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OPTIMIZATION PROBLEMS OVER NONNEGATIVE TRIGONOMETRIC POLYNOMIALS WITH INTERPOLATION CONSTRAINTS [1]

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Abstract: In this article, optimization problems over the cone of nonnegative trigonometric polynomials are described. We focus on linear constraints on the coefficients that represent interpolation constraints. For these problems, the complexity of solving the dual problem is shown to be almost independent of the number of constraints, provided that an appropriate preprocessing has been done. These results can be extended to other curves of the complex plane (real axis, imaginary axis), to nonnegative matrix polynomials and to interpolation constraints on the derivatives.

Keywords: convex optimisation, nonnegative polynomials, interpolation constraints.

1. INTRODUCTION

Nonnegative polynomials are natural objects to model various engineering problems. Among the most representative applications are the filter design problems (Alkire and Vandenberghe, 2001; Davidson *et al.*, 2000; Genin *et al.*, 2000*a*). Recently, self-concordant barriers for several cones of nonnegative polynomials have been proposed in the literature (Nesterov, 2000). They are usually based on results dating back to the beginning of the 20th century. In fact, these cones and their properties were extensively studied by several well-known mathematicians (Fejér, Toeplitz, ...), see e.g. (Karlin and Studden, 1966).

Nowadays, convex optimization techniques allow us to efficiently deal with these cones, which are parametrized by semidefinite matrices (Nesterov and Nemirovskii, 1994; Vandenberghe and Boyd, 1996). Although general semidefinite programming solvers could be used to solve the associated problems, the inherent structure of these polynomial problems must be exploited to derive much more efficient algorithms (Alkire and Vandenberghe, 2001; Genin *et al.*, 2000*b*). They are usually based on the matrix structure which shows up in the dual problem. In particular, solving the standard conic formulation on cones of nonnegative polynomials using the dual matrix structure has been studied in (Genin *et al.*, 2000*b*).

In this article, we consider a particular conic formulation where the linear constraints are interpolations constraints. Indeed, the natural linear constraints on the coefficients of a polynomial are the ones obtained as interpolation conditions on the polynomial or its derivatives : each of them has an unambiguous interpretation. We show that solving the associated optimization problems can be done very efficiently in a number of flops almost independent of the polynomial degree. Moreover, these formulations have some interesting properties that are worth pointing out.

Notation. Hereafter all optimization problems are assumed to be stated in terms of appropriate scalar products defined over the space of complex matrices. For any couple of matrices X and Y let us set their Frobenius scalar product as follows

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$$\langle X, Y\rangle_F \doteq \mathrm{Re}(\mathrm{Trace}\, XY^*) \equiv \mathrm{Re}\sum_{i=1}^{m}\sum_{j=1}^{n} x_{i,j}\overline{y}_{i,j},$$

where $\{x_{i,j}\}_{i,j}$ and $\{y_{i,j}\}_{i,j}$ are the scalar entries of the matrices X and Y, respectively. Both matrices must have the same dimension $m \times n$, but they are not necessarily square. The above definition can thus be applied to vectors and it also follows from the definition that

$$\langle X, Y\rangle_F = \langle \mathrm{Re}(X), \mathrm{Re}(Y)\rangle + \langle \mathrm{Im}(X), \mathrm{Im}(Y)\rangle$$

where $\langle\cdot,\cdot\rangle$ stands for the standard scalar product of matrices, i.e. $\langle X, Y\rangle = \mathrm{Trace}\, XY^*$.

Positive semidefiniteness of a matrix Y is denoted by $Y \succeq 0$. Since we deal with polynomials, the vector of powers $\pi_n(z) = \begin{bmatrix}1 & z & \cdots & z^n\end{bmatrix}^T$ is often used to represent a polynomial by its coefficients. The diagonal matrix defined by the vector y is denoted by $D(y)$. The complex unit is written as $\jmath$, i.e. $\jmath^2 = -1$. The elements of the canonical basis are written as $\{e_k\}_k$, i.e. $I_n = \begin{bmatrix}e_1 & \ldots & e_n\end{bmatrix}$ is the identity matrix.

