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ON ADJOINTS OF HAMILTONIAN SYSTEMS

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Abstract: This paper is concerned with state-space realizations of the adjoints of the variationals of Hamiltonian control systems. It will be shown that the variational systems of a class of Hamiltonian systems have self-adjoint state-space realizations, that is, the variational system and its adjoint have the same state-space realizations. This implies that the input-output mapping of the adjoint of the variational system of a given Hamiltonian system can be calculated by only using the input-output mapping of the original system. Furthermore, this property is applied to adjoint based iterative learning control with optimal control type cost functions.

Keywords: nonlinear control, physical models, iterative improvement, learning control.

1. INTRODUCTION

Hamiltonian control systems are the systems described by well known Hamilton's canonical equations with controlled Hamiltonians (Crouch and van der Schaft, 1987). They are introduced mainly to characterize variational properties of dynamical systems and is used for optimal control, see also (Young, 1969). Those systems were also utilized to describe physical systems, and the related geometric methods of controlling this class of systems supplied fruitful results in control engineering (van der Schaft, 2000; Marsden and Ratiu, 1999). Furthermore, this control framework was generalized in order to handle electro-mechanical systems as well as conventional mechanical ones (Maschke and van der Schaft, 1992), and several control methods are proposed for them, e.g. (Maschke and van der Schaft, 1992; Fujimoto and Sugie, 2001; van der Schaft, 2000). Therefore a scope of this paper contains control of a class of physical systems such as mechanical and electrical systems.

On the other hand, adjoint operators play important roles in linear control systems theory. They provide duality between inputs and outputs which is useful in a variety of control problems, see e.g. (Zhou *et al.*, 1996). Furthermore its nonlinear extension (Batt, 1970) and related works, e.g. (Gray and Scherpen, 1999; Fujimoto *et al.*, 2000), provide useful analysis tools for nonlinear systems. In particular, in (Fujimoto and Scherpen, 2000), the adjoint of the variational systems were utilized in order to characterize the critical points of the Hankel operators and this characterization gives a new balancing and model reduction method. Similar ideas are often used in optimal control (Crouch and van der Schaft, 1987; Young, 1969) and optimization (Walsh, 1975).

In this paper, we discuss state-space realizations of the adjoints of the variational systems of Hamiltonian systems based on the framework developed in (Crouch and van der Schaft, 1987; Fujimoto *et al.*, 2000; Fujimoto and Scherpen, 2000). It will be shown that the variational systems of a class of Hamiltonian systems have self-adjoint state-space realizations, that is, the variational system and its adjoint have the same state-space realizations. This implies that the input-output mapping of the adjoint of the variational system of a given Hamiltonian system can be calculated by only using the input-output mapping of the original system. Furthermore, this property can be utilized for adjoint based iterative learning control (with optimal control type criterion), e.g. (Yamakita and Furuta, 1991), without using the plant model, because we can obtain the adjoint mapping by the input-output data of the original system. This will provide a basis of a new iterative learning control scheme.

2. SELF-ADJOINT STATE-SPACE REALIZATIONS OF HAMILTONIAN SYSTEMS

This section derives the main results, self-adjoint state-space realizations of Hamiltonian systems.

2.1 *Variational systems*

Consider an operator $\Sigma : X\times U \to X\times Y$ with Hilbert spaces X, U and Y with a state-space realization

$$(x^1,y)=\Sigma(x^0,u):\begin{cases}\dot{x} &= f(x,u,t),\ x(t^0)=x^0\\ y &= h(x,u,t)\\ x^1 &= x(t^1)\end{cases} \tag{1}$$

defined on a time interval $t\in[t^0,t^1]$. Typically, $X=\mathbb{R}^n$, $U=L_2^m[t^0,t^1]$ and $Y=L_2^r[t^0,t^1]$. A simpler notation $\Sigma^{x^0}:U\to Y$ with

$$y=\Sigma^{x^0}(u):\begin{cases}\dot{x} = f(x,u,t)\ x(t^0)=x^0\\ y = h(x,u,t)\end{cases}$$

is also employed.

Here let us recall Fréchet derivative of nonlinear operators.

Definition 1. Consider an operator $\Sigma : X\to Y$ with Banach spaces X and Y. Σ is said to be *Fréchet differentiable* at $x\in X$ if there exists an operator $\mathrm{d}\Sigma : X\times X\to Y$ such that $\mathrm{d}\Sigma(x,\xi)$ is linear in ξ and that

$$\lim_{\|\xi\|_X\to 0}\frac{\|\Sigma(x+\xi)-\Sigma(x)-\mathrm{d}\Sigma(x,\xi)\|_Y}{\|\xi\|_X}=0.$$

Under these circumstances, $\mathrm{d}\Sigma(x,\cdot)$ is called the *Fréchet derivative* of Σ at x.

