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OPTIMUM PORTFOLIO CHOICE FOR A CLASS OF JUMP STOCHASTIC MODELS

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Abstract: This paper addresses the problem of choosing the optimum portfolio in the context of continuous time jump stochastic models. The aim is to maximize the wealth of a small risk-averse investor that operates in this financial market. When the case of complete observation is considered, the optimal control problem is formulated and the Hamilton-Jacobi-Bellman equation is solved to yield the solution. On the other hand, to deal with the case of partial observation, the filter equations of the non-observed process are calculated and a sub-optimal control law is presented.

Keywords: Financial systems, optimal control, optimal filtering, stochastic jumps processes.

1. INTRODUCTION

The optimal stochastic control methods have been among the most useful recent techniques developed to deal with problems in economics and finance. It arises from the fact that many economical and financial problems present simultaneously the necessity of decision based on a performance criterion and the presence of uncertainty. In this context, this paper addresses the problem of choosing the optimum portfolio under complete and partial observations.

The celebrated Black-Scholes model (Black and Scholes, 1973) represents one of the most significant contributions in the finance literature. However, the assumption that the volatility of the stock price is constant received considerable criticism (Figlewski, 1989; Schwert, 1989; Stein, 1989). There is evidence that the volatility changes over time because of many factors. For instance, the volatility increases during recessions, for brief periods during and immediately following panic situations, and mainly, when new information arrives.

This work considers a financial market modelled as in the Black-Scholes approach (Black and Scholes, 1973) with just two assets – a risk-free asset and a stock. However, in contrast to the Black-Scholes model (Black and Scholes, 1973), the volatility of the stock price is modelled as a finite continuous time Markov chain, i.e., the stock price is modelled as a switching diffusion model. It means that the volatility of the stock price is piecewise constant. This model tries to take into account these changes of volatility that may affect the stock prices significantly.

Switching diffusions have been used to model a large class of systems with random changes in their structures. These may be consequences of abrupt phenomena, for instance, econometric systems (Blair Jr. and Sworder, 1975), manufacturing systems (Ghosh et al., 1997; Ghosh et al., 1993) and others. Although most of these works deal with linear models (Dufour and Elliott, 1998; Ji and Chizeck, 1992; Fragoso and Hemerly, 1991; Ji and Chizeck, 1990; Blair Jr. and Sworder, 1975; Sworder, 1969), there are some results for non-linear systems (Ghosh et al., 1997; Ghosh et al., 1993).

This paper concerns the problem of wealth maximization of a small risk-averse investor. This problem has been considered in different settings by many authors, for example, Aase (1984), Varaiya (1975), Merton (1969) etc. Although the stock is not modeled as a linear system, an approach similar to Ji and Chizeck (1992) or Fragoso and Hemerly (1991) may be adapted to deal with this problem. Two situations are considered here: the case where the intensity of the parameter modelled by the Markov chain is known and the case that this information is not available.

This paper is organized as follows. In section 2, the problem described above is formally stated. In section 3, some properties of the switching diffusions are presented. In section 4, the problem is solved. Finally, section 5 presents some conclusions of this work.

Notation: Stochastic process will be denoted by omitting the argument $\omega \in \Omega$. For instance, $X(t)$ instead of $X(t,\omega)$. The integrals with respect to $dB(t)$ are taken in the sense to Itô. Almost surely is abbreviated a.s.