2. NONNEGATIVE POLYNOMIALS

Let us now briefly summarize a few facts about nonnegative polynomials. First of all, the characterization of such polynomials depends on the curve of the complex plane on which they are defined. These curves are typically the real axis $\mathbb{R}$, the unit circle $e^{\jmath\mathbb{R}}$ or the imaginary axis $\jmath\mathbb{R}$. The set of nonnegative polynomials on any of these three curves is clearly a convex cone $\mathcal{K}$, i.e.

$$\alpha\mathcal{K} \subset \mathcal{K},\ \forall\alpha \geq 0 \qquad (1)$$
$$\mathcal{K} + \mathcal{K} \subset \mathcal{K} \qquad (2)$$

In this article, this special structure is used to formulate various optimization problems in conic form, based on interpolation constraints.

Let us now examine the cone of nonnegative trigonometric polynomials and its dual. Similar results can be derived for nonnegative polynomials on the real line or on the imaginary axis, see (Genin *et al.*, 2000*b*; Nesterov, 2000). Further details will be given in a forthcoming paper.

On the unit circle, the nonnegative polynomials of interest are the trigonometric polynomials. Remember that a trigonometric polynomial of degree n has the form

$$p(\theta) = \sum_{k=0}^{n}[a_k\cos(k\theta) + b_k\sin(k\theta)],\ \theta \in [0, 2\pi). \qquad (3)$$

where $\{a_k\}_{k=0}^n$ and $\{b_k\}_{k=0}^n$ are two sets of real coefficients. Without loss of generality, we can assume that $b_0 = 0$.

If we define the complex coefficients $\{p_k\}_{k=0}^n$ as

$$p_k = a_k + \jmath b_k,\ k = 0, \ldots, n, \qquad (4)$$

the pseudo-polynomial

$$p(z) = \langle p, \pi_n(z)\rangle_F, \qquad (5)$$

evaluated on the unit circle is equal to the trigonometric polynomial (3). Therefore, we can use either (3) or (5) to represent the *same* mathematical object.

Denote the cone of trigonometric polynomials (of degree n) nonnegative on the unit circle by

$$\mathcal{K}_{\mathbb{C}} = \{p \in \mathbb{R}\times\mathbb{C}^n : \langle p, \pi_n(z)\rangle_F \geq 0,\ z = e^{\jmath\theta}, \theta \in [0, 2\pi)\}. \qquad (6)$$

and define the inner product between two vectors $p = (p_0, \ldots, p_n)^T \in \mathbb{R}\times\mathbb{C}^n$ and $q = (q_0, \ldots, q_n)^T \in \mathbb{R}\times\mathbb{C}^n$ by $\langle p, q\rangle_F$.

The cone of nonnegative polynomials on the unit circle can then be characterized as follows (Nesterov, 2000).

Theorem 1. A trigonometric polynomial $p(z) = \langle p, \pi_n(z)\rangle_F$ is nonnegative on the unit circle if and only if there exists a positive semidefinite Hermitian matrix $Y = \{y_{i,j}\}_{i,j=0}^n$ such that ($y_{i,j} = 0$ for i or j outside their definition range) :

$$p_k = \begin{cases} \displaystyle\sum_{i-j=0} y_{i,j}, & k = 0 \\ \displaystyle 2\sum_{i-j=k} y_{i,j}, & k = 1, \ldots, n \end{cases} \qquad (7)$$

This statement is a direct consequence of Fejér's Theorem (Fejér, 1915; Nesterov, 2000). Note that (7) can be rewritten using the vector $\pi_n(z)$, i.e. $p(z) = \langle Y\pi_n(z), \pi_n(z)\rangle$.

By definition, the cone dual to $\mathcal{K}_{\mathbb{C}}$ is the set of vectors $q = (q_0, \ldots, q_n)^T \in \mathbb{R}\times\mathbb{C}^n$ satisfying the inequality

$$\langle p, q\rangle_F \geq 0,\ \forall p \in \mathcal{K}_{\mathbb{C}}. \qquad (8)$$

If $T(s)$ is the Hermitian Toeplitz matrix defined by the vector $s \in \mathbb{R}\times\mathbb{C}^n$, i.e.

