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A PARAMETER SPACE APPROACH FOR FIXED-ORDER ROBUST CONTROLLER SYNTHESIS BY SYMBOLIC COMPUTATION

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Abstract: In this paper we propose a new method of parameter space design for robust control synthesis, in particular in terms of real stability radius, using quantifier elimination (QE). We also aim at practicality by employing the scheme to combine the sign definition condition (SDC) and a special QE algorithm using Sturm-Habicht sequence. The validity of our approach is confirmed by some concrete examples.

Keywords: parameter space design, robust control design, stability radius, sign definite condition, quantifier elimination, Sturm-Habicht sequence

1. INTRODUCTION

The parameter space approach is known to be one of the effective methods for robust control synthesis and multi-objective design, The approach can be utilized to determine the set of certain parameters which satisfies the given specifications in a parameter space. For robust control synthesis and multi-objective design, recently, the parameter space design accomplished by using quantifier elimination (QE) has been proposed (Dorato *et al.*, 1997; Jirstrand, 1998): The robust control problems are reduced to first-order formula descriptions, then it can be solved by applying general QE. However, *naive* reduction of the control problems to the QE problems, in general, complicated to achieve QE computation efficiently. This is a serious issue in view of efficiency because the worst-case complexity of general QE algorithm based on cylindrical algebraic decomposition (CAD) algorithm has doubly exponential behavior.

While, fortunately, many important design specifications for robustness can be reduced to so called *sign definite conditions (SDC)*:

$$\forall x > 0 \ (f(x) > 0)$$

Moreover, we can use a special QE algorithm using Sturm-Habicht sequence which is much more efficient than the general one for the SDC. This scheme of combining reduction of the specifications to the SDC and usage of a special QE was first successfully introduced to solve robust control design problems in (Anai and Hara, 2000).

The robust controller synthesis problems, which are the problems of finding an appropriate fixed-order controller to achieve stability and a prescribed level of parameter stability margin (stability radius) for a plant, is as yet unsolved. Currently, in an engineering sense, mainly, the techniques for exact computation of stability radius can itself be used in an interactive loop to adjust target parameters to robustify the system. In this paper we propose a systematic approach to such robust controller synthesis problem using quantifier elimination and also aim at practicality. This is realized by utilizing the scheme for robust control design shown in (Anai and Hara, 2000).

The organization of the rest of the paper is as follows: The idea of robust control synthesis based on SDC and special QE algorithm is explained in §2. §3 is devoted to our QE approach to robust control analysis. §4 provides our QE approach to various synthesis problems. Several concrete analysis and synthesis examples are presented for demonstrating the validity of our approach in §5. §6 addresses the concluding remarks.

2. PARAMETRIC APPROACH TO ROBUST CONTROL DESIGN VIA QE

Consider a feedback control system shown in Fig 1, where $\mathbf{p} = [p_1, p_2, \cdots, p_s]$ is the vector of uncertain real parameters in the plant G and $\mathbf{x} = [x_1, x_2, \cdots, x_t]$ is the vector of real parameters of the controller C. Assume that the controller considered here is of fixed order.

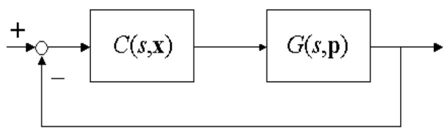

Fig. 1. A standard feedback system

The performance of the control system can often be characterized by a vector $\mathbf{a} = [a_1, \cdots, a_l]$ which are functions of the plant and controller parameters $\mathbf{p}$ and $\mathbf{x}$:

$$a_i = a_i(\mathbf{x}, \mathbf{p}), \quad i = 1, \cdots, l, \tag{1}$$

and the target specifications are usually given as follows:

$$a_i(\mathbf{x}, \mathbf{p}) < \tau_i, \quad i = 1, \cdots, l. \tag{2}$$

The goal is to find the region in the parameter space which satisfies the design specifications. Several concrete robust synthesis problems will be provided in §4.

