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COMPANION FORMS AND CYCLIC MATRICES FOR DISCRETE-TIME PERIODIC SYSTEMS

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Abstract: By means of a proper notion of *periodic cyclic matrix*, we study the possibility of transforming a given periodic system into a canonical companion form. The passage from such form to an input-output periodic representation is straightforward. We characterize the structural properties of a system in canonical form in terms of coprimeness of the two periodic polynomials appearing in the input-output representation. Only single-input single-output discrete-time systems are considered.

Keywords: Periodic Systems, Periodic Cyclic Matrix, Companion Form, Coprimeness of Periodic Polynomials, Reachability and Observability

1. INTRODUCTION AND PROBLEM POSITION

In the last decades, periodic control has raised increasing interest in the system and control community [1]. Yet, some basic issues have been solved only in recent years, and other are still to be clarified. As an example, we mention the celebrated Floquet theorem. Although the original statement by Gaston Floquet goes back to 1883 [2], the discrete-time version of the theorem has been established only recently [3]. In this paper, we clarify some long standing questions related to companion forms for periodic matrices and periodic systems in discrete-time. For, we first introduce the concept of *periodic cyclic matrix* by means of which we address the issue of transforming a given periodic matrix into a companion form (with periodic coefficients). Then, we pass to consider a single-input single-output periodic system, and we discuss when a state space representation is equivalent to an input-output (PARMA) representation. Among other things we consider the periodic transfer function representation, and we show that a periodic system given in a right-coprime fractional representation can be given a state-space form by means of a reachable canonical form. This form enjoys the "n-reachability property", namely any state can be reached in n steps at most, where n is the dimension of the state-space. This is a peculiar feature, since, in general, the reachability transition of a reachable periodic systems requires a time interval with nT time points. Analogously, a periodic system given in a left-coprime fractional input-output representation can be realized in terms of the observable canonical form. The achieved results clarify a number of issues concerning the periodic transfer function and the structural properties of periodic systems. For recent survey paper, see [4] and [5].

2. PROBLEM POSITION

Consider the system

$$x(t+1) = A(t)x(t) \tag{1}$$

where $A(\cdot)$: $t \rightarrow I\!R^n$ is a periodic matrix of period T. The associate transition matrix is

$$\Psi_A(t,\tau) = \begin{cases} A(t)A(t-1)\cdots A(\tau) & , \ t > \tau \\ I_n & , \ t = \tau \end{cases}$$

When $\tau = t - T$, then the transition matrix takes the name of *monodromy matrix*, denoted as $\Phi_A(t)$:

$$\Phi_A(t) = \Psi_A(t+T, t)$$

Notice in passing that, although $\Phi_A(t)$ changes with time, the characteristics multipliers are time-invariant, [6]. System (1), or, equivalently, $A(\cdot)$, is stable if and only if the multipliers are all located inside the unit disk in the complex plane.

Under the action of a state-space Lyapunov transformation $z(t) = Q(t)x(t)$, where $Q(\cdot)$ is a T-periodic matrix invertible for each t, the transformed system takes on the form

$$x(t+1) = \tilde{A}(t)x(t)$$

where

$$\tilde{A}(t) = Q(t+1)A(t)Q(t)^{-1} \tag{2}$$

The two systems with matrices $A(\cdot)$ and $\tilde{A}(\cdot)$ are said to be algebraically equivalent each other.

A basic system-theoretic issue is whether it is possible to work out a change of basis in order to represent in a simplified way a dynamical matrix $A(\cdot)$. This leads to the definition of the companion forms defined in the following section. Precisely, we will introduce the notion of cyclic periodic matrix and we will show that a periodic system is algebraically equivalent to a system in a companion form iff it is cyclic. In Section 3 we analize the structural properties of a system in the reachable (observable) canonical form, by resorting to the notion of strong coprimeness of periodic polynomials. Partial results on the realizability of a periodic system in the canonical observability form can be found in [7] and [8].

3. COMPANION FORMS

In this section we introduce some canonical forms for the dynamical matrix $A(\cdot)$. This leads to the definition of the so-called *companion forms*. Precisely we will focus on the following two forms.

1) The $n \times n$ matrix

$$A_{hc}(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_n(t) & -\alpha_{n-1}(t) & -\alpha_{n-2}(t) & \cdots & -\alpha_1(t) \end{bmatrix}$$

where $\alpha_i(t)$, $i = 1, 2, \cdots, n$, are T-periodic coefficients, is said to be in *h-companion form*

2) The $n \times n$ matrix

$$A_{vc}(t) = \begin{bmatrix} 0 & 0 & \cdots & 0 & -\beta_n(t) \\ 1 & 0 & \cdots & 0 & -\beta_{n-1}(t) \\ 0 & 1 & \cdots & 0 & -\beta_{n-2}(t) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -\beta_1(t) \end{bmatrix}$$

where $\beta_i(t)$, $i = 1, 2, \cdots, n$, are T-periodic coefficients, is said to be in *v-companion form*

We now characterize the conditions under which $A(\cdot)$ is algebraically equivalent to a matrix in h-companion or v-companion form. To this end, the following definition is in order.