The following lemma proves that the Fréchet derivative of an operator with the state-space realization as in (1) is given by its variational system (Crouch and van der Schaft, 1987).

Lemma 2. (Fujimoto and Scherpen, 2000) The state-space realization of the Fréchet derivative of an operator Σ with a state space realization (1) is given by the variational system of Σ defined by $(x_v^1,y_v)=\mathrm{d}\Sigma((x^0,u),(x_v^0,u_v))$:

$$\begin{cases}\dot{x} &= f(x,u,t), \qquad\qquad x(0)=x^0\\ \begin{pmatrix}\dot{x}_v\\ y_v\end{pmatrix} &= \dfrac{\partial}{\partial(x,u)}\begin{pmatrix}f(x,u,t)\\ h(x,u,t)\end{pmatrix}\begin{pmatrix}x_v\\ u_v\end{pmatrix},\ x_v(0)=x_v^0\ .\\ x_v^1 &= x_v(x^1)\end{cases}$$

By its construction in Definition 1, the Fréchet derivative $\mathrm{d}\Sigma(x,\mathrm{d}x)$ is a locally linear approximation to $\Sigma(x)$, that is

$$\mathrm{d}\Sigma(x,\mathrm{d}x)\approx\Sigma(x+\mathrm{d}x)-\Sigma(x)$$

holds when $\mathrm{d}x$ is small.

2.2 *Adjoints of the variational of Hamiltonian systems*

Here we consider a Hamiltonian system Σ_H with a controlled Hamiltonian $H(x,u,t)$ as $(x^1,y)=\Sigma_H(x^0,u)$:

$$\begin{cases}\dot{x} &= J\,\dfrac{\partial H(x,u,t)}{\partial x}^{\mathrm{T}},\ x(t^0)=x^0\\ y &= -\dfrac{\partial H(x,u,t)}{\partial u}^{\mathrm{T}}\\ x^1 &= x(t^1)\end{cases} \tag{2}$$

with a skew-symmetric matrix $J\in\mathbb{R}^{n\times n}$. In classical mechanics, we treat the class of those systems satisfying

$$x=(q,p)\in\mathbb{R}^{2m},\ \ J=\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}\in\mathbb{R}^{2m\times 2m}.$$

The variational system and its adjoint of Σ_H is given by the following theorem.

Theorem 3. Consider the Hamiltonian system with the controlled Hamiltonian Σ_H in (2). Suppose that J is constant. Then the Fréchet derivative of Σ_H is described by another Hamiltonian system $(x_v^1,y_v)=\mathrm{d}\Sigma_H((x^0,u),(x_v^0,u_v))$:

$$\begin{cases}\dot{x} &= J\,\dfrac{\partial H(x,u,t)}{\partial x}^{\mathrm{T}}, \qquad\qquad x(t^0)=x^0\\ \dot{x}_v &= J\,\dfrac{\partial H_v(x,u,x_v,u_v,t)}{\partial x_v}^{\mathrm{T}},\ x_v(t^0)=x_v^0\\ y_v &= -\dfrac{\partial H_v(x,u,x_v,u_v,t)}{\partial u_v}^{\mathrm{T}}\\ x_v^1 &= x_v(t^1)\end{cases} \tag{3}$$

with a controlled Hamiltonian

$$H_v(x,u,x_v,u_v,t)=\frac{1}{2}\begin{pmatrix}x_v\\ u_v\end{pmatrix}^{\mathrm{T}}\frac{\partial^2 H(x,u,t)}{\partial(x,u)^2}\begin{pmatrix}x_v\\ u_v\end{pmatrix}. \tag{4}$$

Furthermore, the adjoint of the variational system with zero initial state $u_a\mapsto y_a=(\mathrm{d}\Sigma^{x^0}(u))^*(u_a)$ is given by $y_a=(\mathrm{d}\Sigma_H^{x^0}(u))^*(u_a)$:

$$\begin{cases}\dot{x} &= J\,\dfrac{\partial H(x,u,t)}{\partial x}^{\mathrm{T}}, \qquad\qquad x(t^0)=x^0\\ \dot{x}_v &= J\,\dfrac{\partial H_v(x,u,x_v,u_a,t)}{\partial x_v}^{\mathrm{T}},\ x_v(t^1)=0\\ y_a &= -\dfrac{\partial H_v(x,u,x_v,u_a,t)}{\partial u_a}^{\mathrm{T}}\end{cases} \tag{5}$$

which has the same state-space realization as $\mathrm{d}\Sigma_H^{x^0}(u)$. Suppose moreover that J is nonsingular. Then the adjoint $(x_a^1,u_a)\mapsto(x_a^0,y_a)=(\mathrm{d}\Sigma(x^0,u))^*(x_a^1,u_a)$ is given by the state-space realization (5) with the initial and final states

$$x_v(t^1)=J\,x_a^1,\ \ x_a^0=J^{-1}x_v(t^0).$$

Proof. First of all, let us calculate the variational system of Σ_H according to Lemma 2.

