, limitations of this approach have been shown (Edwards and Spurgeon, 2000), the most severe limitation being that in some cases the choice of the sliding manifold is no longer independent of that of the control law. In order to overcome this drawback, one must resort to dynamic output feedback (Kwan, 1996).

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The contribution of this paper is a dynamic output feedback strategy based on a high order sliding manifold approach (see (Friedman and Levant, 1994), (Levant, 1997)). The proposed approach uses a high-gain strategy based on the Singular Perturbation Theory (Kokotovic *et al.*, 1986). Moreover, the main drawback affecting high-gain techniques, namely the peaking phenomenon, is avoided by defining a suitable time-varying sliding manifold. In the basic approach, presented in (Cavallo *et al.*, 1993), and developed in (Cavallo *et al.*, 1999*a*), (Cavallo *et al.*, 1999*b*), a state feedback control was presented, leading to a "fast" control variable, while the system state was "slow". Thus, the feedback system exhibited no reaching phase in the state, without peaking, and only a quick transient in the control signal. When using full state feedback, first order sliding manifold approaches suffice for defining the behavior of the closed-loop system. However, when considering output feedback, high order sliding strategies must be investigated. Due to the high gain strategy employed, the assumption of minimum phase plant is required.

A case study is presented to show the effectiveness of the proposed strategy to control a distillation column.

2. CONTROL STRATEGY

Consider a MIMO plant with r inputs and r outputs and Smith-McMillan degree n

$$\dot{x} = Ax + Bu + \chi \qquad (1)$$

$$y = Cx. \qquad (2)$$

where $y \in \mathbb{R}^r$ is the system output, $u \in \mathbb{R}^r$ is the control input, $x \in \mathbb{R}^n$ is the system state and $\chi \in \mathbb{R}^n$ is a term taking into account disturbances and model errors. Let H_k be the k-th Markov parameter of the system, i.e., $H_k = CA^{k-1}B$.

For a given integer ρ, a sliding manifold based on the system output is defined as

$$= \left\{ (x,t) \in \mathbb{R}^n \times \mathbb{R}_+ \; : \; \sigma^{(k)}(y,t) = 0, k = 0, \ldots, \rho - 1 \right\} \qquad (3)$$

where $\sigma : \mathbb{R}^r \times \mathbb{R}_+ \to \mathbb{R}^r$ and $\eta(t)$ are

$$\sigma(y,t) = -y + \eta(t), \qquad (4)$$

$$\eta(t) = e^{Wt} \sum_{i=0}^{\rho-1} c_i \frac{t^i}{i!} \qquad (5)$$

and W is a Hurwitz $r \times r$ real matrix to be suitably selected, while $c_i, i = 0, \ldots, \rho - 1$ are real vectors given by the recursive equation

$$c_k = y^{(k)}(0) - \sum_{i=0}^{k-1} \binom{k}{i} W^{k-i} c_i, \quad k = 1, \ldots, \rho - 1, \quad c_0 = y(0). \qquad (6)$$

The objective of the control strategy is to steer the system state towards the manifold so that

$$\lim_{t \to \infty} \left(y^{(k)}(t) - \eta^{(k)}(t) \right) = 0, \; k = 0, \ldots, \rho - 1. \qquad (7)$$

The following theorem addresses the problem.

Theorem 1. Consider a MIMO plant with minimal realisation

$$\dot{x} = Ax + Bu + \chi \qquad (8)$$

$$y = Cx \qquad (9)$$

where $\chi \in \mathbb{R}^n$ is a state disturbance.

Let the control law be defined by the differential equation

$$\varepsilon^{\nu} D_{\nu} u^{(\nu)} + \varepsilon^{\nu-1} D_{\nu-1} u^{(\nu-1)} + \cdots + \varepsilon D_1 \dot{u} = N_{\rho} \sigma^{(\rho)} + N_{\rho-1} \sigma^{(\rho-1)} + \cdots + N_1 \dot{\sigma} + N_0 \sigma, \qquad (10)$$

where $\varepsilon > 0$ is a "small" real constant, and $D_i, i = 1, \ldots, \nu$, $N_i, i = 0, \ldots, \rho$ are real constant $r \times r$ matrices to be selected, with ν and ρ integers such that $\nu \geq \rho$.