Definition 3.1. A $n \times n$ T-periodic matrix $A(\cdot)$ is said to be T-cyclic if there exists a T-periodic vector $x(\cdot)$, $x(t) \neq 0$, $\forall t$, such that the following $n \times n$ matrix

$$R(t) = \begin{bmatrix} x_1(t) & x_2(t) & x_3(t) & \cdots & x_n(t) \end{bmatrix} \tag{3}$$

where

$$x_i(t) = \Psi_A(t, t-i+1)x(t), \quad i = 1, 2, \cdots n$$

is invertible, for all t. If such a vector $x(\cdot)$ exists, it is said to be a periodic cyclic generator.

To the best knowledge of the authors, the above definition is given herein for the first time.

It is not difficult to show that for a T-cyclic matrix $A(\cdot)$ it is possible to find a set of T-periodic coefficients $\alpha_i(\cdot)$, $i = 1, 2, \cdots n$ such that

$$\begin{aligned} &\alpha_n(t-1)x(t) + \alpha_{n-1}(t-2)\Psi_A(t,t-1)x(t-1) + \\ &\alpha_{n-2}(t-3)\Psi_A(t,t-2)x(t-2) + \cdots + \\ &\alpha_1(t-n)\Psi_A(t,t-n+1)x(t-n+1) + \\ &\Psi_A(t,t-n)x(t-n) = 0 \end{aligned} \tag{4}$$

Indeed, being matrix $R(t)$ invertible, any vector of R^n, and, in particular $\Psi_A(t, t-n)x(t-n)$, can be seen as a linear combination of the columns of $R(t)$. The periodicity of the coefficients is a consequence of the periodicity of $R(t)$ and $\Psi_A(t, t-n)x(t-n)$.

It is also important to stress that a T-periodic matrix in h-companion form or in v-companion form is indeed T-cyclic. Precisely, the cyclic generator associated with a h-companion form is $x_{hc} = \left[\, 0 \; 0 \; \cdots \; 1 \,\right]'$, whereas the one associated with a v-companion form is $x_{vc} = \left[\, 1 \; 0 \; \cdots \; 0 \,\right]'$.

We are now in the position to characterize the class of T-cyclic matrices in terms of companion forms.

Theorem 3.1. With reference to an $n \times n$ T-periodic matrix $A(\cdot)$, the following statements are equivalent each other

(i) $A(\cdot)$ is T-cyclic.
(ii) $A(\cdot)$ is algebraically equivalent to a T-periodic matrix in v-companion form.
(iii) $A(\cdot)$ is algebraically equivalent to a T-periodic matrix in h-companion form.

Proof (i) $\leftrightarrow$ (ii)

Assume that (i) holds. Then there exists a T-periodic vector $x(\cdot)$ such that $R(t)$ defined in (3) is invertible for each t. Then it is immediate to see that for any $i = 1, 2, \cdots, n-1$, the $i-th$ column of $A(t)R(t)$ is equal to the $i+1$-th column of $R(t+1)$. As for the last column of $A(t)R(t)$, consider equation (4) with t replaced by $t+1$. It follows that the last column of $A(t)R(t)$ is a linear combination of the preceding columns of $R(t+1)$, the $i-th$ being weighted by coefficient $\alpha_{n-i+1}(t-i+1)$. Hence,

$$A(t)R(t) = R(t+1)A_{vc}(t)$$

where the parameters $\beta_i(t)$ appearing in the expression of $A_{vc}(t)$ are given by $\beta_{n-i}(t) = \alpha_{n-i}(t-i)$, $i = 0, 1, \cdots, n-1$. This proves point (ii). Vice versa, if point (ii) holds, then there exists a T-periodic invertible matrix $S(\cdot)$ such that $S(t+1)A(t)S(t)^{-1} = A_{vc}(t)$. Hence, a cyclic generator for $A(\cdot)$ is $x(t) = S(t)^{-1}x_{vc}$, as it is easy to verify.

