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NONSINGULAR TERMINAL SLIDING MODE CONTROL OF A CLASS OF NONLINEAR DYNAMICAL SYSTEMS

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Abstract: In this paper, a nonsingular terminal sliding mode concept is introduced for the control of a class of nonlinear dynamical systems. This nonsingular terminal sliding mode control not only enables the £nite time convergence of the system equilibrium, but also eliminates the singularity problem associated with conventional terminal sliding mode control. Simulations are presented to show the effectiveness of the design.

Keywords: Sliding modes, £nite time, nonlinear systems, stability

1. INTRODUCTION

Variable structure systems (VSS) are well known for their robustness in system parameter variations and external disturbances (Utkin 1992), (Zinober 1993). VSS have been successfully used in many applications, such as robots, aircrafts, DC and AC motors, power system, process control and so on. Of particular interest in VSS is the so called sliding mode control, which is designed to drive and then constrain the system states to lie in a set of prescribed switching manifolds. When in the sliding, the closed-loop response becomes totally insensitive to both internal parameter uncertainties and external disturbances.

Conventional switching manifolds, which represent the desired dynamic performance of VSS, are usually linear hyperplanes, which result in asymptotical convergence. For some applications requiring high precision, these manifolds may not be able to deliver fast convergence without imposing strong control. To overcome this problem, recently, a terminal sliding mode (TSM) control has been developed (Yu and Man 1996), (Yu and Man 1998), (Man and Yu 1997), (Wu et al 1998). It offers some superior properties such as fast £nite time convergence and less steady state errors. However, the conventional TSM controller design methods have a common drawback, that is there exists a singularity problem in the TSM control. Existing methods to address this problem adopt an indirect way to avoid the singularity.

In this paper, we presents a nonsingular terminal sliding mode concept and use it for the control design of a class of nonlinear dynamical systems. We will show that this new terminal sliding mode does not incur the singularity problem while maintaining the major advantages of the conventional TSM control. Simulation results are presented to show the effectiveness of the control design.

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2. THE NONSINGLUAR TERMINAL SLIDING MODE CONCEPT

To illustrate the concept of the nonsingular terminal sliding mode concept, we consider the second order dynamical system

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= f(x) + b(x)u \end{aligned} \tag{1}$$

where $x = (x_1, x_2)^T$ is the state, $f(x)$ and $b(x)$ $(\neq 0)$ are smooth nonlinear functions, and u is the scalar control.

The conventional TSM is described by the following nonlinear switching line

$$s = x_2 + \lambda\, x_1^{q/p} \tag{2}$$

where $\lambda > 0$, and p, q are odd positive integers $(p > q)$. The sliding mode $s = 0$, i.e.

$$x_2 + \lambda\, x_1^{q/p} = 0 \tag{3}$$

can be achieved by an appropriate design of the control u using, for example, the criterion

$$s\dot{s} \leq -K|s| \quad K > 0.$$

One commonly used control design satisfying this criterion is

$$\begin{aligned}u = -b^{-1}(x)(f(x) + g(x) + \frac{\lambda q}{p} x_1^{(q-p)/p} x_2 \\ + K\text{sign}(s))\end{aligned} \tag{4}$$

Under this control, it can be easily proved that the state will reach the sliding mode $s = 0$ in t_r, where

$$t_r = \frac{|s(0)|}{K}$$

When the sliding mode $s = 0$ is reached, the second order system will enter the prescribed sliding mode

$$x_2 + \lambda\, x_1^{q/p} = \dot{x}_1 + \lambda\, x_1^{q/p} = 0 \tag{5}$$

The time taken from x_1 from $x_1(t_r) \neq 0$ to 0, t^s, is determined by

$$t^s = \frac{p}{\lambda(p-q)} |x_1(t_r)|^{(p-q)/p} \tag{6}$$

As seen from (4), the singularity may occur at the third term containing $x_1^{(q-p)/p} x_2$ if $x_2 \neq 0$ while $x_1 = 0$. This situation will not occur in the sliding mode because $s = 0$ means $x_2 = -\lambda x_1^{q/p}$ hence as long as $2q > p > q$, the term $x_1^{(q-p)/p} x_2$ will be equivalent to $x_1^{(2q-p)/p}$ which will be nonsingular. The singularity problem may occur in the reaching phase when there is little control to enforce $x_1 \neq 0$ while $x_2 \neq 0$. Indirect approach can be used to avoid this (Man and Yu 1997, Wu et al 1998).