The QE based approach is one of the effective tools if the specifications (2) are reduced to first-order formula descriptions. However, the general QE algorithm has its inherent undesired computational complexity of doubly exponential. It is therefore important to employ the strategy combining the reduction of (2) to a "*simple*" QE problem and an efficient specialized algorithm for a particular type of inputs aiming at practical applicability. A successful example of such attempts is the scheme of combining the SDC and a special QE algorithm using Sturm-Habicht sequence presented in (Anai and Hara, 2000). Many control design specifications such as H_∞-norm constraints, frequency restricted norms, phase/gain margins, and $\mathcal{D}$-stability constraint for robustness can be recast as SDC

$$\forall x > 0 \ (f(x, \mathbf{x}, \mathbf{p}) > 0). \tag{3}$$

See (Anai and Hara, 2000) for a concrete comparison on the efficiency to solve the SDC by the general QE algorithm based on CAD algorithm and a special one using Sturm-Habicht sequence. The computational complexity of Sturm-Habicht sequence that is the dominant part of the algorithm for low order cases is still bounded by single exponential. The computational experiments, however, show in the case of the moderate number of parameters (2 or 3 parameters) that the algorithm works in an efficient way for the rather high order system (more than ten-th order) that is considered to be practical in an engineering sense.

In this paper we focus on the real stability radius. Though there are many results about the explicit formula to compute the stability radius so far (*e.g.* (Kokame and Mori, 1993; Hitz and Kaltofen, 1998)), the problem of determining the fixed-order controller to achieve stability and a desired level of parameter stability margin has not been solved yet. Therefore, here we propose a new synthesis method of fixed-order robust controllers by using the above scheme. This work also extends the applicability of the scheme.

3. REAL STABILITY RADIUS

In many control systems plant parameters may vary over a wide range from the nominal value $\mathbf{p}^0 = [p_1^0, \cdots, p_\ell^0]$. If the controller $\mathbf{x}$ is given, the maximal range of variation of the parameter $\mathbf{p} = [p_1, \cdots, p_\ell]$, measured in a suitable norm for which the stability is preserved is called the parametric stability margin (radius of stability) with the controller $\mathbf{x}$. It is defined by

$$\rho_m = \sup\{ \ r \mid g(s, \mathbf{x}, \mathbf{p}) \ \textit{stable}, \ ||\mathbf{p} - \mathbf{p}^0|| < r\},$$

where $g(s, \mathbf{x}, \mathbf{p})$ is the characteristic polynomial of the closed-loop system shown in Fig 1.

We consider the following type of characteristic polynomial, *i.e.* its coefficients are linear function of the plant parameter $\mathbf{p}$:

$$g(s, \mathbf{x}, \mathbf{p}) = a_1(s, \mathbf{x})p_1 + \cdots + a_\ell(s, \mathbf{x})p_\ell + b(s, \mathbf{x}),$$

where $a_i(s)$ and $b(s)$ are polynomials over the reals $\mathbb{R}$ and $p_i \in \mathbb{R}$. We refer to this as the *linear case.* We assume that the degree of $g(s, \mathbf{x}, \mathbf{p})$ is fixed.

3.1 *An explicit formula*

Here we employ the results from (Bhattacharyya *et al.*, 1995) and show their results briefly: Since we consider the linear case, the characteristic polynomial of the system of Fig.1 can be written as follows:

$$g(s, \mathbf{p}^0 + \Delta\mathbf{p}) = g(s, \mathbf{p}^0) + \sum_{i=0}^{\ell} a_i(s)\Delta p_i, \tag{4}$$

where

$$\Delta \mathbf{p} = \mathbf{p} - \mathbf{p}^0 = [\Delta p_1, \cdots, \Delta p_\ell].$$

Let s^* be a point on the stability boundary $\partial\mathcal{D}$. Suppose s^* is a root of $g(s, \mathbf{p}^0 + \Delta\mathbf{p})$. Then we have

$$g(s^*, \mathbf{p}^0) + \sum_{i=0}^{\ell} a_i(s^*)\Delta p_i = 0. \quad (5)$$

Taking account of weighted perturbations, we can rewrite above equation as follows:

$$g(s^*, \mathbf{p}^0) + \sum_{i=0}^{\ell} \frac{a_i(s^*)}{w_i} w_i \Delta p_i = 0, \quad (6)$$

where $w_i > 0$. If s^* is real, (6) can be expressed as

$$\mathbf{A}(s^*)\mathbf{u}(s^*) = \mathbf{b}(s^*), \quad (7)$$

where

$$\mathbf{A}(s^*) = \left[\frac{a_1(s^*)}{w_1} \cdots \frac{a_\ell(s^*)}{w_\ell}\right],$$
$$\mathbf{u}(s^*) = [w_1\Delta p_1, \cdots, w_\ell \Delta p_\ell]^T,$$
$$\mathbf{b}(s^*) = -g(s^*, \mathbf{p}^0).$$