$$T(s) = \begin{bmatrix} s_0 & \bar{s}_1 & \cdots & \bar{s}_n \\ s_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \bar{s}_1 \\ s_n & \ddots & s_1 & s_0 \end{bmatrix}, \qquad (9)$$

the cone dual to $\mathcal{K}_{\mathbb{C}}$ is characterized by $T(s) \succeq 0$, i.e.

$$\mathcal{K}_{\mathbb{C}}^* = \{s \in \mathbb{R}\times\mathbb{C}^n : T(s) \succeq 0\}. \qquad (10)$$

Using the operator dual to $T(\cdot)$, equation (7) can also be written as $p = T^*(Y)$, which means that

$$p_k = \langle Y, T_k\rangle,\ k = 0, \ldots, n \qquad (11)$$

where the matrices $\{T_k\}_{k=0}^n$ are defined by the identity $T(s) = \frac{1}{2}\sum_{k=0}^n (T_k s_k + T_k^T\overline{s}_k), \forall s \in \mathbb{R}\times\mathbb{C}^n$.

3. THE OPTIMIZATION PROBLEM

The problem of optimizing over the cone of nonnegative polynomials, subject to linear constraints on the coefficients of these polynomials, has already been studied by the authors in a wider framework (Genin *et al.*, 2000*b*). Remember that this class of problems is exactly the standard *conic formulation* introduced in (Nesterov and Nemirovskii, 1994). In this section, we now focus on the particular case of trigonometric polynomials constrained by interpolation constraints. The consequent structures of the primal and dual problems lead to efficient algorithms for solving such problems.

Several important optimization problem on the unit circle can be formulated as the following *primal* problem

$$\begin{array}{ll} \min & \langle c, p\rangle_F \\ \mathrm{s.t.} & \langle a_i, p\rangle_F = b_i \;\; i = 1, \dots, m \, . \\ & p \in \mathcal{K}_{\mathbb{C}} \end{array} \tag{12}$$

where the linear constraints are independent. From a computational point of view, the problem dual to (12) has again a considerable advantage over its primal counterpart. This dual problem reads as follows

$$\begin{array}{ll} \max & \langle b, y\rangle \\ \mathrm{s.t.} & s + \sum_{i=1}^{m} y_i a_i = c \\ & s \in \mathcal{K}_{\mathbb{C}}^* \end{array} \tag{13}$$

Since its constraints are equivalent to $T(c - A^* y) \succeq 0$, the Toeplitz structure can be used to efficiently solve this dual problem (Genin *et al.*, 2000*b*). Using Theorem 1, the primal optimization problem (12) can also be recast as the semidefinite programming problem

$$\begin{array}{lll} \min & \langle T(c), Y\rangle & \\ \mathrm{s.t.} & \langle T(a_i), Y\rangle = b_i, & i = 1, \dots, m \, . \\ & 0 \preceq Y = Y^* \in \mathbb{C}^{(n+1)\times(n+1)} & \end{array} \tag{14}$$

An interpolation constraint on the nonnegative trigonometric polynomial $p(\theta)$, i.e. $p \in \mathcal{K}_{\mathbb{C}}$, corresponds to

$$p(\theta_i) = \sum_{k=0}^{n} [a_k \cos(k\theta_i) + b_k \sin(k\theta_i)] = b_i \geq 0, \tag{15}$$

with $\theta_i \in [0, 2\pi]$ and it is equivalent to the linear constraint

$$\langle a_i, p\rangle_F \doteq p(z_i) = \langle p, \pi_n(z_i)\rangle_F = b_i, \; z_i = e^{\jmath \theta_i}. \tag{16}$$

Note that $T(\pi_n(z)) = \pi_n(z)\pi_n(z)^*, \forall |z| = 1$. If all the linear constraints of (12) are interpolation constraints, the dual can thus be written as

$$\begin{array}{ll} \max & \langle b, y\rangle \\ \mathrm{s.t.} & T(c) - V D(y) V^* \succeq 0 \end{array} \tag{17}$$

where the Vandermonde matrix V is defined by the points $\{z_1, \dots, z_m\}$, i.e.

$$V = \begin{bmatrix} 1 & \dots & 1 \\ z_1 & \dots & z_m \\ \vdots & & \vdots \\ z_1^n & \dots & z_m^n \end{bmatrix}. \tag{18}$$

Assumption 1. The components of the vector b are strictly positive, i.e. $b_i > 0, \forall i$.