Assume the following:

- the system is minimum phase
- the disturbances are "matched", i.e. there exists $g \in \mathbb{R}^r$ such that

$$\chi = Bg \qquad (11)$$

- the integer ρ and the matrices N_k, $k = 1, \ldots, \rho$ are such that the algebraic equations

$$\sum_{i=1}^{\rho} N_i H_i = U \qquad (12)$$

$$\sum_{i=k}^{\rho} N_i H_{i-k+1} = 0, \quad k = 2, \ldots, \rho \qquad (13)$$

  are satisfied with U invertible $r \times r$ real matrix;
- the polynomial

$$\det \left(D_{\nu} s^{\nu} + \cdots + D_1 s + U \right) \qquad (14)$$

  is strictly Hurwitz;
- there exists a real $\mu < 0$ such that the roots z_i, $i = 1, \ldots, \rho$ of the polynomial

$$\det \left(N_{\rho} s^{\rho} + N_{\rho-1} s^{\rho-1} + \cdots + N_0 \right) \qquad (15)$$

  satisfy

$$Re(z_i) < \mu, \quad i = 1, \ldots, \rho \qquad (16)$$

- there exists a real $\gamma \leq 0$ such that

$$Re\lambda_{\max}(W) < \gamma \leq 0 \qquad (17)$$

  where $Re\lambda_{\max}(X)$ denotes the largest real part of the eigenvalues of the matrix X.

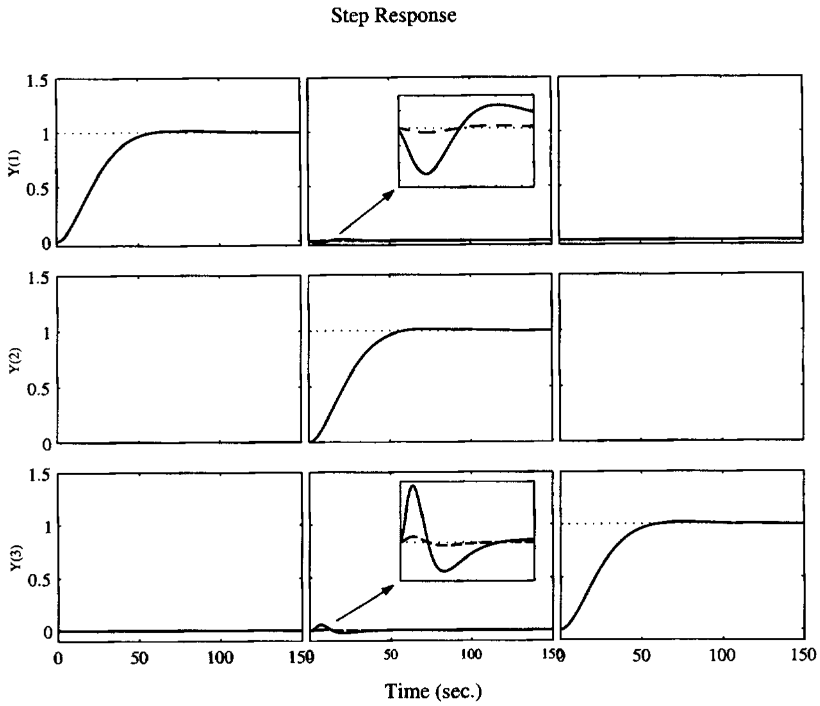

Fig. 1. Step response of the closed-loop system for $\varepsilon = 10^{-2}$ (–) and $\varepsilon = 10^{-3}$ (– –).

Then, there exist $\varepsilon_0 > 0$, $\delta > 0$, $\lambda < 0$, with $\max(\gamma, \mu) < \lambda$, such that for any $\varepsilon \in (0, \varepsilon_0]$, the solution $(x(t,\varepsilon), u(t,\varepsilon))$ of (8), (10), is such that

$$\sum_{k=0}^{\rho-1} \|y^{(k)}(t,\varepsilon)\| \le \delta + ae^{\lambda t}, \forall t \in [0, +\infty) \quad (18)$$

where $a > 0$ is a constant depending on the data.

PROOF. The first step is to compute the total time derivatives of the function $\sigma(y,t)$:

$$\sigma^{(k)} = -CA^k x - \sum_{i=1}^{k} H_i\left(u^{(k-i)} + g^{(k-i)}\right) + \eta^{(k)},$$
$$k = 1, \ldots, \rho \quad (19)$$

hence, after some manipulations and using (12), (13), the RHS of (10) can be written as

$$-\sum_{k=0}^{\rho} N_k CA^k x - U(u+g) + \sum_{k=0}^{\rho} N_k \eta^{(k)}. \quad (20)$$

Letting $\tau = t/\varepsilon$, the controller equation in the "fast" time scale becomes

$$D_\nu \frac{d^\nu u}{d\tau^\nu} + \cdots + D_1 \frac{du}{d\tau} + Uu = \text{const.} \quad (21)$$

Then, Hurwitzness of (14) is necessary and sufficient for the stability of the boundary layer.