If s^* is complex, (6) can be written as the same equation (7) with

$$\mathbf{A}(s^*) = \begin{bmatrix} \frac{a_{R,1}(s^*)}{w_1} & \cdots & \frac{a_{R,\ell}(s^*)}{w_\ell} \\ \frac{a_{I,1}(s^*)}{w_1} & \cdots & \frac{a_{I,\ell}(s^*)}{w_\ell} \end{bmatrix},$$
$$\mathbf{u}(s^*) = [w_1\Delta p_1, \cdots, w_\ell \Delta p_\ell]^T,$$
$$\mathbf{b}(s^*) = [-g_R^0, -g_I^0]^T,$$
$$a_{R,k} = Re(a_k(s^*)),\ a_{I,k} = Im(a_k(s^*)),$$
$$g_R^0 = Re(g(s^*, \mathbf{p}^0)),\ g_I^0 = Im(g(s^*, \mathbf{p}^0)).$$

Since the minimum norm solution of (6) given by

$$\rho(s^*) = \inf\{||\Delta\mathbf{p}|| \mid \Delta\mathbf{p}\ satisfies\ (6)\} \quad (8)$$

is a candidate of the stability radius, the equation (7) determines the parametric stability margin in any norm: Let $u^*(s^*)$ be the minimum norm solution of (7) for s^*. Then

$$\rho = \inf_{s^* \in \partial\mathcal{D}} ||u^*(s^*)||.$$

If (7) has no solution, then ρ is set equal to ∞.

3.2 *QE approach to robust control analysis*

We now consider the ℓ^2-norm case. Assume that $\mathbf{A}$ has full rank $= 2$ for simplicity, Then the minimum norm solution $u^*(s^*)$ is given by

$$u^*(s) = \mathbf{A}^T(s)[\mathbf{A}(s)\mathbf{A}^T(s)]^{-1}\mathbf{b}(s). \quad (9)$$

First we consider the case where s^* is real. Let a finite set of intersection points between $\partial\mathcal{D}$ and real axis be $\{r_1, \cdots, r_k\}$. For each r_i we can compute the $\rho(r_i) \equiv ||u^*(r_i)||_2$ from (9) immediately.

For the complex s^*, we use an appropriate parameterization $\alpha(t)$ of the stability domain boundary $\partial\mathcal{D}$, where $t \in I = [t_s, t_e] \subset \mathbb{R} \cup \{\pm\infty\}$. We allow only polynomial descriptions for the parameterization , *e.g.* Hurwitz case $\alpha(t) = \mathbf{i}t$ where $\mathbf{i}$ is an imaginary unit. Substituting the parameterization $\alpha(t)$ for the indeterminate s^* in the formula (9) leads to an expression $u^*(\alpha(t))$. We simply denote $u^*(\alpha(t))$ by $u^*(t)$.

Having an explicit formula $u^*(t)$ enables us to compute the *exact* minimum of $||u^*(t)||_2$ with respect to $t \in I$ symbolically. Actually we can compute the minimum

$$\mathcal{F}_m := \inf_{t \in \mathcal{I}} ||u^*(t)||_2^2 \quad (10)$$

by a quantifier elimination as will be shown later.

Note that we must deal with the case $\mathbf{A}$ is not full rank. If $rank(\mathbf{A}) = 0$, (7) has no solution, so $\rho = \infty$. If $rank(\mathbf{A}) = 1$, (7) is consistent iff $rank[\mathbf{A}, \mathbf{b}] = 1$, otherwise (7) has no solution, hence $\rho = \infty$. Therefore, for the case of $rank[\mathbf{A}, \mathbf{b}] = 1$, we simply replace two equations with a single equation and can proceed as before. Let $\{d_1, \cdots, d_m\}$ be the values of t for which the rank drops and $rank[\mathbf{A}, \mathbf{b}] = 1$. Consequently, the stability radius is given by

$$\rho_m = \min\{\sqrt{\mathcal{F}_m},\ \rho(r_i),\ \rho(\mathbf{i}d_j)\}. \quad (11)$$