Remark 2. Since we work with nonnegative polynomials, this assumption on the vector b is not restrictive. If there exists an integer i such that $b_i = 0$, one can factorize $p(z)$ as $p(z) = \tilde{p}(z)(z - z_i)$ and rewrite the optimization problem using the polynomial $\tilde{p}(z)$.

4. SOLVING THE OPTIMIZATION PROBLEM

4.1 *Strict feasibility*

The standard assumption on the primal and dual problems is the so-called "strict feasibility" assumption. This assumption is necessary in order to properly define the primal and dual central-path and thus to solve our pair of primal and dual problems (Nesterov, 1996). Moreover, it ensures that the optimal values of both problems coincide, which is an important property to solve our class of problem efficiently.

Assumption 2. (Strict feasibility). There exist points $\hat{p} \in \operatorname{int} \mathcal{K}$, $\hat{s} \in \operatorname{int} \mathcal{K}^*$ and $\hat{y} \in \mathbb{R}^m$ that satisfy the following linear system

$$A\hat{p} = b, \tag{19}$$

$$\hat{s} + A^T \hat{y} = b. \tag{20}$$

Our particular problem class allows us to further discuss the interpretation of the previous assumption. More specifically, we could get some information about the strict feasibility of our problem in advance.

First, we analyze the strict feasibility of the primal constraints. If the number of interpolation points is less or equal to $n+1$, i.e. $m \leq n+1$, it is clear that there exists a strictly positive polynomial $\tilde{p}$ such that $A\tilde{p} = b$. Indeed, if $m = n+1$, let $\{l_i(z)\}_{i=1}^{n+1}$ be the set of Lagrange polynomials of degree n associated to the interpolation points. By definition, these polynomials satisfy the identities

$$l_i(z_j) = \delta_{ij}, \; 1 \leq i, j \leq n+1 \tag{21}$$

where δ_{ij} is the well-known Kronecker delta. The polynomial $\tilde{p}(z) = \sum_{i=1}^{n+1} b_i |l_i(z)|^2$ clearly satisfies all our interpolation constraints and belongs to $\operatorname{int} \mathcal{K}_{\mathbb{C}}$. If $m < n+1$, we can add $n+1-m$ "extra" interpolation constraints and check that the (original) primal problem is always strictly feasible. If the number of interpolation points is strictly greater that $n+1$, we cannot say anything in advance about the primal strict feasibility.

Let us now analyze the strict feasibility of the dual constraints. The structure of our interpolation constraints allows us to specifically characterize the interior of the dual space. It is composed by the set of vector s such that

$$T(s) = T(c - A^* y) = T(c) - \sum_{i=1}^{m} y_i \pi_n(z_i) \pi_n(z_i)^* \succ 0. \tag{22}$$

If $m \geq n+1$, we conclude from this inequality that there always exists $s \in \text{int}\,\mathcal{K}_{\mathbb{C}}^*$. Another simple situation arises when $c \in \text{int}\,\mathcal{K}_{\mathbb{C}}^*$, i.e $T(c) \succ 0$. Then the dual problem is always strictly feasible. Such a situation occurs when we want to minimize an integral of the polynomial $p(\theta)$ on a finite interval $I \subset [0, 2\pi)$:

$$\langle c, p \rangle = \int_I p(\theta) d\theta \tag{23}$$

subject to interpolation constraints. This situation is frequent in practice and one easily checks that in this case $c \in \text{int}\,\mathcal{K}_{\mathbb{R}}^*$, that is $T(c) \succ 0$.

Let us point out a remarkable property of our class of problems. If the number of constraints is equal to $n+1$, both primal and dual problems are strictly feasible and this property is *independent* of the data. Except for that particular case, there usually exists a trade-off between strict primal and dual feasibility.

Therefore, the largest class of interpolation problems on nonnegative trigonometric polynomials of degree n, for which strict feasibility holds and does not depend on the interpolation points, satisfies the following assumption.

Assumption 3. The number m of interpolation constraints is less or equal to $n+1$ and the objective vector c satisfies $T(c) \succ 0$.