In this paper, we propose a very simple dynamics which is able to avoid this problem completely. The simple nonsingular terminal sliding mode (NTSM) is based on the following modified TSM model

$$s = \lambda x_1 + x_2^{\gamma} \tag{7}$$

where $\gamma = p/q > 0$ with $p,\ q$ and λ being defined as before and we assume $1 < \gamma < 2$. One can easily see that when $s = 0$, the NTSM (7) is equivalent to (2) so the time to reach the equilibrium $x_1 = 0$ in the sliding mode is the same (6).

The key point of using (7) is that the derivative of s along the system dynamcis does not result in terms with negative (fractional) powers.

To design the NTSM control, we use the Lyapunov function

$$V(x) = \frac{1}{2}s^2$$

which derivative along the dynamics (1) is

$$\begin{aligned}\dot{V} &= s\dot{s} \\ &= s(\lambda\dot{x}_1 + \gamma x_2^{\gamma-1}\dot{x}_2) \\ &= s(\lambda x_2 + \gamma x_2^{\gamma-1}(f(x) + b(x)u))\end{aligned} \tag{8}$$

If the control is taken as

$$\begin{aligned}u = -b^{-1}(x)(f(x) + \lambda\gamma^{-1}x_2^{2-\gamma} \\ + K\text{sign}(s))\end{aligned} \tag{9}$$

then we have

$$s\dot{s} = -K|x_2|^{\gamma-1}|s| < 0$$

It is easily seen that the control (9) does not contain any terms with negative (fractional) power due to $1 < \gamma < 2$, meaning there will be no singularity. One question remains is whether the NTSM $s = 0$ will be reached in £nite time. The answer is yes. Indeed, substituting the control (9) into the second equation of (1) yields

$$\dot{x}_2 = -\lambda\gamma^{-1}x_2^{2-\gamma} - K\text{sign}(s) \tag{10}$$

It can be easily seen that if $x_2 = 0$, then (10) becomes

$$\dot{x}_2(t) = -K\text{sign}(s)$$

which suggests that $x_2 = 0$ while $x_1 \neq 0$ is not an attractor. For the cases of $s > 0$ and $s < 0$, we can obtain $\dot{x}_2 < -K$ and $\dot{x}_2 > K$ respectively. It means there exists a vicinity of $x_2 = 0$, $|x_2(t)| < \delta$ for a small $\delta > 0$, so that we have $\dot{x}_2 < -K$ for $s > 0$ and $\dot{x}_2 > K$ for $s < 0$ respectively. Therefore the crossing of trajectory from one boundary of the vicinity $x_2 = \delta$ to the other boundary $x_2 = -\delta$ for $s > 0$ and from $x_2 = -\delta$ to $x_2 = \delta$ for $s < 0$ is £nite time. For the region outside the $|x_2(t)| < \delta$, the time to reach the boundaries of the vicinity is £nite. Indeed, we can easily show that

$$s\dot{s} < -\delta K|s|,$$

meaning the £nite time reachability of the boundaries. Therefore we can conclude that the TSM $s = 0$ will be reached from anywhere in the state space in £nite time.

In practice, the switching function sign will result in chattering in system response. One practical solution is to replace the switching function by a saturation function, such as

$$\mathrm{sat}(z) = \begin{cases} \mathrm{sign}(z) & \text{if } |z| > \phi \\ z/\phi & \text{if } |z| \leq \phi \end{cases} \quad (11)$$

We can easily prove that if $|s| < \phi$, then $|x_1| < \phi$ and furthermore, $|\dot{x}_1| < (2\phi)^{1/\gamma}$. This gives a guide of how to choose the width of the saturation function in order to ensure satisfactory steady state error due to the introduction of the approximation of the switching function.

In the following section, we shall discuss the sliding mode control design of a class of nonlinear dynamical systems using the proposed NTSM.