$$\begin{cases}\dot{x} &= J\,\dfrac{\partial H(x,u,t)}{\partial x}^{\mathrm{T}}\\ \begin{pmatrix}\dot{x}_v\\ y_v\end{pmatrix} &= \dfrac{\partial}{\partial(x,u)}\begin{pmatrix}J\,\dfrac{\partial H(x,u,t)}{\partial x}^{\mathrm{T}}\\ -\dfrac{\partial H(x,u,t)}{\partial u}^{\mathrm{T}}\end{pmatrix}\begin{pmatrix}x_v\\ u_v\end{pmatrix}\ .\\ x_v^1 &= x_v(t^1)\end{cases}$$

We obtain

$$\begin{aligned}\begin{pmatrix}\dot{x}_v\\ y_v\end{pmatrix} &= \begin{pmatrix}J & 0\\ 0 & -I\end{pmatrix}\frac{\partial^2 H(x,u,t)}{\partial(x,u)^2}^{\mathrm{T}}\begin{pmatrix}x_v\\ u_v\end{pmatrix}\\ &= \begin{pmatrix}J & 0\\ 0 & -I\end{pmatrix}\frac{\partial H_v(x,u,x_v,u_v,t)}{\partial(x_v,u_v)}^{\mathrm{T}}\\ &= \begin{pmatrix}J\,\dfrac{\partial H_v(x,u,x_v,u_v,t)}{\partial x_v}^{\mathrm{T}}\\ -\dfrac{\partial H_v(x,u,x_v,u_v,t)}{\partial u_v}^{\mathrm{T}}\end{pmatrix}\end{aligned}$$

which equals to (3). Next we calculate its adjoint

$$\begin{cases}\dot{x} &= J\,\dfrac{\partial H(x,u,t)}{\partial x}^{\mathrm{T}}\\ \begin{pmatrix}\dot{x}_a\\ y_a\end{pmatrix} &= \begin{pmatrix}-I & 0\\ 0 & I\end{pmatrix}\left(\begin{pmatrix}J & 0\\ 0 & -I\end{pmatrix}\dfrac{\partial^2 H(x,u,t)}{\partial(x,u)^2}\right)^{\mathrm{T}}\begin{pmatrix}x_a\\ u_a\end{pmatrix}.\\ x_a^0 &= x_a(t^0)\end{cases}$$

Here let us define a (possibly singular) coordinate transformation $\bar{x}_a = Jx_a$ and use the fact $J^{\mathrm{T}} = -J$, then we obtain

$$\begin{aligned}\begin{pmatrix}\dot{\bar{x}}_a\\ y_a\end{pmatrix} &= \begin{pmatrix}J & 0\\ 0 & I\end{pmatrix}\begin{pmatrix}\dot{x}_a\\ y_a\end{pmatrix}\\ &= \begin{pmatrix}-J & 0\\ 0 & I\end{pmatrix}\left(\begin{pmatrix}J & 0\\ 0 & -I\end{pmatrix}\frac{\partial^2 H(x,u,t)}{\partial(x,u)^2}\right)^{\mathrm{T}}\begin{pmatrix}x_a\\ u_a\end{pmatrix}\\ &= \begin{pmatrix}-J & 0\\ 0 & I\end{pmatrix}\frac{\partial^2 H(x,u,t)}{\partial(x,u)^2}\begin{pmatrix}J & 0\\ 0 & -I\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}x_a\\ u_a\end{pmatrix}\\ &= \begin{pmatrix}-J & 0\\ 0 & I\end{pmatrix}\frac{\partial^2 H(x,u,t)}{\partial(x,u)^2}\begin{pmatrix}-\bar{x}_a\\ -u_a\end{pmatrix}\\ &= \begin{pmatrix}J & 0\\ 0 & -I\end{pmatrix}\frac{\partial^2 H(x,u,t)}{\partial(x,u)^2}\begin{pmatrix}\bar{x}_a\\ u_a\end{pmatrix}.\end{aligned}$$