From now on, we only focus on problems which fulfill this assumption. The hypothesis on c could be relaxed sometimes but we have kept it in order not to shadow the inherent simplicity of our problems. When appropriate, we shall point out the possible extensions and leave them to the reader.

4.2 *One interpolation constraint*

Let us now solve the primal problem

$$\min\{\langle c, p\rangle : p(\bar{z}) = \langle p, \pi_n(\bar{z})\rangle = b, p \in \mathcal{K}_{\mathbb{C}}\}. \tag{24}$$

Both primal and dual optimal solutions can easily be computed in an explicit way using Assumption 3. They are given by the expressions :

$$y = \frac{1}{\langle T(c)^{-1}\pi_n(\bar{z}), \pi_n(\bar{z})\rangle}, \tag{25}$$

$$p = T^*(qq^*),\ q = \frac{T(c)^{-1}\pi_n(\bar{z})}{\langle T(c)^{-1}\pi_n(\bar{z}), \pi_n(\bar{z})\rangle}. \tag{26}$$

The following example shows a well-known application of this result.

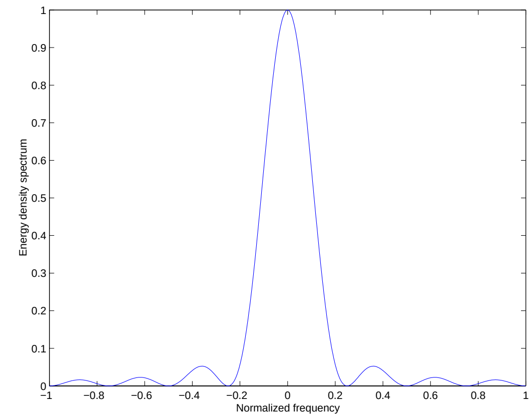

Fig. 1. Energy density spectrum ($|\mathcal{H}(e^{\jmath\omega})|^2$ – n=7)

Example 3. (Moving average system). Let $h[n]$ be a discrete time signal and $\mathcal{H}(e^{\jmath\omega})$ be its Fourier transform. The function $|\mathcal{H}(e^{\jmath\omega})|^2$ is known as the energy density spectrum because it determines how the energy is distributed in frequency (Oppenheim and Schafer, 1989). Let us compute the signal which has the minimum energy

$$2\pi E = \int_{-\pi}^{\pi} |\mathcal{H}(e^{\jmath\omega})|^2 d\omega \tag{27}$$

and satisfies $|\mathcal{H}(e^{\jmath 0})| = 1$.

This is exactly an example of the problem class (24). Since $p(e^{\jmath\omega}) = |\mathcal{H}(e^{\jmath\omega})|^2$ is a trigonometric polynomial, $\int_{-\pi}^{\pi} p(e^{\jmath\omega})d\omega = p_0$. The vector c which defines the objective function is thus equal to $c = [1, 0, \ldots, 0]^T$. The interpolation constraint is obviously defined by $\bar{z} = \pi_n(e^{\jmath 0})$ and $b = 1$.

Therefore, the optimal primal solution is given by

$$p = T^*(qq^*),\ q = \frac{[1, \ldots, 1]^T}{n+1}. \tag{28}$$

and the corresponding Fourier transform $\mathcal{H}(e^{\jmath\omega})$ can be set to

$$\mathcal{H}(e^{\jmath\omega}) = \sum_{i=0}^{n} \frac{1}{n+1} e^{-\jmath\omega}. \tag{29}$$

As shown in Figure 1, $|\mathcal{H}(e^{\jmath\omega})|^2$ is an approximation of a low-pass filter. The corresponding signal is exactly the impulse response of the *moving average system* :

$$h[k] = \begin{cases} \frac{1}{n+1} & 0 \leq k \leq n+1 \\ 0 & \text{otherwise} \end{cases}. \tag{30}$$

Since convolution of a discrete signal $x[n]$ with $h[n]$ returns a signal $y[n]$ such that

$$y[k] = \frac{1}{n+1} \sum_{l=0}^{n} x[k-l], \tag{31}$$

$y[n]$ is the “moving average” of $x[n]$.

4.3 *Two interpolation constraints*

Before investigating problems with two interpolation constraints, we state the following proposition, which gives the optimal solution of a 2-dimensional SDP problem.