2. THE MODEL AND THE PROBLEM STATEMENT

This work addresses the problem of optimizing the wealth of a small investor in the generalized Black-Scholes model where the risk asset price is modeled as a switching diffusion. In this context, one should consider the following statistically mutually independent objects:

a) Brownian motion $B = \{B(t), \mathcal{F}_t^B; 0 \le t < \infty\}$ defined on some probability space $(\Omega^B, \mathcal{F}^B, P^B)$;

b) A homogeneous continuous time Markov chain $\theta = \{\theta(t), \mathcal{F}_t^\theta; 0 \le t < \infty\}$ defined on some probability space $(\Omega^\theta, \mathcal{F}^\theta, P^\theta)$, with right continuous trajectories, and taking values on the finite set $\mathsf{S} = \{1,2,\ldots,n\}$. One should also assume that $p(t) \overset{\Delta}{=} \{p_1(t),\ldots,p_n(t)\}$, with $p_i(t) = P^\theta(\theta(t)=i)$, where $i \in \mathsf{S}$, satisfies the following Kolmogorov forward equation $dp/dt = \Lambda p(t)$ where $\Lambda = [\lambda_{ij}]$ is the stationary $n \times n$ transition rate matrix of θ with $\lambda_{ij} \ge 0$ for $i \ne j$, and $\lambda_{ii} = -\sum_{i \ne j} \lambda_{ij}$, i.e., the process is supposed to be conservative.

Remark 2.1: One may denote a suitable complete probability space $(\Omega, \mathcal{F}, P)$, where $\mathcal{F} = \mathcal{F}^B \times \mathcal{F}^\theta$ denotes the σ-algebra generated by rectangles $A^B \in \mathcal{F}^B$ and $A^\theta \in \mathcal{F}^\theta$. Thus, according to the Fubini's theorem, since P^B and P^θ are σ-finite then P is unique and given by $P(A^B \times A^\theta) = P^B(A^B) P^\theta(A^\theta)$. For details, see Bartle (1966).

On the other hand, according to the generalized Black-Scholes model (Black and Scholes, 1973), one may suppose that the prices X_0 of the risky-free asset and X_1 of the stock defined on the suitable complete probability space $(\Omega, \mathcal{F}, P)$ are given by

$$dX_0(t) = \rho X_0(t)dt \qquad (1)$$

and

$$dX_1(t) = \mu X_1(t)dt + \sigma(t) X_1(t) dB(t) \qquad (2)$$

where ρ and μ are constants and $\sigma(t) \overset{\Delta}{=} \theta(t)$ is a continuous time markov chain.

Remark 2.2: It is obvious that one must require $\mu > \rho$ a.s., since otherwise there would be no sense in investing in the risky stock.

Remark 2.3: Since Lipschitz continuity and linear growth conditions are satisfied then equations (1) and (2) have one unique strong solution. The proof follows the same lines of theorem 4.6 on page 128 in Liptser and Shiryayev (1977).

One may also denote the wealth of an agent $W(t)$ defined on a suitable complete probability space $(\Omega, \mathcal{F}, P)$ as

$$dW(t) = W(t)\left\{(1-u)\frac{dX_0}{X_0} + u\frac{dX_1}{X_1}\right\} \qquad (3)$$

where u is the fraction of the agent's wealth that is invested in the risky asset, thereby investing the fraction $(1-u)$ in the safe one. Therefore, the evolution of the process $(W(t), \theta(t)) \in R \times \mathsf{S}$ is described by

$$\begin{aligned} dW(t) &= u(\mu W(t)dt + \sigma W(t) dB(t)) \\ &+ (1-u)(\rho W(t)dt) = \\ &(u\mu + (1-u)\rho) W(t)dt + \sigma u W(t) dB(t) \end{aligned} \qquad (4)$$

and

$$P(\theta(t+\Delta t = j / \theta(t) = i) = \lambda_{ij}\Delta t + o(\Delta t) \qquad (5)$$

where $\lim_{\Delta t \to 0} \dfrac{o(\Delta t)}{\Delta t} = 0$.