3. NTSM CONTROL OF A CLASS OF NONLINEAR DYNAMICAL SYSTEMS

Consider the nonlinear dynamical system

$$\dot{\mathbf{x}}_1 = \mathbf{f}_1(\mathbf{x}_1, \mathbf{x}_2) \quad (12)$$
$$\dot{\mathbf{x}}_2 = \mathbf{f}_2(\mathbf{x}_1, \mathbf{x}_2) + \mathbf{g}(\mathbf{x}_1, \mathbf{x}_2) + B(\mathbf{x}_1, \mathbf{x}_2)\mathbf{u} \quad (13)$$

where $\mathbf{x}_1 = (x_{11}, x_{12}, \ldots, x_{1n})^T \in R^n$, $\mathbf{x}_2 = (x_{21}, x_{22}, \ldots, x_{2n})^T \in R^n$, $\mathbf{f}_1$ and $\mathbf{f}_2$ are smooth vector functions and $\mathbf{g}$ represents the uncertainties and disturbances satisfying $\|\mathbf{g}(\mathbf{x}_1, \mathbf{x}_2)\| < l_g$ where $l_g > 0$, B is a nonsingular matrix and $\mathbf{u} \in R^n$ is the control vector. We further assume that $(\mathbf{x}_1, \mathbf{x}_2) = (\mathbf{0}, \mathbf{0})$ if and only if $(\mathbf{x}_1, \dot{\mathbf{x}}_1) = (\mathbf{0}, \mathbf{0})$. Also

$$\frac{\partial \mathbf{f}_1}{\partial \mathbf{x}_2} B(\mathbf{x}_1, \mathbf{x}_2)$$

is assumed to be nonsingular. Note that many practical dynamical systems satisfy this condition, for example, the mechnical systems. Robotic systems are certainly a special case of (12) and (13).

We now construct the following NTSM for the design:

$$\mathbf{s} = \Lambda \mathbf{x}_1 + \dot{\mathbf{x}}_1^{\Gamma} \quad (14)$$

where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, $(\lambda_i > 0)$, $\Gamma = \mathrm{diag}(\gamma_1, \ldots, \gamma_n)$ $(1 < \gamma_i < 2)$ for $i = 1, \ldots, n$, and $\mathbf{x}_1^{\Gamma}$ is represented as

$$\mathbf{x}_1^{\Gamma} = (x_{11}^{\gamma_1}, \ldots, x_{1n}^{\gamma_n})^T$$

We also adopt the notion that

$$\frac{d(\mathbf{x}_1^{\Gamma})}{dt} = \Gamma \mathrm{diag}(x_{11}^{\gamma_1 - 1}, \ldots, x_{1n}^{\gamma_n - 1})\dot{\mathbf{x}}_1 \quad (15)$$

which can be easily verified.

We now design the NTSM control for the system (12) and (13). Consider the Lyapunov function

$$V = \frac{1}{2}\mathbf{s}^T\mathbf{s} \quad (16)$$

The time derivative of (16) along the dynamics (12) and (13) is

$$\begin{aligned} \dot{V} &= \mathbf{s}^T\dot{\mathbf{s}} = \mathbf{s}^T(\Lambda\dot{\mathbf{x}}_1 + \frac{d(\dot{\mathbf{x}}_1)}{dt}) \\ &= \mathbf{s}^T(\Lambda\dot{\mathbf{x}}_1 + \Gamma \mathrm{diag}(\dot{x}_{11}^{\gamma_1 - 1}, \ldots, \dot{x}_{1n}^{\gamma_n - 1})\ddot{\mathbf{x}}_1) \end{aligned} \quad (17)$$

Since

$$\begin{aligned} \ddot{\mathbf{x}}_1 &= \frac{\partial \mathbf{f}_1}{\partial \mathbf{x}_1}\dot{\mathbf{x}}_1 + \frac{\partial \mathbf{f}_1}{\partial \mathbf{x}_2}\dot{\mathbf{x}}_2 \\ &= \frac{\partial \mathbf{f}_1}{\partial \mathbf{x}_1}\dot{\mathbf{x}}_1 + \frac{\partial \mathbf{f}_1}{\partial \mathbf{x}_2}(\mathbf{f}_2 + \mathbf{g} + B\mathbf{u}) \end{aligned} \quad (18)$$