Proposition 4. Let $b_1, b_2 \in \operatorname{int} \mathbb{R}_+$ and $\alpha, \gamma \in \mathbb{R}$ and $\beta \in \mathbb{C}$. The optimal value of the optimization problem

$$\begin{array}{ll} \max & b_1 y_1 + b_2 y_2 \\ \text{s.t.} & \begin{bmatrix} \alpha & \overline{\beta} \\ \beta & \gamma \end{bmatrix} \succeq \begin{bmatrix} y_1 & 0 \\ 0 & y_2 \end{bmatrix} \end{array} \tag{32}$$

is reached at the optimal point

$$y_1 = \alpha - |\beta| \sqrt{\frac{b_2}{b_1}}, \qquad y_2 = \gamma - |\beta| \sqrt{\frac{b_1}{b_2}} \tag{33}$$

and it is equal to $b_1 \alpha + b_2 \gamma - 2|\beta|\sqrt{b_1 b_2}$.

If the number of interpolation constraints is equal to 2, the dual problem (13) is now given by

$$\begin{array}{ll} \max & \langle b, y \rangle \\ \text{s.t.} & T(c) \succeq y_1 \pi_n(z_1)[\pi_n(z_1)]^* + y_2 \pi_n(z_2)[\pi_n(z_2)]^* \end{array} . \tag{34}$$

The dual constraint is thus given by

$$T(c) - \begin{bmatrix} \pi_n(z_1) & \pi_n(z_2) \end{bmatrix} \begin{bmatrix} y_1 & 0 \\ 0 & y_2 \end{bmatrix} \begin{bmatrix} \pi_n(z_1) & \pi_n(z_2) \end{bmatrix}^* \succeq 0. \tag{35}$$

Define the matrix $M_T(c; z_1, z_2) = (m_{i,j})_{1 \le i,j \le 2}$ by $m_{i,j} = \langle T(c)^{-1} \pi_n(z_j), \pi_n(z_i) \rangle, \forall i, j$. Using a Schur complement approach, the previous linear matrix inequalities can be rewritten as

$$M_T(c; z_1, z_2)^{-1} \succeq D(y). \tag{36}$$

Using Proposition 4, the optimal dual solution is thus given by

$$\begin{aligned} y_1 = {} & \frac{1}{\det(M_T)} [\langle T(c)^{-1} \pi_n(z_2), \pi_n(z_2) \rangle \\ & - |\langle T(c)^{-1} \pi_n(z_1), \pi_n(z_2) \rangle| \sqrt{\frac{b_2}{b_1}}], \\ y_2 = {} & \frac{1}{\det(M_T)} [\langle T(c)^{-1} \pi_n(z_1), \pi_n(z_1) \rangle \\ & - |\langle T(c)^{-1} \pi_n(z_1), \pi_n(z_2) \rangle| \sqrt{\frac{b_1}{b_2}}]. \end{aligned}$$

Let us define the vector $\begin{bmatrix} v_1 & v_2 \end{bmatrix}^T$ as the solution of the linear system

$$\begin{bmatrix} 1 & 0 \\ 0 & \sigma \end{bmatrix}^* M_T(c; z_1, z_2) \begin{bmatrix} 1 & 0 \\ 0 & \sigma \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \sqrt{b_1} \\ \sqrt{b_2} \end{bmatrix} \tag{37}$$

where σ is equal to $e^{-\jmath \arg \langle T(c)^{-1} \pi_n(z_2), \pi_n(z_1) \rangle}$. The vector

$$q = T(c)^{-1} \begin{bmatrix} \pi_n(z_1) & \pi_n(z_2) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \sigma \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{38}$$

corresponds to a trigonometric polynomial $p(z) = |\langle q, \pi_n(z) \rangle|^2$ which satisfies our interpolation constraints and such that $\langle c, p \rangle = \langle b, y \rangle$. The vector $p = T^*(qq^*)$ is thus the (primal) optimal one.

4.4 *More interpolation constraints ($m \le n+1$)*

Under Assumption 3, the above analysis can always be carried out. The key step is to use the spectral factorization of nonegative polynomials.