Moreover, by letting $\varepsilon = 0$ in (10) and using (20), the equivalent control $\tilde{u}$ is computed

$$\tilde{u} = -U^{-1}\left(\sum_{k=0}^{\rho} N_k CA^k \tilde{x} - \sum_{k=0}^{\rho} N_k \eta^{(k)}\right) - g \quad (22)$$

where $\tilde{x}$ is the state of the reduced order system. Hence, the reduced order closed-loop system becomes

$$\dot{\tilde{x}} = \left(A - BU^{-1}\sum_{k=0}^{\rho} N_k CA^k\right)\tilde{x} + BU^{-1}\sum_{k=0}^{\rho} N_k \eta^{(k)}. \quad (23)$$

Note that the disturbance has been completely rejected. In order to discuss the stability of the reduced order system, we must compute the roots of the polynomial

$$\det\left(sI - A + BU^{-1}\sum_{k=0}^{\rho} N_k CA^k\right) =$$
$$\frac{1}{\det U} \det\begin{pmatrix} sI - A & B \\ -\sum_{k=0}^{\rho} N_k CA^k & U \end{pmatrix}. \quad (24)$$

By noticing that the matrix in the RHS of the above expression is the Rosenbrock system matrix of the system with transfer matrix

$$\sum_{k=0}^{\rho} N_k \left(CA^k (sI - A)^{-1} B + H_k\right) \quad (25)$$

and using hypothesis (13), it is possible to prove that the reduced order closed loop system poles are the invariant zeros of the transfer matrix

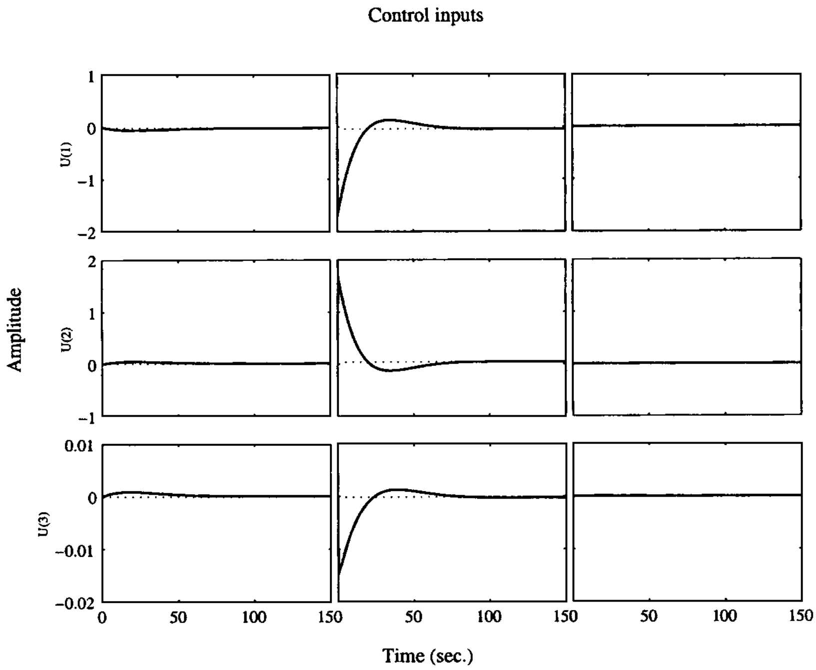

Fig. 2. Control input of the closed-loop system for $\varepsilon = 10^{-2}(-)$ and $\varepsilon = 10^{-3}(--)$.

$$\sum_{k=0}^{\rho} N_k s^k C(sI-A)^{-1}B. \quad (26)$$

Thus, the minimum phase assumption and condition (16) guarantee the stability of the reduced order system.

Moreover, equation

$$N_\rho \sigma^{(\rho)} + N_{\rho-1}\sigma^{(\rho-1)} + \cdots + N_0\sigma = 0. \quad (27)$$

along with initial conditions $\sigma^{(k)}(0) = 0, k = 0, \ldots, \rho - 1$ deduced from (4)–(6), implies that the reduced order system state slides on the manifold for all time instants. Hence, applying Tikhonov's Theorem on the infinite time horizon (Hoppensteadt, 1966), it follows that, as $\varepsilon \to 0$, the system state uniformly approaches the reduced order system state, which in turn slides on the manifold. □

Note that, even if conditions (6) can not be exactly fulfilled, due to unknown initial system output derivatives, stability of (27) still assures that the reduced order system state asymptotically approaches the manifold.