This proves (5). Furthermore, if J is nonsingular, then the behavior of the state $x_a(t)$ can be recovered by $x_a(t) = J^{-1}\bar{x}_a(t)$. This completes the proof. □

This theorem proves the fact that the adjoint of the variational system of the Hamiltonian system Σ_H in (2) has a self-adjoint state-space realization. Note that the input-output mapping of the variational system is given by

$$\mathrm{d}\Sigma((x^0,u),(\mathrm{d}x^0,\mathrm{d}u)) \approx \Sigma(x^0+\mathrm{d}x^0,u+\mathrm{d}u) - \Sigma(x^0,u) \quad (6)$$

for a small $(\mathrm{d}x^0,\mathrm{d}u)$. This implies that the input-output mapping of the adjoint $(\mathrm{d}\Sigma_H(x^0,u))^*$ can also be produced by (6) under certain initial conditions. This theorem will be utilized for iterative learning control in the next section.

Remark 4. Strictly speaking, the operator $\mathrm{d}\Sigma$ is *not* self-adjoint because $\mathrm{d}\Sigma$ and $(\mathrm{d}\Sigma)^*$ have different boundary conditions (initial and final states). To obtain self-adjoint operators in a strict sense, we need to assign both initial and final states of the original system (2) as the state-space realizations in (Fujimoto *et al.*, 2000). Also it is noted that a class of port-controlled Hamiltonian systems have a similar structure to that in Theorem 3, and it can be utilized to derive self-adjoint nonlinear Hilbert adjoints (Scherpen and Gray, 2001).

Example 5. Consider a mechanical system

$$\begin{pmatrix}\dot{q}\\ \dot{p}\end{pmatrix} = \begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}\begin{pmatrix}\dfrac{\partial H_0(q,p)}{\partial q}^{\mathrm{T}}\\ \dfrac{\partial H_0(q,p)}{\partial p}^{\mathrm{T}}\end{pmatrix} + \begin{pmatrix}0\\ I\end{pmatrix}u$$

with

$$H_0(q,p) = \frac{1}{2}p^{\mathrm{T}}M(q)^{-1}p + V(q).$$

This system can be described by the Hamiltonian system (2) with a controlled Hamiltonian

$$H(q,p,u) = H_0(q,p) - u^{\mathrm{T}}q.$$

For this system the output function defined in (2) is

$$y = -\frac{\partial H}{\partial u}^{\mathrm{T}} = q.$$

Therefore the adjoint of the variational system with respect to the above output can be calculated by using the input-output mapping of the variational system using (6).

2.3 *Adjoints of the variational of Hamiltonian systems with dissipation*

Next we consider a Hamiltonian system Σ_H with a controlled Hamiltonian $H(x,u,t)$ with dissipation $(x^1,y) = \Sigma_H(x^0,u)$:

$$\begin{cases}\dot{x} &= (J-R)\,\dfrac{\partial H(x,u,t)}{\partial x}^{\mathrm{T}},\ x(t^0) = x^0\\ y &= -\dfrac{\partial H(x,u,t)}{\partial u}^{\mathrm{T}}\\ x^1 &= x(t^1)\end{cases} \quad (7)$$

with a skew-symmetric matrix $J \in \mathbb{R}^{n\times n}$ and a semi-positive definite one $R \in \mathbb{R}^{n\times n}$. The additional term R represents dissipative elements such as friction of mechanical systems and resistance of electric circuits. For this system, the following theorem holds.

Theorem 6. Consider the Hamiltonian system with dissipation and the controlled Hamiltonian Σ_H in (7). Suppose that J and R are constant and that there exist nonsingular matrices $T_x \in \mathbb{R}^{n\times n}$ and $T_u \in \mathbb{R}^{m\times m}$ satisfying

$$\begin{aligned}J &= -T_x J\, T_x^{-1}\\ R &= T_x R\, T_x^{-1}\\ \frac{\partial^2 H(x,u,t)}{\partial(x,u)^2} &= \begin{pmatrix}T_x & 0\\ 0 & T_u\end{pmatrix}\frac{\partial^2 H(x,u,t)}{\partial(x,u)^2}\begin{pmatrix}T_x & 0\\ 0 & T_u\end{pmatrix}^{-1}.\end{aligned} \quad (8)$$