(i) $\leftrightarrow$ (iii)

Assume again that (i) holds and let $x(\cdot)$ be such that $R(t)$ defined in (3) is invertible for each t. Moreover, consider again the coefficients $\alpha_i(\cdot)$ defined in (4). Finally, set

$$v_0(t) = x(t) \tag{5}$$
$$v_1(t) = A(t-1)v_0(t-1) + \alpha_1(t-1)x(t) \tag{6}$$
$$v_2(t) = A(t-1)v_1(t-1) + \alpha_2(t-1)x(t) \tag{7}$$
$$\vdots$$
$$v_{n-1}(t) = A(t-1)v_{n-2}(t) + \alpha_{n-1}(t-1)x(t) \tag{8}$$

It it apparent that the vectors $v_i(t)$, $i = 1, 2, \cdots, n-1$, constitute a triangular combination of the columns of $R(t)$. As such, they span the entire space R^n, for each t. It is then possible to define the invertible state-space transformation

$$S(t) = \left[\, v_{n-1}(t) \;\; v_{n-2}(t) \;\; \cdots \;\; v_0(t) \,\right]^{-1} \tag{9}$$

From the definition of $v_i(t)$, it is straightforwardly seen that the last $n-1$ columns of $A(t)S(t)^{-1}$ are given by

$$\begin{aligned}
A(t)v_0(t) &= v_1(t+1) - \alpha_1(t)x(t+1) \\
A(t)v_1(t) &= v_2(t+1) - \alpha_2(t)x(t+1) \\
A(t)v_2(t) &= v_3(t+1) - \alpha_3(t)x(t+1) \\
&\vdots \\
A(t)v_{n-2}(t) &= v_{n-1}(t+1) - \alpha_{n-1}(t)x(t+1)
\end{aligned}$$

The first column of $A(t)S(t)^{-1}$ can be computed by means of a back propagation of the sequence (5)-(8), and by taking into account (4). It follows

$$\begin{aligned}
&A(t)v_{n-1}(t) = \\
&A(t)A(t-1)v_{n-2}(t) + \alpha_{n-1}(t-1)A(t)v_0(t) = \\
&\cdots = \Psi_A(t+1, t-n+1)v_0(t-n+1) + \\
&\sum_{k=0}^{n-2} \alpha_{n-k-1}(t-k-1)\Psi_A(t+1, t-k)v_0(t-k) = \\
&-\alpha_n(t)v_0(t+1)
\end{aligned}$$

Hence $A(t)S(t)^{-1} = S(t+1)A_{hc}(t)$ so yielding point (ii). Conversely, if statement (iii) holds, then $S(t+1)A(t)S(t)^{-1} = A_{hc}(t)$ for some invertible T-periodic state-space transformation $S(\cdot)$. Then, $x_{hc}(t)$ is a cyclic generator for $A_{hc}(t)$ and, consequently, vector $x(t) = S(t)^{-1}x_{hc}(t)$ is a periodic cyclic generator for $A(\cdot)$, as can be easily verified.

4. STRUCTURAL PROPERTIES

In this section we point out the relation between cyclicity and the structural properties of a SISO periodic system described by

$$\begin{aligned}
x(t+1) &= A(t)x(t) + B(t)u(t) \\
y(t) &= C(t)x(t)
\end{aligned}$$

where $B(\cdot)$ and $C(\cdot)$ are T-periodic as well. We first consider systems in the*reachable canonical form*, i.e. systems described by

$$A(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_n(t) & -\alpha_{n-1}(t) & -\alpha_{n-2}(t) & \cdots & -\alpha_1(t) \end{bmatrix}$$
$$B(t) = \left[\, 0 \ 0 \ \cdots \ 0 \ 1 \,\right]'$$
$$C(t) = \left[\, \gamma_1(t) \ \gamma_2(t) \ \cdots \ \gamma_{n-1}(t) \ \gamma_n(t) \,\right]$$

where $\alpha_i(\cdot)$ and $\gamma_i(\cdot)$ are T-periodic functions, $\forall i$. The dynamic matrix of this form is a matrix in *h-canonical form.*

The system in the canonical reachable form can be equivalently rewritten by resorting to the two T-periodic polynomials

$$d(\sigma, t) = \sigma^n + \alpha_1(t)\sigma^{n-1} + \alpha_2(t)\sigma^{n-2} + \cdots + \alpha_n(t)$$
$$n(\sigma, t) = \gamma_1(t)\sigma^{n-1} + \gamma_2(t)\sigma^{n-2} + \cdots + \gamma_n(t)$$

which completely define the system. The variable σ represents the one step-ahead operator. It is worth noticing that this operator does not commute with a periodic polynomial, say $r(\sigma, t)$. However, the following skew commutative rule holds:

$$\sigma^k r(\sigma, t) = r(\sigma, t + r)\sigma^k$$

Hence, commutation occurs iff k is a multiple of the period T.