Now consider to compute the minimum $\mathcal{F}_m$ of $\mathcal{F}(t)$. In general, $\mathcal{F}(t)$ is a rational polynomial, say $\mathcal{F}(t) = N(t)/D(t)$ for polynomials N and D. Finding the minimum of $\mathcal{F}(t)$, which is a type of optimization called *hyperbolic optimization*, can be solved as the following QE problem:

$$\exists t \in I\ ((D > 0 \wedge N \leq zD) \vee (D < 0 \wedge N \geq zD))$$

where z is newly introduced variable corresponding to $\mathcal{F}(t)$. Since the denominator of $\mathcal{F}(t)$ is strictly positive, *i.e.*, $D > 0$ is true, the above formula can be reduced further to

$$\exists t \in I\ (N \leq zD). \quad (12)$$

By performing QE for (12) we have an equivalent quantifier-free formula $\Psi(z)$ which presents the possible range of z, in particular, stating the minimal value of $\mathcal{F}(t)$. Equivalently, we can solve (12) by solving the following QE problem

$$\forall t \in I\ (N - zD > 0). \quad (13)$$

Performing QE for (13) gives a quantifier-free formula $\Phi(z)$ equivalent to (13). $\Phi(z)$ presents the possible range of z, in particular, stating the maximum value of z which corresponds to the minimum of $\mathcal{F}(t)$. In other words, $\Phi(z) = \neg\Psi(z)$.

The first-order formula of the type (13) can be reduced to the following SDC:

$$\forall y > 0\ (h(y) > 0) \quad (14)$$

by a bilinear transformation $y = -\frac{(t-t_s)}{(t-t_e)}$, where $h(y)$ is a polynomial. A special QE algorithm using Sturm-Habicht sequence introduced in (Anai and Hara, 2000) can be utilized for the SDC. This is why we employ the reduction (13) instead of (12). Let the resulting formula after applying QE to (14) be $\Pi(z)$. Then $\neg\Pi(z)$ shows the possible range of z stating the minimum of $\mathcal{F}(t)$.

4. SYNTHESIS PROBLEMS

For the synthesis problems, the control parameters $\mathbf{x}$ remains as free parameters during the procedures in the previous section. Here we illustrate how we solve several concrete synthesis problems. First we consider the following basic problem:

<u>Problem</u> *1.* Consider the control system depicted in Fig.1. Given a specific value of stability radius ρ. Let $g(s, \mathbf{x}, \mathbf{p})$ be a characteristic polynomial of the closed-loop system with fixed degree and $\mathbf{p}^0$ be a vector of nominal values of plant parameters such that $g(s, \mathbf{x}, \mathbf{p}^0)$ is $\mathcal{D}$-stable. Then the problem is to find the feasible region of control parameters $\mathbf{x}$ to achieve the desired level ρ of stability radius.

$\mathcal{D}$-stability condition: First, we should mention the condition of parameters $\mathbf{x}$ so that $g(s, \mathbf{x}, \mathbf{p}^0)$ is $\mathcal{D}$-stable. The condition would be given as a semialgebraic set. For Hurwitz stability, such condition of $\mathbf{x}$ is given by the well-known Liénard-Chipart criterion immediately. For Schur stability and wedge shape regions, *i.e.* the domain $\mathcal{D}$ of which the complementary set $\overline{\mathcal{D}} = \mathbb{C} - \mathcal{D}$ is of the form $\overline{\mathcal{D}} = \{x(\omega, t) + \mathbf{i}y(\omega, t) \in \mathbb{C} | \omega \in \mathbb{R}, t \in [t_s, t_e]\}$, the pole location problem can also be reduced to check a sign definite condition, see (Kimura and Hara, 1993). Therefore, $\mathcal{D}$-stability condition is also solved efficiently by a special QE using Sturm-Habicht sequence.

Algorithm: Problem 1 is solved by the same procedure shown in §3.2: For the real case $s^* = r_i$ $(i = 1, \cdots, k)$ is real, immediately from (9), we have $\rho(r_i) = ||u^*(r_i)||_2$ as formulas in $\mathbf{x}$. Let $\delta = \rho^2$, then we have

$$\psi_i(\mathbf{x}) \equiv (||u^*(r_i)||_2^2 \geq \delta), \quad for\ i = 1, \cdots, k$$

In the case where s^* is complex, the formula (13) is of the same form containing the parameters $\mathbf{x}$:

$$\forall t \in I\ (N(\mathbf{x}, t) - \delta \cdot D(\mathbf{x}, t) > 0). \tag{15}$$