Then the Fréchet derivative of Σ_H is described by another Hamiltonian system $(x_v^1, y_v) = \mathrm{d}\Sigma_H((x^0,u),(x_v^0,u_v))$:

$$\begin{cases} \dot{x} &= (J-R)\dfrac{\partial H(x,u,t)}{\partial x}^{\mathrm{T}}, \qquad x(t^0)=x^0 \\ \dot{x}_v &= (J-R)\dfrac{\partial H_v(x,u,x_v,u_v,t)}{\partial x_v}^{\mathrm{T}}, \ x_v(t^0)=x_v^0 \\ y_v &= -\dfrac{\partial H_v(x,u,x_v,u_v,t)}{\partial u_v}^{\mathrm{T}} \\ x_v^1 &= x_v(t^1) \end{cases} \tag{9}$$

with a controlled Hamiltonian $H_v(x,u,x_v,u_v,t)$ in (4). Furthermore, the adjoint of the variational system with zero initial state $u_a \mapsto y_a = (\mathrm{d}\Sigma^{x^0}(u))^*(u_a)$ is given by $y_a = (\mathrm{d}\Sigma_H^{x^0}(u))^*(u_a)$:

$$\begin{cases} \dot{x} &= (J-R)\dfrac{\partial H(x,u,t)}{\partial x}^{\mathrm{T}}, \qquad x(t^0)=x^0 \\ \dot{x}_v &= -(J-R)\left.\dfrac{\partial H_v(x,u,x_v,u_v,t)}{\partial x_v}^{\mathrm{T}}\right|_{u_v=T_u u_a}, \ x_v(t^1)=0 \\ y_a &= -T_u^{-1}\left.\dfrac{\partial H_v(x,u,x_v,u_v,t)}{\partial u_v}^{\mathrm{T}}\right|_{u_v=T_u u_a} \end{cases}. \tag{10}$$

Suppose moreover that $J-R$ is nonsingular. Then the adjoint $(x_a^1, u_a) \mapsto (x_a^0, y_a)(\mathrm{d}\Sigma(x^0,u))^*(x_a^1,u_a)$ is given by the state-space realization (10) with the initial and final states

$$x_v(t^1) = -(J-R)T_x x_a^1, \ \ x_a^0 = -((J-R)T_x)^{-1}x_v(t^0).$$

Proof. First of all, let us calculate the variational system of Σ_H according to Lemma 2.

$$\begin{cases} \dot{x} &= (J-R)\dfrac{\partial H(x,u,t)}{\partial x}^{\mathrm{T}} \\ \begin{pmatrix}\dot{x}_v\\ y_v\end{pmatrix} &= \dfrac{\partial}{\partial(x,u)}\begin{pmatrix}(J-R)\dfrac{\partial H(x,u,t)}{\partial x}^{\mathrm{T}} \\ -\dfrac{\partial H(x,u,t)}{\partial u}^{\mathrm{T}}\end{pmatrix}\begin{pmatrix}x_v\\ u_v\end{pmatrix}. \\ x_v^1 &= x_v(t^1) \end{cases}$$

We obtain

$$\begin{pmatrix}\dot{x}_v\\ y_v\end{pmatrix} = \begin{pmatrix}J-R & 0\\ 0 & -I\end{pmatrix}\frac{\partial^2 H(x,u,t)}{\partial(x,u)^2}^{\mathrm{T}}\begin{pmatrix}x_v\\ u_v\end{pmatrix} = \begin{pmatrix}(J-R)\dfrac{\partial H_v(x,u,x_v,u_v,t)}{\partial x_v}^{\mathrm{T}} \\ -\dfrac{\partial H_v(x,u,x_v,u_v,t)}{\partial u_v}^{\mathrm{T}}\end{pmatrix}$$

which equals to (9). Next we calculate its adjoint

$$\begin{cases} \dot{x} &= (J-R)\dfrac{\partial H(x,u,t)}{\partial x}^{\mathrm{T}} \\ \begin{pmatrix}\dot{x}_a\\ y_a\end{pmatrix} &= \begin{pmatrix}-I & 0\\ 0 & I\end{pmatrix}\left(\begin{pmatrix}J-R & 0\\ 0 & -I\end{pmatrix}\dfrac{\partial^2 H(x,u,t)}{\partial(x,u)^2}\right)^{\mathrm{T}}\begin{pmatrix}x_a\\ u_a\end{pmatrix}. \\ x_a^0 &= x_a(t^0) \end{cases}$$