Then (17) becomes

$$\begin{aligned} \dot{V} = \mathbf{s}^T(\Lambda\dot{\mathbf{x}}_1 &+ \Gamma \mathrm{diag}(\dot{x}_{11}^{\gamma_1 - 1}, \ldots, \dot{x}_{1n}^{\gamma_n - 1}) \times \\ &(\frac{\partial \mathbf{f}_1}{\partial \mathbf{x}_1}\dot{\mathbf{x}}_1 + \frac{\partial \mathbf{f}_1}{\partial \mathbf{x}_2}(\mathbf{f}_2 + \mathbf{g} + B\mathbf{u})) \end{aligned} \quad (19)$$

If the control $\mathbf{u}$ is chosen as

$$\begin{aligned} \mathbf{u} = &-\left(\frac{\partial \mathbf{f}_1}{\partial \mathbf{x}_2}B(\mathbf{x}_1, \mathbf{x}_2)\right)^{-1}\left(K\frac{\mathbf{s}}{\|\mathbf{s}\|} + \right. \\ &\frac{\partial \mathbf{f}_1}{\partial \mathbf{x}_2}\mathbf{f}_2(\mathbf{x}_1, \mathbf{x}_2) + \frac{\partial \mathbf{f}_1}{\partial \mathbf{x}_1}\mathbf{f}_1(\mathbf{x}_1, \mathbf{x}_2) + \frac{\partial \mathbf{f}_1}{\partial \mathbf{x}_2}\mathbf{g}(\mathbf{x}_1, \mathbf{x}_2) \\ &\left.\Gamma^{-1}\Lambda\dot{\mathbf{x}}_1^{(2I-\Gamma)}\right) \end{aligned} \quad (20)$$

then (17) becomes

$$\dot{V} = -K\sum_{i=1}^{n}|\dot{x}_{i1}|^{\gamma_i - 1}\frac{s_i^2}{\|\mathbf{s}\|} < 0$$

which can be shown to indicate the £nite time convergence of $\mathbf{s} = \mathbf{0}$ using similar arguments in Section 2.

One can easily see that the NTSM control (20) does not involve any terms which has negative powers. When in the sliding mode $\mathbf{s} = \mathbf{0}$, we have

$$\lambda_i x_{1i} + \dot{x}_{1i}^{\gamma_i} = 0$$

which is equivalent to

$$\dot{x}_{1i} + \lambda_i^{1/\gamma_i}x_{1i}^{1/\gamma_i} = 0$$

which £nite time convergence is well understood (see Section 2 and references (Yu and Man 1996, Yu and Man 1998, Wu et al 1998)). Hence we can claim the the NTSM control can deliver £nite time convergence without any singularity.

In the following, we shall use a simulation to demonstrate the effectiveness of the controller proposed.

4. SIMULATIONS

Consider the nonlinear dynamical system as

$$\begin{aligned} \dot{x}_1 &= x_1^2 + x_2 \\ \dot{x}_2 &= x_1 \cos(x_2) + u \end{aligned} \tag{21}$$

The switching manifold is chosen as

$$s = x_1 + \dot{x}_1^{5/3} \tag{22}$$

Apparently (22) is equivalent to $x_1^{3/5} + \dot{x}_1 = 0$ if $s = 0$. According to Section 3, if we design the controller as

$$u = -x_1 \cos(x_2) - 2x_1(x_1^2 + x_2) - \frac{3}{5}\dot{x}_1^{1/3} - \frac{3}{5}\mathrm{sign}(s)$$

then

$$s\dot{s} = -|x_2^{2/3}||s|$$

which, as demonstrated in Sections 2 and 3, will result in £nite time convergence toward $s = 0$. Therefore, following the NTSM in $s = 0$, the system state will converge to zero in £nite time.

Several simulations were done to demonstrate the effectiveness of the control design. The initial condition is $x(0) = (1,\ 1)^T$. We £rst use the pure switching function in the controller. Figure 1 shows the phase plane response of x_1–x_2. Figure 2 depicts the responses of the states $x_1, x_2, \dot{x}_1$, the switching function s and the control signal. One can easily see that no singularity occurs. We also replaced the pure switching function by a saturation function with width $\phi = 0.1$. Figure 3 shows the phase plane response of x_1–x_2. Figure 4 depicts the responses of the states $x_1, x_2, \dot{x}_1$, the switching function s and the control signal. One can easily see that no singularity occurs and also no discontinuous switching occurs.