Consequently, we lead to the following SDC;

$$\forall y > 0\ (h_p(\mathbf{x}, y) > 0), \tag{16}$$

where h_p is a polynomial. After performing QE for (16) we have an equivalent quantifier-free formula $\phi(\mathbf{x})$ showing the possible range of $\mathbf{x}$ which satisfies the given stability radius condition. $\phi(\mathbf{x})$ is also a semialgebraic set in $\mathbf{x}$. Moreover, for the non full rank case, from (9), we have

$$\kappa(\mathbf{x}) \equiv (||u^*(\mathbf{i}d_j)||_2^2 \geq \delta) \quad for\ j = 1, \cdots, m$$

Finally, the formula

$$\Gamma(\mathbf{x}) \equiv\ \phi(\mathbf{x})\ \vee\ (\bigvee_i \psi_i(\mathbf{x}))\ \vee (\bigvee_j \kappa_j(\mathbf{x}))$$

gives the possible region of $\mathbf{x}$ which satisfies the given stability radius specification.

Next we show some advanced synthesis problems including multi-objective problems and optimization which can be solved naturally by using our parameter space approach based on QE presented in this paper and (Anai and Hara, 2000).

<u>Problem</u> *2.* Find the maximum attainable stability radius ρ by a fixed-order $\mathcal{D}$-stable controller $C(s, \mathbf{x})$.

<u>Problem</u> *3.* Find the best achievable nominal performance by a fixed-order $\mathcal{D}$-stable controller under a stability radius constraint.

Problem 2: The attainable stability radius can be obtained by the same procedure as before if we leave ρ as a free parameter, resulting in the semialgebraic expression $\Gamma'(\rho, \mathbf{x})$. Moreover, the stability condition, say $\mathcal{S}(\mathbf{x})$, of the controller $C(s, \mathbf{x})$ is obtained by the same way shown above. Then the attainable stability radius condition is given by

$$\varphi_2(\rho, \mathbf{x}) \equiv \Gamma'(\rho, \mathbf{x}) \wedge \mathcal{S}(\mathbf{x}).$$

The maximum attainable stability radius can be obtained by solving the optimization problem:

$$Maximize\ \rho\ \ subject\ to\ \varphi_2(\rho, \mathbf{x}). \tag{17}$$

Problem 3: Consider the system with a given stability radius ρ_0. Let $T_i(s, \mathbf{x}, \mathbf{p}^0)$ be the transfer functions and

$$||T_i(s, \mathbf{x}, \mathbf{p}^0)||_{[\underline{\omega_i}, \overline{\omega_i}]} < \gamma_i \tag{18}$$

be the nominal performance specifications with frequency restrictions. Here $\gamma_i > 0$ are free parameters. The frequency restricted norm constraints (18) can be reduced to SDCs and solved by our approach, resulting in the formulas $\mathcal{N}_i(\gamma_i, \mathbf{x})$, respectively. Therefore, the achievable nominal performances condition are given as

$$\varphi_3(\gamma_i, \mathbf{x}) \equiv \Gamma'(\rho_0, \mathbf{x}) \wedge \mathcal{S}(\mathbf{x}) \wedge \textstyle\bigwedge_i \mathcal{N}_i(\gamma_i, \mathbf{x}).$$

The maximum achievable nominal performance obtained by solving the following optimization problem:

$$Maximize\ \gamma_i\ \ subject\ to\ \varphi_3(\gamma_i, \mathbf{x}). \tag{19}$$

Optimization (17)(19): In general, both the optimization problems are nonlinear and nonconvex. Since φ_2, φ_3 are polynomial constraints, they can be solved by using QE exactly. However, we have to use general QE algorithm because generally the reduced QE problems are considered not to have a specific structure desirable in the computation. Hence, this is practical only for modest size of problems. Methods of numerical optimizations could be utilized for large size problems.

5. ANALYSIS AND SYNTHESIS EXAMPLES

This section provides analysis and synthesis problems to confirm the validity of our approach*.