Here let us define a (possibly singular) coordinate transformation $\bar{x_a} = -(J-R)T_x x_a$ and input and output transformations $\bar{u_a} = T_u u_a$ and $\bar{y_a} = T_u y_a$, then we obtain

$$\begin{aligned} \begin{pmatrix}\dot{\bar{x}}_a\\ \bar{y}_a\end{pmatrix} &= \begin{pmatrix}-(J-R)T_x & 0\\ 0 & T_u\end{pmatrix}\begin{pmatrix}\dot{x}_a\\ y_a\end{pmatrix} \\ &= \begin{pmatrix}(J-R)T_x & 0\\ 0 & T_u\end{pmatrix}\left(\begin{pmatrix}J-R & 0\\ 0 & -I\end{pmatrix}\frac{\partial^2 H(x,u,t)}{\partial(x,u)^2}\right)^{\mathrm{T}}\begin{pmatrix}x_a\\ u_a\end{pmatrix} \\ &= \begin{pmatrix}(J-R)T_x & 0\\ 0 & T_u\end{pmatrix}\frac{\partial^2 H(x,u,t)}{\partial(x,u)^2}\begin{pmatrix}-J-R & 0\\ 0 & -I\end{pmatrix}\begin{pmatrix}x_a\\ u_a\end{pmatrix} \\ &= \begin{pmatrix}J-R & 0\\ 0 & I\end{pmatrix}\begin{pmatrix}T_x & 0\\ 0 & T_u\end{pmatrix}\frac{\partial^2 H}{\partial(x,u)^2}\begin{pmatrix}T_x & 0\\ 0 & T_u\end{pmatrix}^{-1}\begin{pmatrix}(J-R)T_x x_a\\ -T_u u_a\end{pmatrix} \\ &= -\begin{pmatrix}J-R & 0\\ 0 & I\end{pmatrix}\frac{\partial^2 H(x,u,t)}{\partial(x,u)^2}\begin{pmatrix}\bar{x}_a\\ \bar{u}_a\end{pmatrix}. \end{aligned}$$

This proves (10). Furthermore, if $J-R$ is nonsingular then the behavior of the state $x_a(t)$ can be recovered by $x_a(t) = -T_x^{-1}(J-R)^{-1}\bar{x_a}(t)$. This completes the proof. □

Remark 7. Note that the dynamics of x_a in (10) is the reverse-time version of that of x_v in (9). For example, we can utilize Theorem 6 in the following two cases.
(i) Suppose the input u is given such that the time history of the Hessian of the Hamiltonian with respect to (x,u) is symmetrical with respect to the time t, i.e.,

$$\frac{\partial^2 H(x,u,t)}{\partial(x,u)^2}(t-t^0) = \frac{\partial^2 H(x,u,t)}{\partial(x,u)^2}(t^1-t)$$

for all $t \in [t^0,t^1]$. Then $\mathrm{d}\Sigma_H$ has a pseudo self-adjoint state-space realization. This condition can occur in a PTP control of robot manipulators.
(ii) Suppose $H(x,u,t)$ is linear in u and consider a *round trip* type trajectory, that is, consider two inputs u_1 and u_2 such that

$$\varphi(t-t^0,t^0,x^0,u_1) = \varphi(t^1-t,t^0,\varphi(t^1,t^0,x^0,u_1),u_2)$$

for all $t \in [t^0,t^1]$, where $\varphi(t,t^0,x^0,u)$ denotes the solution of the state $x(t)$ of the system $\Sigma_H(x^0,u)$. Then the state-space realizations of $(\mathrm{d}\Sigma^{x^0}(u_1))^*$ and $\mathrm{d}\Sigma^{\varphi(t^1,t^0,x^0,u_1)}(u_2)$ coincide with each other, that is, $(\mathrm{d}\Sigma^{x^0}(u_1))^*$ can be calculated by the input-output data of $\Sigma^{\varphi(t^1,t^0,x^0,u_1)}(u_2)$ and vice versa. This condition can hold when we perform iterative learning control with respect to a *round trip* type desired trajectory.

Example 8. Consider an LCR-circuit depicted in Figure 1. Let φ_1 and φ_2 denote the flux linkages, H_L denotes the inductance energy (a nonlinear function of φ_1 and φ_2), R_1 denotes the resistance, H_C denotes the stored energy of capacitance (a nonlinear function of Q), Q denotes the charge, and V denote the input voltage. Let us definite the input $u=V$ and the state $x=(Q,\varphi_1,\varphi_2)$. Then we obtain the Hamiltonian system (7) with

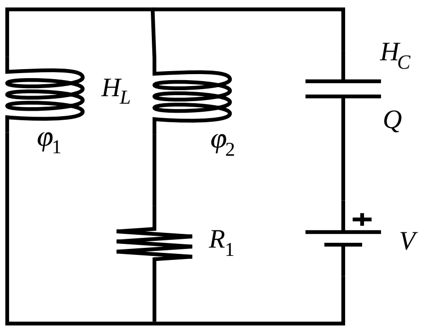

Fig. 1. LCR-circuit

$$H(Q,\varphi_1,\varphi_2,u) = H_C(Q) + H_L(\varphi_1,\varphi_2) + Q\,u$$
$$J = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & R_1 \end{pmatrix}.$$