Remember that the optimization problem of interest reads as follows :

$$\begin{array}{ll} \min & \langle c, p \rangle \\ \text{s.t.} & \langle p, \pi_n(z_i) \rangle_F = b_i \ \ i = 1, \ldots, m \, . \\ & p \in \mathcal{K}_{\mathbb{C}} \end{array} \tag{39}$$

If we use an arbitrary spectral factor $q(z)$ of the nonnegative trigonometric polynomial $p(z)$, i.e. $p(z) = |q(z)|^2$ or $p = T^*(qq^*)$, the primal optimization problem can be rewritten as

$$\begin{array}{ll} \min & \langle T(c) q, q \rangle \\ \text{s.t.} & \langle q, \pi_n(z_i) \rangle = \sqrt{b_i} e^{\jmath \theta_i} \ \ i = 1, \ldots, m \end{array} . \tag{40}$$

where $\{\theta_i\}_{i=1}^m$ is a set of phases.

Define $\tilde{b}$ as the component-wise square root of b, the signatures $\{\sigma_i\}_{i=1}^m$ by $\sigma_i = e^{\jmath \theta_i}, \forall i$ and the matrix M_T by

$$M_T(c; z_1, \ldots, z_m) = V^* T(c)^{-1} V \tag{41}$$

where $V = \begin{bmatrix} \pi_n(z_1) & \cdots & \pi_n(z_m) \end{bmatrix}$

The optimal solution of (40) is then equal to

$$q = T(c)^{-1} V M_T(c; z_1, \ldots, z_m)^{-1} D(\tilde{b}) \sigma. \tag{42}$$

The corresponding optimal value is thus a function of the signature vector σ :

$$\langle T(c) q, q \rangle = \sigma^* D(\tilde{b}) M_T(c; z_1, \ldots, z_q)^{-1} D(\tilde{b}) \sigma \tag{43}$$

and the optimal solution of problem (40) is given by

$$\begin{array}{ll} \min & \sigma^* D(\tilde{b}) M_T(c; z_1, \ldots, z_m)^{-1} D(\tilde{b}) \sigma \\ \text{s.t.} & |\sigma_i| = 1, \ i = 1, \ldots, m \end{array} . \tag{44}$$

In general this class of problems is NP-hard. If $m > 2$, an explicit solution would be difficult to obtain easily from this new problem. However, we can now derive an algorithm based on the following relaxation of problem (44) :

$$\begin{array}{ll} \min & \langle M_T^{-1}(z_1, \ldots, z_m), X \rangle, \\ \text{s.t.} & d(X) = b \\ & 0 \preceq X = X^* \in \mathbb{C}^{(m+1) \times (m+1)} \end{array} . \tag{45}$$

where $d(X)$ is the vector defined by the diagonal elements of X and e is the all-ones vector.

Theorem 5. If Assumption 3 holds, relaxation (45) is exact.

Remark 6. The proof shows that it would be sufficient to assume that $W_1 T(c) W_1^* \succ 0$ for some W_1 in order to get an exact relaxation. For simplicity reasons, we leave the exact reformulation of our statement in that case to the reader.

The optimal coefficients p can be retrieved from the solution X of (45) via the identity

$$p = T^*(T(c)^{-1} V M_T(c; z_1, \ldots, z_m)^{-1} X \\ M_T(c; z_1, \ldots, z_m)^{-*} V^* T(c)^{-1}). \quad (46)$$

The complexity of solving relaxation (45) is only a function of the desired accuracy ϵ and the number of interpolation constraints m. If Assumption 3 holds and if the original problem has been pre-processed, it can be solved in a number of iterations that do *not* depend on the degree n. Indeed, solving the dual of (45) using a *standard* path-following scheme requires $\mathcal{O}(\sqrt{m} \log \frac{1}{\epsilon})$ Newton steps. At each step computing the gradient and the Hessian of the barrier function

$$f(y) = -\log\det(M_T^{-1}(z_1, \ldots, z_m) - D(y)) \quad (47)$$

requires $\mathcal{O}(m^3)$ flops. Note that the pre-processing can be done via fast Toeplitz solvers, see (Kailath and Sayed, 1999).