In order to apply Theorem 1, eqns. (12), (13) and (16) must be simultaneously fulfilled. It can be proved that this is always possible with a suitable choice of the order of sliding ρ, as shown in (Cavallo and Natale, 2001), and then by solving first the system

$$\underbrace{\begin{pmatrix} H_1^T & H_2^T & \cdots & H_\rho^T \\ 0 & H_1^T & \cdots & H_{\rho-1}^T \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & H_1^T \end{pmatrix}}\underbrace{\begin{pmatrix} N_1^T \\ N_2^T \\ \vdots \\ N_\rho^T \end{pmatrix}} = \underbrace{\begin{pmatrix} U^T \\ 0 \\ \vdots \\ 0 \end{pmatrix}} \quad (28)$$

and a multilinear optimisation problem of pole location to satisfy condition (16).

3. CASE STUDY

The case study tackles the first benchmark problem proposed in (IFAC, 1990), namely the control of a binary distillation column, whose model includes the effects of pressure variation.

3.1 *Plant description*

The linearised model for a 8 plates distillation column is

$$\dot{x} = Ax + Bu + E\omega \quad (29)$$

$$y = Cx \quad (30)$$

where the inputs $u = \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix}^T$ are the reboiler steam temperature, the condenser coolant temperature and the controlled reflux, respectively; the outputs $y = \begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix}^T$ are the composition of more volatile component in bottom product, the composition of more volatile component in top product and the pressure, respectively; the disturbance input ω is the change of input feed concentration. All the other state variables are the compositions of more volatile component in the various plates. Rather than measuring all of the 11 state variables, resorting to an output feedback is less expensive in terms of sensors. All the numerical values can be found in (IFAC, 1990), and it is possible to verify that the system is asymptotically stable and minimum phase. Moreover, the disturbance is not matched with the input, hence condition 11 is not fulfilled. However the proposed controller can effectively reject all the components of the disturbance along the (B), which constitutes the main contribution. The requirements are to design a controller to regulate the 3 outputs against the unmeasurable disturbance with as fast a settling time as possible, subject to the constraints on the control inputs $|u_1| \leq 2.5$, $|u_2| \leq 2.5$, $|u_3| \leq 0.30$, $\forall t$ for a disturbance $|\omega| \leq 1, \forall t$.

3.2 *Controller design*

The first step of the design procedure is to compute the Markov parameters of the plant and construct the matrix and the related system of linear equations (28) in the unknowns N_k. The integer ρ is chosen as the minimum index of Markov parameters sufficient to guarantee the solvability of the system = , i.e., k is increased until the condition

$$\text{rank} \quad = \text{rank}\left(\qquad \right)$$

is satisfied. However, this procedure is not general and it does not always lead to a finite number ρ defining the order of the sliding strategy. This problem can be solved in general, as described in (Cavallo and Natale, 2001). In the present case study, it is easy to find that $\rho = 2$. Since there are infinite solutions, the one which maximise the number of null elements in the matrix N_2 is selected, so that the degree of the polynomial (15) is minimised. By applying a pseudo-inversion algorithm and choosing $U = I$, it results

$$N_1 = \begin{pmatrix} -1.492 & 0 & 0.136 \\ 1.492 & 0 & 0.081 \\ 0.027 & 0 & 0.001 \end{pmatrix}, N_2 = \begin{pmatrix} 0 & -347.102 & 0 \\ 0 & 347.102 & 0 \\ 0 & -3.054 & 0 \end{pmatrix}.$$

With this choice only 4 arbitrary stable roots of $\det N(s)$ have to be selected. By locating the desired roots not too far from the open-loop poles of the plant (so as to limit the control effort) e.g., in $\{-0.4 \pm j0.2, -0.3 \pm j0.1\}$, and using a standard optimisation algorithm, N_0 is

$$N_0 = \begin{pmatrix} 0.915 & 0.699 & 0.542 \\ 0.593 & 0.193 & 0.225 \\ -0.078 & -0.573 & -0.043 \end{pmatrix}$$

The second step of the procedure is to design the denominator of the controller by designing the polynomial matrix $D(s)$. This can be done by assigning the roots of $\det D(s)$ compatibly with the choice of the matrix U. A diagonal polynomial matrix $D(s)$ is imposed, whose determinant has roots $\{-0.5, -1, -2\}$. The resulting controller has transfer matrix

$$C(s) = \frac{N_2 s^2 + N_1 s + N_0}{\varepsilon^3 s^3 + \varepsilon^2 7/2 s^2 + \varepsilon 7/2 s}. \qquad (31)$$