Assumptions 2.4: Let U be the set of admissible controls. Such a process $u : \mathsf{R} \times \mathsf{S} \to \mathsf{U}$ is called an admissible policy if $u \in [0,1]$ and u satisfies the following conditions:

a) Restriction on growth condition

$$\begin{aligned} & u(x(t),\varepsilon(t))^2 \le L_1 \int_0^t (1 + x(s)^2) dK(s) \\ & + L_2(1 + \varepsilon(t)^2 + x(t)^2) \end{aligned} \qquad (6)$$

b) Lipshitz condition

$$\begin{aligned} & |u(x(t),\varepsilon(t)) - u(y(t),\varepsilon(t))|^2 \le \\ & L_1 \int_0^t (x(t)-y(t))^2 dK(s) + L_2(x(t)-y(t))^2 \end{aligned} \qquad (7)$$

where L_1 and L_2 are positive constants, $K(\cdot)$ is a non-decreasing right continuous function, $0 \le K(\cdot) \le 1$, $x(\cdot)$ and $y(\cdot)$ are continuous measurable functions, $\varepsilon(t) \in \mathsf{S}$ and $0 \le t \le T$.

Remark 2.5: With the assumptions 2.4, the equation (4) has one unique strong solution. The proof follows the same lines of theorem 4.6 on page 128 and theorem 4.9 on page 142 in Liptser and Shiryayev (1977).

Assumption 2.6: In this work, the jump sizes are considered predictable in $\theta = \{\theta(t), \mathcal{F}_t^\theta; 0 \le t < \infty\}$, i.e., one does not know if a jump will occur, but if it does, its intensity is known.

Remark 2.7: Assumption 2.6 is necessary to prove the main results of this paper. On the other hand, that this assumption is not followed is the reason to be only possible to get a sub-optimal control law in section 4.3.

On the other hand, one may define a function $\Phi : \mathsf{R}^{+} \times \mathsf{R} \times \mathsf{S} \to \mathsf{R}$ defined on a vector space endowed with the product topology. Thus, again according to the Fubini's theorem and assumption 2.4, one may get $E_W[\Phi] = E_{B\times\theta}[\Phi] = \int_{\Omega^B}\int_{\Omega^\theta} \phi dP^B dP^\theta$.

For details, see Bartle (1966).

The problem considered in this paper is described as follows:

Problem 2.8: Suppose that, starting with the wealth $W(0) = w$ at time $t = 0$, a small investor wants to maximize the expected utility of the wealth $U(W(T))$ at some future date T. If no borrowing is allowed and the utility function of this agent is increasing and concave (the agent is risk averse – see Mas-colell *et al.*, 1995, for details), then the problem is to find the value function $\Phi(s, w, i)$ and a Markov control $u^* = u^*(t, W(t), i)$, $u^* \in [0,1]$, such that $\Phi(s, w, i)$ is given by

$$\Phi(s, w, i) = \sup_{0\le u\le 1}\{J(s, w, i, u)\} \tag{8}$$

where

$$\begin{aligned} &J(s, w, i, u) \\ &= E_W[U(W^u(T)) / W(s) = w, \theta(s) = i] \\ &= E_{B\times\theta}[U(W^u(T)) / W(s) = w, \theta(s) = i] \end{aligned} \tag{9}$$

Remark 2.9: The problem 2.8 was first considered by Merton (1969) who solved this problem when considering ρ, μ, σ are constants.

3. SWITCHING DIFFUSIONS: BASIC PROPERTIES

This section intends to review some basic properties of switching diffusions. It follows the same steps of Fragoso and Hemerly (1991).

Proposition 3.1: The process $(W(t), \theta(t)) \in \mathsf{R} \times \mathsf{S}$ is a Markov process.

Sketch of proof: Consider the following statements:

a) $\theta(t)$ is a Markov process according to its definition. For details, see Heyman and Sobel (1982);

b) $B(t)$ is also a Markov Process, since its increments are independent;

c) The sources of uncertainty in $W(t)$ are $\theta(t)$ and $B(t)$ and they are independent. On the other hand, the process $(W(t), \theta(t))$ is defined on σ-algebra, generated by rectangles $A^B \in \mathsf{F}^B$ and $A^\theta \in \mathsf{F}^\theta$, $\mathsf{F} = \mathsf{F}^B \times \mathsf{F}^\theta$.

Thus, $(W(t), \theta(t))$ is a Markov process.