5.1 *Stability radius computation*

Consider the continuous time control system with the plant

$$G(s,\mathbf{p}) = \frac{2s+3-\frac{1}{3}p_1-\frac{5}{3}p_2}{s^3+(4-p_2)s^2+(-2-2p_1)s+(-9+\frac{5}{3}p_1+\frac{16}{3}p_2))}$$

controlled by a PI controller

$$C(s) = 5+\frac{3}{s}.$$

See (Bhattacharyya *et al.*, 1995). The characteristic polynomial $g(s,\mathbf{p})$ of the closed-loop system is given by

$$s^4+(4-p_2)s^3+(8-2p_1)s^2+(12-3p_2)+(9-p_1-5p_2)$$

Let suppose $\mathbf{p}^0 = [p_1^0, p_2^0] = [0,0]$, which guarantees the nominal stability. We now compute the ℓ^2-stability margin with weights $w_1 = w_2 = 1$ in terms of Hurwitz sense, *i.e.*, $\alpha(t) = \mathbf{i}t$ and $\{r_1 = 0\}$. For $r_1 = 0$, from (9), we have $\rho(0) = ||u^*(0)||_2 = \frac{9\sqrt{26}}{26}$. For $d_1 = \sqrt{3}$, (7) is consistent, and we have $\rho(\mathbf{i}\sqrt{3}) = \frac{3\sqrt{2}}{5}$. Then

$$\mathcal{F}(t) = ||u^*(t)||_2^2 = \frac{t^8-16t^6-22t^4+240t^2+105}{(2t^2-1)^2}.$$

The QE problem corresponding to (13) is

$$\forall t > 0\ (t^8-16t^6+106t^4+112t^2+137-z(4t^4-4t^2+1) > 0).$$

By applying QE, we have $z-16 < 0$, which leads to $\sqrt{\mathcal{F}_m} = 4$. Consequently, $\rho_m = \min(4, \frac{9\sqrt{26}}{26}, \frac{3\sqrt{2}}{5}) = \frac{3\sqrt{2}}{5}$. This result coincides with that in (Bhattacharyya *et al.*, 1995).

5.2 *Robust stability synthesis*

Consider the feedback system shown in Fig.1 with

$$G(s,\mathbf{p}) = \frac{1}{s^2+p_1 s+p_2}, \quad C(s,\mathbf{x}) = x_1+\frac{x_2}{s},$$

where the characteristic polynomial is given by

$$g(s,\mathbf{x},\mathbf{p}) = s^3+p_1 s^2+(x_1+p_2)s+x_2. \quad (20)$$

Suppose the nominal values of the plant parameters are $\mathbf{p}^0 = [p_1^0, p_2^0] = [1,1]$ and the weights are $w_1 = w_2 = 1$. The synthesis problem is to find the feasible set of parameter values of $\mathbf{x}$ in the PI controller $C(s,\mathbf{x})$ for the system to achieve a given level of stability margin $\delta (= \rho^2)$ in Hurwitz sense. Let $\delta = 0.5$. By Liénard-Chipart criterion, $g(s,\mathbf{x},\mathbf{p}^0)$ is Hurwitz iff

$$\theta(\mathbf{x}) = (x_2 > 0\ \wedge\ x_1-x_2+1 > 0)$$

holds. The QE problem corresponding to (15) is

$$\forall t > 0$$
$$t^8+(-2x_1-2)t^6+(x_1^2+2x_1+\tfrac{3}{2})t^4-2x_2t^2+x_2^2 > 0.$$

By applying QE, we have

$$\phi(\mathbf{x}) = (x_2 < 0\ \vee\ (P_1 > 0 \wedge x_2 \neq 0)),$$

where

$P_1(\mathbf{x}) = 256x_2^4-768x_1x_2^3-768x_2^3+16x_1^4x_2^2+64x_1^3x_2^2+736x_1^2x_2^2+1344x_1x_2^2+480x_2^2-32x_1^5x_2-160x_1^4x_2-464x_1^3x_2-752x_1^2x_2-528x_1x_2-112x_2+8x_1^6+48x_1^5+124x_1^4+176x_1^3+142x_1^2+60x_1+9.$

For $r_1 = 0$, (7) is inconsistent, hence we have $\rho(0) = \infty$. For non full rank case, (7) is inconsistent. Consequently, the formula

$$\theta(\mathbf{x})\ \wedge\ \phi(\mathbf{x}) \quad (21)$$

provides the feasible set of parameters $\mathbf{x}$ for the system to achieve the desired level of stability radius. The shaded region in Fig.2 corresponds to (21).