The last step of the design is the definition of the function $\eta(t)$ so that $\sigma(0) = \dot{\sigma}(0) = 0$. Moreover, the regulation of the output to a desired value y_d can be simply achieved by replacing y with the regulation error $y - y_d$ in σ. In the case of a step reference input with null initial conditions on the plant, this is obtained by filtering the reference signal with a second order filter, i.e.

$$F(s) = \text{diag}\,\{f(s), f(s), f(s)\}$$
$$\text{with} \quad f(s) = \frac{1}{1 + 2\zeta/\omega_n s + s^2/\omega_n^2}$$

In this way, for $\varepsilon \to 0$, the bandwidth of the closed-loop system is the bandwidth of $F(s)$. As requested by the benchmark, ζ and ω_n have been selected to have as fast a settling time as possible, compatibly with the control input limits. The values $\zeta = 0.8$, $\omega_n = 0.07$ rad/s guarantee a settling time of approximately 82 s. The performance of the controller can be improved by decreasing the parameter ε. For the values of $\varepsilon = 10^{-2}$ and $\varepsilon = 10^{-3}$, the step response is reported in Fig. 1. First, it can be recognised how the actual settling time is very close to the desired one for both values of ε. Then, it is evident how for the lowest value of ε the channels of the closed-loop system are very well decoupled, which is confirmed also by the analysis of the singular values of the closed-loop system depicted in Fig. 3, where the dashed lines ($\varepsilon = 10^{-3}$) are much closer to each other than the solid lines ($\varepsilon = 10^{-2}$). In any case, this good performance is obtained respecting the bounds imposed on the control inputs, in fact in Fig. 2 all the three control signals are below the limit values, for all the time and, although "fast", they do not exhibit peaks. To test the ability of the closed-loop system to reject disturbances, a disturbance signal $\omega(t)$ has been applied to the system, as a square wave with a period of 500 s and the maximum foreseen amplitude. The resulting output signals (reported in Fig. 4), have amplitudes much lower than the amplitudes of the same outputs in the open-loop case, which are of the order of 0.1 (graphically not reported due to lack of space).

4. CONCLUSION

In this paper a new MIMO robust output feedback control strategy has been presented. The proposed controller is based on a high order sliding manifold approach and guarantees strong robustness properties that are typical of high-gain control systems. However, the main problem affecting the latter, namely the "peaking phenomenon" is avoided by using a time-varying sliding manifold. The effectiveness of the proposed strategy is shown on an IFAC benchmark of the control of a binary distillation column.

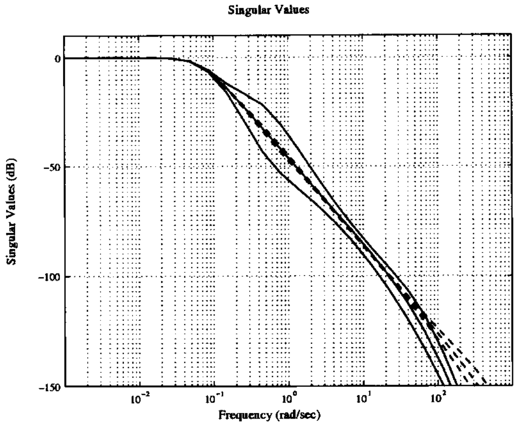

Fig. 3. Singular values of the closed-loop system for $\varepsilon = 10^{-2}\,(-)$ and $\varepsilon = 10^{-3}\,(--)$.

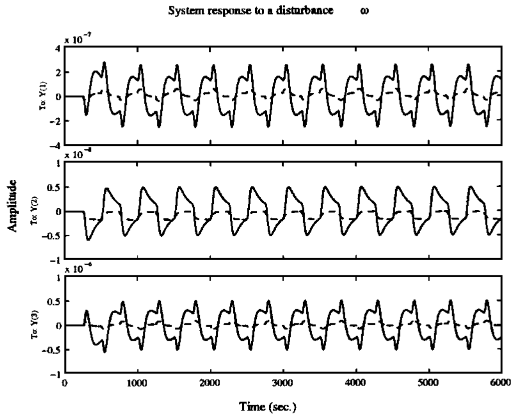

Fig. 4. Closed-loop system response to a disturbance for $\varepsilon = 10^{-2}\,(-)$ and $\varepsilon = 10^{-3}\,(--)$.

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