■

Proposition 3.2: The process $\{(W(t), \theta(t)); s \le t < \infty\}$ has sample paths that are continuous from the right.

Proof: It is obvious from equation (4) and the definition of $\theta(t)$.

■

Proposition 3.3: The process $\{(W(t), \theta(t)); s \le t < \infty\}$ has a stochastically continuous transition probability. Therefore, is uniquely defined by its infinitesimal generator.

Proof: It follows from proposition 3.1, proposition 3.2 and the Dynkin's formula.

■

Definition 3.4: Let T_h be the operator defined on the space of $\mathsf{B}(\mathsf{R} \times \mathsf{R} \times \mathsf{S})$ of bounded measurable scalar functions Φ defined on $\mathsf{R} \times \mathsf{R} \times \mathsf{S} = \mathsf{X}$ and equipped with the norm $\|\Phi\| = \sup_{x\in\mathsf{X}}|\Phi(x)|$ as follows

$$\begin{aligned} &T_h\Phi(s, w, i) = \\ &E_W[\Phi(s+h, W(s+h), \theta(s+h)) \\ &/ W(s) = w, \theta(s) = i] \end{aligned} \tag{10}$$

Thus, one may define the infinitesimal generator L of a family of transition probabilities of the Markov process $\{(W(t), \theta(t)); s \le t < \infty\}$ as

$$\begin{aligned} &L\Phi(s, w, i) = \\ &\lim_{h\to 0}\frac{T_h\Phi(s, W(s), i) - T_0\Phi(s, W(s), i)}{h} \end{aligned} \tag{11}$$

where the limit is the uniform limit in $\mathsf{B}(\mathsf{R} \times \mathsf{R} \times \mathsf{S})$. The domain of definition $\mathsf{D}_L \subset \mathsf{B}(\mathsf{R} \times \mathsf{R} \times \mathsf{S})$ consists of all functions for which limit in (11) exists. For details and examples, see page 36 in Arnold (1974).

Remark 3.5: $L\Phi$ can be interpreted as the "average" change of the function Φ.

Remark 3.6: If $\Phi \in \mathsf{D}_L$ then $\lim_{h\to 0} T_h\Phi(s, W(s), \theta(s)) = \Phi(s, W(s), \theta(s))$.

Now, one should notice that $\Phi \in \mathsf{D}_L$ is the class of functions with continuous derivatives of first order in t on $[s, T]$ and first and second orders in $W(t)$, almost everywhere.

Proposition 3.7: The infinitesimal generator of $\{(W(t), \theta(t)); s \le t < \infty\}$, with $\{(W(t); s \le t < \infty\}$ satisfying the equation (4) and $u \in \mathsf{U}$ is given by

$$\begin{aligned} &L^u\Phi(t, W, i) = \frac{\partial\Phi(t, W, i)}{\partial t} \\ &+ W(u\mu + (1-u)\rho)\frac{\partial\Phi(t, W, i)}{\partial W} \\ &+ \frac{1}{2}\sigma_i^2(t)u^2W^2\frac{\partial^2\Phi(t, W, i)}{\partial W^2} + \sum_{j=1}^{n}\lambda_{ij}\Phi(t, W(t), j) \end{aligned} \tag{12}$$

Sketch of Proof: From equation (11), one may write

$$\begin{aligned} &L^u\Phi(s, W(s), i) = \lim_{h\to 0}\frac{T_h\Phi(s, W(s), i) - T_0(s, W(s), i)}{h} = \\ &\lim_{h\to 0}\frac{1}{h}\{E_{B\times\theta}[\Phi(s+h, W(s+h), \theta(s+h)) / W(s) = w, \\ &\theta(s) = i] - T_0\Phi(s, W(s), i)\} \end{aligned}$$