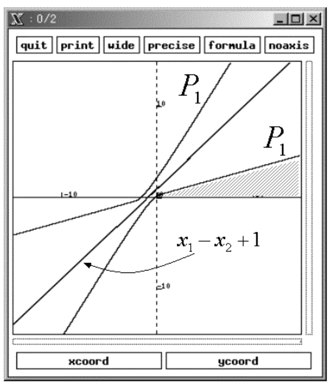

Fig. 2. The possible region of $\mathbf{x}$ described by (21)

5.3 *Robust stability with sensitivity*

We can add any design constraint, which can be reduced to a SDC, in the robust stabilization in §5.2. A typical example is to add finite frequency H_∞ norms of interested closed-loop transfer functions such as sensitivity function $S(s)$. Let us consider the robust stabilization for the same system in §5.2 under a sensitivity constraint

$$||S(s)||_{[0,1]} \equiv \max_{0\le\omega\le 1} ||S(\mathbf{i}\omega)|| < 0.1 \quad (22)$$

where

$$S(s) = \frac{s^3+s^2+s}{s^3+s^2+(x_1+1)s+x_2}.$$

* All the symbolic computation and feasible region plots were done by using a computer algebra system RISA/ASIR (cf. `http://www.math.kobe-u.ac.jp/Asir/asir.html`).

We can see from a simple symbolic computation that the frequency restricted H_∞-norm constraint (22) is reduced to the following SDC:

$\forall z > 0$
$(x_2^2 z^3 + (x_1^2 + 2x_1 + 3x_2^2 - 2x_2 - 99)z^2 + (2x_1^2 + 2x_1 + 3x_2^2 - 4x_2 - 99)z + x_1^2 + x_2^2 - 2x_2 - 99) > 0.$

Performing QE to this gives the following condition in $\mathbf{x}$:

$$(P_3 > 0 \wedge P_4 \geq 0) \vee (P_2 \geq 0 \wedge P_4 \geq 0), \qquad (23)$$

where

$P_2(\mathbf{x}) = 3x_2^2 - 2x_2 + x_1^2 + 2x_1 - 99,$

$P_3(\mathbf{x}) = 264627x_2^4 + 7128x_1x_2^3 - 349668x_2^3 - 3596x_1^3x_2^2 + 169274x_1^2x_2^2 + 462528x_1x_2^2 - 13152942x_2^2 + 2392x_1^4x_2 + 7952x_1^3x_2 - 426492x_1^2x_2 - 705672x_1x_2 + 19405980x_2 - 400x_1^6 - 1996x_1^5 + 105419x_1^4 + 352836x_1^3 - 9467766x_1^2 - 15524784x_1 + 288178803,$

$P_4(\mathbf{x}) = x_2^2 - 2x_2 + x_1^2 - 99.$

The shaded region in Fig.3 corresponds to (23).

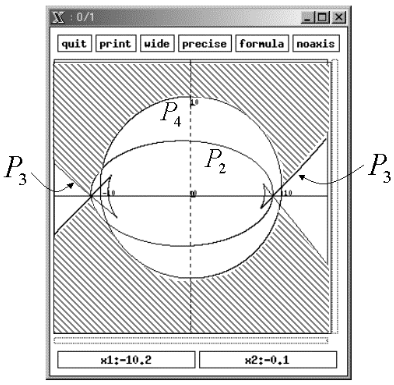

Fig. 3. The possible region of $\mathbf{x}$ described by (23)

Finally, we can obtain the admissible region of $\mathbf{x}$ which meets the all requirements given in §5.2 and 5.3 as illustrated in Fig.4 by superposing (21) and (23) in the parameter space.

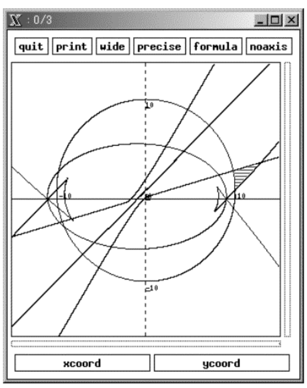

Fig. 4. The possible region of $\mathbf{x}$ by (21) $\wedge$ (23)

6. CONCLUSION

In this paper we proposed a new method of parameter space design for robust control synthesis, particularly in terms of stability radius, based on the scheme of combining of the SDC and a special QE. The validity of our approach has been confirmed by several concrete examples. We can accommodate our method naturally to independent perturbation setting among the polynomial coefficients just by using an explicit formula for the case (*e.g.* (Hitz and Kaltofen, 1998)). In this sense, the framework presented in this paper would provide a unifying platform for further research along this direction.

The advantages of using QE is to be able to resolve many control synthesis problems that are difficult to solve in view of numerical methods. Moreover, QE can be also useful for building up the mathematical modeling (in particular for optimization) of the problems that has no appropriate formularization.

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