5. CONCLUSION

In this paper, a nonsingular terminal sliding mode control has been proposed for a class of nonlinear dynamical systems. This control nonsingular terminal sliding mode control not only enables the £nite time convergence of the system equilibrium, but also eliminates the singularity problem associated with conventional terminal sliding mode control. Simulations have shown the effectiveness of the design. Further work will be pursued in the connections of this class of terminal sliding mode to other £nite time mechanism such as 2–sliding modes.

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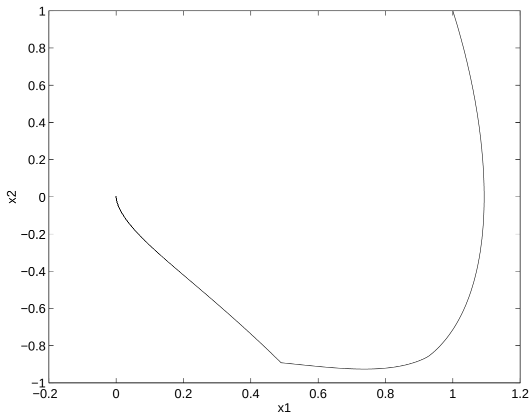

Figure 1. Phase Plane response.

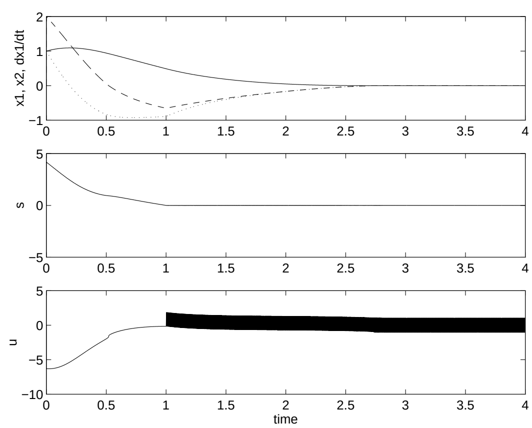

Figure 2. (a) State response (solid – x_1, dotted – x_2, dashed – $\dot{x}_1$). (b). Switching function. (c). Control signal.

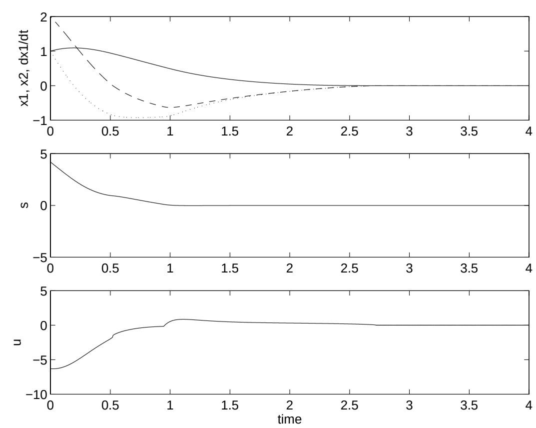

Figure 4. (a) State response (solid – x_1, dotted – x_2, dashed – $\dot{x}_1$). (b). Switching function. (c). Control signal.

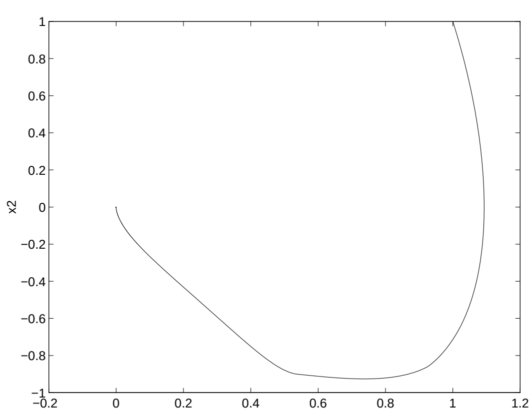

Figure 3. Phase Plane responses.