Additionally, from equation (5), one may get

$$L^u\Phi(s,W(s),i)=$$

$$\lim_{h\to 0}\frac{1}{h}\{\sum_{j=1}^{n} E_B[\Phi(s+h,W(s+h),j)/W(s)=w]\cdot$$

$$P^{\theta}[\theta(s+h)=j/\theta(s)=i]-T_0\Phi(s,W(s),i)\}=$$

$$=\lim_{h\to 0}\frac{1}{h}\{E_B[\Phi(s+h,W(s+h),i)/W(s)=w]$$

$$-T_0\Phi(s,W(s),i)\}$$

$$+\lim_{h\to 0}\{\sum_{j=1}^{n} E_B[\Phi(s+h,W(s+h),j)/W(s)=w]\cdot\lambda_{ij}\}$$

And, from definition 3.4 and remark 3.6

$$L^u\Phi(s,W(s),i)=$$

$$\lim_{h\to 0}\frac{1}{h}\{E_B[\Phi(s+h,W(s+h),i)/W(s)=w]$$

$$-\Phi(s,W(s),i)\}+\sum_{j=1}^{n}\lambda_{ij}\Phi(s,W(s),j)$$

Finally, from equation (4) and equations on pages 41 and 42 in Arnold (1974), one can see that

$$\lim_{h\to 0}\frac{1}{h}\{E_B^{s,W(s)}[\Phi(s+h,W(s+h),i)/W(s)=w]$$

$$-\Phi(s,W(s),i)\}=\frac{\partial\Phi}{\partial s}(s,W,i)$$

$$+W(u\mu+(1-u)\rho)\frac{\partial\Phi}{\partial W}(s,W,i)$$

$$+\frac{1}{2}\sigma_i^2u^2W^2\frac{\partial^2\Phi}{\partial W^2}(s,W,i)$$

Thus, the proof is complete.

■

Theorem 3.8: The Hamilton-Jacobi-Bellman equation associated to this problem is given by

$$\sup_u\{L^u\Phi(t,W,i)\}=0 \tag{13}$$

with boundary condition $\Phi(T,W,i)=U(W)$, for $i=1,\ldots,n$.

Proof: It follows from the Dynkin's formula and the Bellman's optimality principle. For details, see section 2 on page 152 of Fleming and Rishel (1975).

■

Theorem 3.9: (Dynamic Programming Verification Theorem) Let Φ be the solution of the dynamic programming equation $\sup_u\{L^u\Phi(t,W,i)\}=0$ with boundary condition $\Phi(t,W,i)=U(W)$, for $i=1,\ldots,n$. Then:

a) $J(t,W,i,u)\le\Phi(t,W,i)$ for any admissible feedback control u and any initial data;

b) If u^* is an admissible feedback control such that $L^{u^*}\Phi=\sup_u[L^u\Phi]$. Thus u^* is optimal.

Proof: This proof follows the same lines of theorem 4.1 on page 159 of Fleming and Rishel (1975).

■

4. PROBLEM SOLUTION

In this section, two different approaches are used to solve problem 2.8. The first approach, which is the simpler one, is to consider the situation of complete access to the intensity of the parameter σ. A model of this type may be useful to perform a sensibility analysis of the possible outcomes of random events that are supposed to present jumps in random future dates, that is, answers "what if" questions. On the other hand, the second approach deals with the problem 2.8 without assuming that the parameter σ is available. This approach is suited to cases in which it is desired to apply optimal control to real financial markets where the volatility σ is not given. It is more complicated than the first one, since it requires the determination of the optimal filter equations for the parameter σ.

4.1 Optimal control with complete observation

The results presented in section 3 will be used to solve problem 2.8. In general, it is difficult to find the explicit solution of the problem 2.8. However, one may circumvent this problem if it is considered that the utility function U is given by a power function $U(W)=W^r$, $0<r<1$.