This system reduces to a port-controlled Hamiltonian system

$$\left\{ \begin{array}{l} \begin{pmatrix} \dot{Q} \\ \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & -R_1 \end{pmatrix} \dfrac{\partial (H_C + H_L)}{\partial (Q,\varphi_1,\varphi_2)}^{\mathrm{T}} + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} u \\ y \quad = -Q \end{array} \right. .$$

This system satisfies the matching condition (8) with

$$T_x = \mathrm{diag}(1,-1,-1), \quad T_u = 1.$$

Therefore, we can calculate the adjoint of the variational system by using the input-output mapping of the original system provided the assumptions in Remark 7 hold.

3. APPLICATION TO ITERATIVE LEARNING CONTROL

This section briefly explains how to apply the results in Section 2 to iterative learning control.

3.1 *General framework*

Let us consider the system Σ in (1) and a cost function $\Gamma : X^2 \times U \times Y \to \mathbb{R}$. The objective is to find the optimal input $(x^0_\star, u_\star)$ minimizing the cost function Γ, that is,

$$(x^0_\star, u_\star) := \arg \min_{(x^0,u)\in X_1 \times U_1} \Gamma(x^0,u,x^1,y) \qquad (11)$$

with $X_1 \times U_1 \subset X \times U$. In general, however, it is difficult to obtain a global minimum since the cost function Γ is not *convex*. Hence we try to obtain a local minimum here, i.e., $X_1 \times U_1 \subsetneq X \times U$. Note that the Fréchet derivative of Γ is

$$\mathrm{d}\Gamma(x^0,u,x^1,y)(\mathrm{d}x^0,\mathrm{d}u,\mathrm{d}x^1,\mathrm{d}y)$$

where

$$\mathrm{d}\Gamma(x^0,u,x^1,y) \in (X^2 \times U \times Y)^*.$$

It follows from well-known Riesz's representation theorem and the linearity of Fréchet derivative that there exists an operator $\Gamma' : X^2 \times U \times Y \to X^2 \times U \times Y$ such that

$$\begin{aligned} &\mathrm{d}\Gamma(x^0,u,x^1,y)(\mathrm{d}x^0,\mathrm{d}u,\mathrm{d}x^1,\mathrm{d}y) \\ &= \langle \Gamma'(x^0,u,x^1,y), (\mathrm{d}x^0,\mathrm{d}u,\mathrm{d}x^1,\mathrm{d}y) \rangle_{X^2\times U\times Y}. \end{aligned} \qquad (12)$$

Since $(x^1,y) = \Sigma(x^0,u)$, the cost function Γ is described by

$$\Gamma(x^0,u,x^1,y) = \Gamma((x^0,u),\Sigma(x^0,u)).$$

Hence a necessary condition for the optimality (11) is characterized via its Fréchet derivative as

$$\mathrm{d}\left(\Gamma((x^0_\star,u_\star),\Sigma(x^0_\star,u_\star))\right)(\mathrm{d}x^0,\mathrm{d}u) = 0$$

for all $(\mathrm{d}x^0,\mathrm{d}u)$. Here we can calculate

$$\begin{aligned} &\mathrm{d}\left(\Gamma((x^0,u),\Sigma(x^0,u))\right)(\mathrm{d}x^0,\mathrm{d}u) \\ &= \mathrm{d}\Gamma((x^0,u),\Sigma(x^0,u))\left((\mathrm{d}x^0,\mathrm{d}u),\mathrm{d}\Sigma(x^0,u)(\mathrm{d}x^0,\mathrm{d}u)\right) \\ &= \left\langle \Gamma'((x^0,u),\Sigma(x^0,u)), \begin{pmatrix} \mathrm{id}_{X\times U} \\ \mathrm{d}\Sigma(x^0,u) \end{pmatrix} (\mathrm{d}x^0,\mathrm{d}u) \right\rangle_{X^2\times U\times Y} \\ &= \left\langle \left(\mathrm{id}_{X\times U}, (\mathrm{d}\Sigma(x^0,u))^*\right) \Gamma'(x^0,u,x^1,y), (\mathrm{d}x^0,\mathrm{d}u) \right\rangle_{X\times U}. \end{aligned}$$