If the number of interpolation constraints is strictly greater than $n+1$, strict feasibility of the primal problem depends on the data. Therefore, a *general* procedure which solves efficiently the primal problem and uses the structure of the interpolation constraints is not likely to exist. An appropriate preprocessing could lead to a dual constraint having the following structure

$$\hat{C} - \hat{V} D(y_2) \hat{V}^* \succeq D(y_1). \quad (48)$$

where $y_1 \in \mathbb{R}^{n+1}$ and $y_2 \in \mathbb{R}^{m-n-1}$. As the Toeplitz structure of the dual constraint is lost, the resulting algorithm cannot use the underlying displacement operator nor a divide-and-conquer strategy to evaluate the gradient and the Hessian of the self-concordant barrier function. This strategy will thus be slower than the one designed in (Genin *et al.*, 2000*b*).

5. EXTENSIONS

Most of the previous results still holds in the context of *nonnegative matrix polynomials*. As before, these nonnegative polynomials could be defined on the real line, on the imaginary axis and on the unit circle. We refer to (Genin *et al.*, 2000*b*) for the parametrization of theses cones using the set of positive semidefinite matrices.

Let $N = D(0, 1, \ldots, n)$ and $p^{(k)}(\theta)$ be the k-th derivative of $p(\theta)$. If all the linear constraints of (12) are *interpolation constraints on the derivatives*, i.e.

$$\langle a_i, p \rangle_F \doteq p^{(k_i)}(\theta_i) = \langle p, (\jmath N)^{k_i} \pi_n(z_i) \rangle_F = b_i, \\ z_i = e^{\jmath \theta_i}, \forall i \quad (49)$$

the dual problem (13) reads now as follows

$$\begin{array}{ll} \max & \langle b, y \rangle \\ \text{s.t.} & T(c) - \sum_{i=1}^{m} y_i T((\jmath N)^{k_i} \pi_n(z_i)) \succeq 0 \end{array} . \quad (50)$$

Since $T((\jmath N)^m \pi_n(z))$ has a special structure, we could use it to get an equivalent optimization problem with a structure similar to one obtained with standard interpolation constraints.

More details on both extensions will be provided in a forthcoming article.

6. REFERENCES

Alkire, B. and L. Vandenberghe (2001). Convex optimization problems involving finite autocorrelation sequences. *Math. Programming*. submitted.

Davidson, T. N., Z.-Q. Luo and J.F. Sturm (2000). Linear matrix inequality formulations of spectral mask constraints. *IEEE Transactions on Signal Processing*. submitted.

Fejér, Leopold (1915). Über trigonometrische Polynome. In: *Leopold Fejér, Gesammelte arbeiten I*. number 146. Birkhäuser.

Genin, Y., Y. Hachez, Yu. Nesterov and P. Van Dooren (2000*a*). Convex optimization over positive polynomials and filter design. In: *Proceedings UKACC International Conference on Control 2000*. CD-ROM Paper SS-41, University of Cambridge, UK, 4-7 September 2000.

Genin, Y., Y. Hachez, Yu Nesterov and P. Van Dooren (2000*b*). Optimization problems over positive pseudo-polynomial matrices. *SIAM Journal on matrix analysis and applications*. submitted.

Kailath, T. and Sayed, A.H., Eds.) (1999). *Fast reliable algorithms for matrices with structure*. SIAM.

Karlin, Samuel and William J. Studden (1966). *Tchebycheff systems : with applications in analysis and statistics*. John Wiley and sons.

Nesterov, Yu (1996). Long-step strategies in interior-point primal-dual methods. *Math. Programming* **76**, 47–94.

Nesterov, Yu (2000). Squared functional systems and optimization problems. In: *High Performance Optimization* (H.Frenk et al., Eds.). Chap. 17, pp. 405–440. Kluwer Academic Publishers.

Nesterov, Yu and A. Nemirovskii (1994). *Interior-Point Polynomial Algorithms in Convex Programming*. Vol. 13 of *SIAM Studies in Applied Mathematics*. Society of Industrial and Applied Mathematics. Philadelphia, PA.

Oppenheim, Alan V. and Ronald W. Schafer (1989). *Discrete-Time Signal Processing*. Prentice Hall Signal Processing Series. Prentice Hall.

Vandenberghe, Lieven and Stephen Boyd (1996). Semidefinite programming. *SIAM Rev.* **38**, 49–95.