Theorem 4.1: If $\frac{\partial^2\Phi(t,W,i)}{\partial W^2}<0$, $\frac{\partial\Phi(t,W,i)}{\partial W}>0$, in equation (12), and $U(W)=W^r$, $0<r<1$, then the optimal control $u^*(t)$ that solves the problem 2.8 is given by

$$u^*(t)=\min\left(\frac{(\mu-\rho)}{\sigma_i^2(1-r)},\quad 1\right),\ \theta(t)=i \tag{14}$$

Proof: Firstly, it is necessary to find $u(t,W,i)$ that maximizes the Hamilton-Jacobi-Bellman equation given by (13). Because $\frac{\partial^2\Phi}{\partial W^2}<0$, $\frac{\partial\Phi}{\partial W}>0$, and noting that equation (13) is a second degree polynomial in u, one gets

$$u^*(t)=\min\left(-\frac{(\mu-\rho)\dfrac{\partial\Phi(t,W,i)}{\partial W}}{\sigma_i^2W\dfrac{\partial^2\Phi(t,W,i)}{\partial W^2}},\quad 1\right) \tag{15}$$

Thus, if one takes into account (12) and substitutes (14) into (13), then

$$\frac{\partial\Phi(t,W,i)}{\partial t}+\rho W\frac{\partial\Phi(t,W,i)}{\partial W}$$

$$-\frac{(\mu-\rho)^2\left(\dfrac{\partial\Phi(t,W,i)}{\partial W}\right)^2}{2\sigma_i^2\dfrac{\partial^2\Phi}{\partial W^2}}+\sum_{j=1}^{n}\lambda_{ij}\Phi(t,W(t),j)=0 \tag{16}$$

Since $U(W)=W^r$, the solution of (5.9) is given by $\Phi(t,W,i)=f(t,i)W^r$, where $f(t,i)$ is the solution of the following system of n ordinary differential equations

$$\frac{df(t,i)}{dt}+\rho rf(t,i)-\frac{(\mu-\rho)^2rf(t,i)}{2\sigma_i^2(r-1)}$$
$$+\sum_{j=1}^{n}\lambda_{ij}f(t,j)=0 \tag{17}$$

where the initial conditions can be calculated from the bounded condition of the partial differential equation. Substituting $\Phi(t,W,i) = f(t,i)W^r$ in (15), one may arrive to equation (14).

■

Remark 4.2: One may see that the control law given by equation (14) requires explicit knowledge the value of parameter σ.

Remark 4.3: It is interesting to note that the system of n ordinary differential equations given by equation (17) is similar to the set of interconnected Riccati equations that arises in continuous time markovian jump linear quadratic control, for instance, see Fragoso and Hemerly (1991), Ji and Chizeck (1992) or Sworder, 1969. On the other hand, the optimal control has the same form of the optimal control calculated for a financial market whose the stock has constant volatility.

Remark 4.4: One may see in equation (14) that there is a trade-off between the risk premium $(\mu-\rho)$ and the volatility σ. If the risk premium $(\mu-\rho)$ is large and the volatility σ is small it is more convenient to invest in the stock. However, if the risk premium $(\mu-\rho)$ is small and the volatility σ is large it is more convenient to invest in the risk-free asset.

4.2 Optimal filtering for the unobservable process

This section concerns the filtering problem for the unobservable process θ. Many authors have considered this problem, for instance, Wonham (1965), Liptser and Shiryayev (1977) and Björk (1980). Here, the approach in Liptser and Shiryayev (1977) is adapted to deal with the filtering problem.

In Liptser and Shiryayev's approach (Liptser and Shiryayev, 1977), the filter equations of the unobservable process θ are calculated by means of the introduction of a suitable process denoted by $\delta(i,\theta)$ for $i,\theta \in \mathsf{S}$, where δ is the Kronecker's symbol. Their work aims at calculating the a posteriori probability given by

$$\pi_i(t) = E[\delta(i,\theta(t))/\mathsf{F}_t^{\xi}] = P(\theta(t) = i/\mathsf{F}_t^{\xi}) \tag{18}$$

where F_t^{ξ} is the σ-algebra generated by the observation process.