Therefore, if the adjoint $(\mathrm{d}\Sigma(x^0,u))^*$ is available, we can reduce the cost function Γ down at least to a local minimum by an iteration law

$$\begin{aligned} &(x^0_{(i+1)},u_{(i+1)}) = (x^0_{(i)},u_{(i)}) \\ &-K_{(i)}\left(\mathrm{id}_{X\times U}, (\mathrm{d}\Sigma(x^0_{(i)},u_{(i)}))^*\right)\Gamma'(x^0_{(i)},u_{(i)},x^1_{(i)},y_{(i)}) \end{aligned} \qquad (13)$$

or, in the case x^0 is fixed, by another one

$$\begin{aligned} &u_{(i+1)} = u_{(i)} \\ &-K_{(i)}\left(0_{UX},\mathrm{id}_U\right)\left(\mathrm{id}_{X\times U},\ (\mathrm{d}\Sigma(x^0_{(i)},u_{(i)}))^*\right) \\ &\times \Gamma'(x^0_{(i)},u_{(i)},x^1_{(i)},y_{(i)}) \end{aligned} \qquad (14)$$

with a small $K_{(i)} > 0$.

The results in Section 2 enable us to execute this procedure without using the parameters of the original operator Σ, provided Σ is a Hamiltonian system Σ_H. More precise discussion will be made in the following subsection.

3.2 *Iterative learning control*

In this subsection, we consider the Hamiltonian system $\Sigma = \Sigma_H$ in (2) and execute the iterative learning procedure (13) with respect to two typical cost functions.

Conventional iterative learning control

A typical problem of iterative learning control (Yamakita and Furuta, 1991) is to produce an input

u^d letting the output y track a given desired trajectory y^d, that is, to reduce the cost function:

$$\Gamma(y) = \int_{t^0}^{t^1} (y(t) - y^d(t))^{\mathrm{T}} \Gamma_y (y(t) - y^d(t)) \mathrm{d}t \quad (15)$$

with a positive definite matrix $\Gamma_y \in \mathbb{R}^{m\times m}$. In this case, Γ' in (12) is given by

$$\Gamma'(y) = 2\left(0,\ 0,\ 0,\ \Gamma_y(y - y^d)\right).$$

Hence the iteration law (14) reduces to

$$u_{(i+1)} = u_{(i)} - K_{(i)} (\mathrm{d}\Sigma_H^{x^0}(u_{(i)}))^* \Gamma_y (y_{(i)} - y^d).$$

The input-output mapping of the adjoint operator $(\mathrm{d}\Sigma_H^{x^0}(u_{(i)}))^*$ can be obtained by that of the original operator Σ_H using (5) and (6).

Optimal control type criterion

A typical optimal control problem is to achieve a given desired final state x^1 while minimizing the norm of u, that is, to reduce the cost function:

$$\Gamma(u, x^1) = \int_{t^0}^{t^1} u^{\mathrm{T}} \Gamma_u u \, \mathrm{d}t + (x^1 - x^{1\,d})^{\mathrm{T}} \Gamma_{x^1} (x^1 - x^{1\,d}) \quad (16)$$

with positive definite matrices $\Gamma_u \in \mathbb{R}^{m\times m}$ and $\Gamma_{x^1} \in \mathbb{R}^{n\times n}$. In this case, Γ' in (12) is given by

$$\Gamma'(u, x^1) = 2\left(0,\ \Gamma_u\, u,\ \Gamma_{x^1}(x^1 - x^{1\,d}),\ 0\right).$$

Hence the iteration law (14) reduces to

$$u_{(i+1)} = u_{(i)} - K_{(i)} \times \left(\Gamma_u u + (0_{UX}, \mathrm{id}_U)(\mathrm{d}\Sigma(x^0_{(i)}, u_{(i)})^* (\Gamma_{x^1}(x^1 - x^{1\,d}), 0)\right).$$

The input-output mapping of the adjoint operator $(\mathrm{d}\Sigma(x^0_{(i)}, u_{(i)}))^*$ can be obtained by that of the original operator Σ_H using (5) and (6).

Thus iterative learning control with respect to the cost functions (15) and (16) can be executed. Of course this procedure can be performed with any cost function $\Gamma(x^0, u, x^1, y)$, provided $\Sigma = \Sigma_H$ as in (2) (or (7) under the circumstances in Remark 7). This result will provide a basis of a new iterative learning control for a class of physical systems in the Hamiltonian formulation.

4. CONCLUSION

This paper has discussed state-space realizations of the adjoints of the variational systems of Hamiltonian control systems. It has been shown that the variational systems of a class of Hamiltonian systems have self-adjoint state-space realizations. Furthermore, this property has been utilized for adjoint based optimal control without using the model of the system, because we can obtain the adjoint mapping by the input-output data of the original system. In the succeeding research, we will investigate a new iterative learning control scheme based on the results in this paper.

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