Remark 4.5: The process $\delta(i,\theta(\cdot))$ for $i,\theta(\cdot) \in \mathsf{S}$ is governed by

$$\delta(i,\theta(t)) = \delta(i,\theta(0)) + \int_0^t \lambda_{\theta_s i} ds + m^i(t) \tag{19}$$

where m^i is a square integrable martingale. For details, see Liptser and Shiryayev (1977), page 332.

Assumption 4.6: Since the objective here is to estimate the volatility $\sigma \overset{\Delta}{=} \theta$ in equation (2), a good choice for the observable process is

$$d\xi(t) = a\theta dt + bdB(t) \tag{20}$$

where ξ is the trading volume of the stock X_1 and a and b are certain constants.

Remark 4.7: According to Schwert (1989) there is an extensive literature that deals with the relation between volatility and trading volume. Moreover, a good model for this relation is provided by equation (20). Besides, the constants a and b can be estimated by means of econometric tools. See also Karpoff (1987).

If one takes into account theorem 9.1 on page 333 in Liptser and Shiryayev (1977), remark 4.5, assumption 4.6 and remark 4.7, the posteriori probabilities π_i are given by the following system of equations

$$\pi_i(t) = p_i(0) + \int_0^t \sum_{\gamma \in \mathsf{L}} \lambda_{\gamma i}(u)\pi_\gamma(u)du$$
$$+\frac{a}{b}\int_0^t \pi_i(u)\left[i - \sum_{\gamma \in \mathsf{L}} \gamma\pi_\gamma(u)\right] d\overline{B}_u \tag{21}$$

where $\overline{B}$ is the innovation process.

4.3 Optimal control with partial observation

In real financial markets, the volatility σ in equation (2) is not accessible to immediate observations. Therefore, the control law given by equation (14) cannot be implemented. One way to circumvent this obstacle is to consider the adaptive version. In this case, one of the following estimates may be chosen:

a)The conditional expectation

$$E[\theta(t)/\mathsf{F}_t^{\xi}] = \sum_{i \in \mathsf{L}} i\pi_i(t) \tag{22}$$

that is the optimal mean square estimative;

b) The estimate obtained from the condition

$$\max_i P[\theta(t) = i/\mathsf{F}_t^{\xi}] = \pi_{i_t(\xi)}(t) \tag{23}$$

that is an estimate that maximizes the a posteriori probability.

From (22) or (23) the control law may be rewrite as

$$u^*(t) = \min\left(\frac{(\mu-\rho)}{\hat{\sigma}^2(1-r)}, \quad 1\right) \tag{24}$$

where $\hat{\sigma}$ is an estimate of the volatility.

Remark 4.8: It is obvious that the control law u^* given by equation (24) is a sub-optimal control law because separation and certainty equivalence were invoked.

Remark 4.9: If one uses the estimation of σ given by equation (22), then an easy way to measure the degree up to which the control law given by (24) is sub-optimal is to observe how much $\max_i P[\theta(t) = i/\mathsf{F}_t^{\xi}]$ differs from 1.

5. CONCLUSIONS

In this paper, a model for the financial market of the Black-Scholes type has been presented. The main contribution of this model with respect to the literature is to consider the volatility of the stock modelled as a continuous time Markov chain. Based on this statement, the problem of wealth maximization has been completely solved by means of dynamic programming arguments when the complete observation case is considered. On the other hand, when there is no access to the jump parameters, a sub-optimal control law is proposed. Moreover, to arrive at the sub-optimal control law, an interesting procedure is introduced to estimate the volatility by using optimal filters equations and the relation between the trading volume and the volatility of the stock as a model for the observable process.

Although there are other works that deal with models with stochastic volatility, this work aims at generalizing the Merton's work (Merton, 1969) without complicating the basic equations.

Finally, this paper can be seen as an extension of some works that deal with linear stochastic equations with jump parameters, for instance, Fragoso and Hemerly (1991), Ji and Chizeck (1992) and Sworder, 1969.

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