By means of the above defined polynomials, one can represent the system given in the canonical reachable form by the input-output polynomial model

$$y(t) = n(\sigma, t)z(t)$$
$$u(t) = d(\sigma, t)z(t)$$

where, as easily checkable, the variable $z(t)$ coincides with the first state variable $x_1(t)$. Hence,

$$y(t) = n(\sigma, t)d(\sigma, t)^{-1}u(t)$$

is the right fractional representation of the system.

There are some papers where the algebra of periodic polynomials is studied. Among them, the paper [9] furnishes some preliminary facts. The concept we are interested in is the definition of *strong right coprimeness* of two periodic polynomials. We say that $(n(\sigma, t), d(\sigma, t))$ are strongly right coprime if there exist two periodic polynomials $x(\sigma, t)$ and $y(\sigma, t)$ such that the following operatorial Bezout identity holds true:

$$x(\sigma, t)n(\sigma, t) + y(\sigma, t)d(\sigma, t) = 1$$

The system in the canonical reachable form defined above is, by construction, a fully reachable periodic system. In addition, thanks to the structure of matrices $(A(\cdot), B(\cdot))$, the reachability interval is no longer than the system order n. Notice that this property does not hold in general since, as well known, the interval of time required to reach the states of a reachable periodic system may be as long as nT steps, see [10] and [11].

Now, we want to assess the observability properties of the system, which, as already said, depends on the properties of the two polynomials above.

Theorem 4.1. The system in the reachable canonical form is observable for each t if and only if the two polynomials $(d(\sigma, t), n(\sigma, t))$ are strongly right coprime.

Proof If the system is not observable, then

$$\begin{bmatrix} \sigma I - A(t) \\ C(t) \end{bmatrix} p(t) = 0 \qquad (10)$$

for some nonzero exponentially modulated vector function $p(t) = \lambda^t \bar{p}(t)$, where λ is a complex number and $\bar{p}(\cdot)$ a periodic function. Hence, taking into account the structure of $A(\cdot)$ and $C(\cdot)$ it follows

$$n(\sigma, t)p_1(t) = 0, \quad d(\sigma, t)p_1(t) = 0 \qquad (11)$$

where $p_1(\cdot)$ is the first entry of $p(\cdot)$. Notice that $p_1(\cdot)$ cannot be identically zero. Hence, the Bezout identity cannot be satisfied by any $x(\sigma, t)$ and $y(\sigma, t)$, i.e. the two polynomials $n(\sigma, t)$, $d(\sigma, t)$ are not strongly right coprime.

Vice-versa, if the polynomials are not strongly right coprime, there exists an exponentially modulated non zero function $p_1(t)$ such that (11) is satisfied. By taking $p_{i+1}(t) = p_i(t+1)$, $i = 1, 2, \cdots n-1$ the conclusion (10) follows, so that the system is not observable for each t. ∎

It is easy to show that a dual result can be given if the system in the canonical observable form is considered, i.e.

$$A(t) = \begin{bmatrix} 0 & 0 & \cdots & 0 & -\beta_n(t) \\ 1 & 0 & \cdots & 0 & -\beta_{n-1}(t) \\ 0 & 1 & \cdots & 0 & -\beta_{n-2}(t) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -\beta_1(t) \end{bmatrix}$$
$$B(t) = \left[\, \delta_1(t) \ \delta_2(t) \ \cdots \ \delta_{n-1}(t) \ \delta_n(t) \,\right]'$$
$$C(t) = \left[\, 0 \ 0 \ \cdots \ 0 \ 1 \,\right]$$

where $\beta_i(\cdot)$ and $\delta_i(\cdot)$ are T-periodic functions, $\forall i$. Now, the fractional polynomial representation of the system in canonical observable form is

$$d(\sigma,t)y(t) = z(t)$$
$$n(\sigma,t)u(t) = z(t)$$

where

$$d(\sigma,t) = \sigma^n + \sigma^{n-1}\beta_1(t) + \sigma^{n-2}\beta_2(t) + \cdots \beta_n(t)$$
$$n(\sigma,t) = \sigma^{n-1}\delta_1(t) + \sigma^{n-2}\delta_2(t) + \cdots \delta_n(t)$$

Hence,

$$y(t) = d(\sigma,t)^{-1}n(\sigma,t)u(t)$$

is the left fractional representation of the system. Notice in passing that this model corresponds to the so-called PARMA representation, widely used for prediction and identification purposes.

The polynomials $d(\sigma,t)$, $n(\sigma,t)$ are said to be strongly left coprime if there exist two periodic polynomials $x(\sigma,t)$ and $y(\sigma,t)$ such that the following identity holds true:

$$n(\sigma,t)x(\sigma,t) + d(\sigma,t)y(\sigma,t) = 1$$

The system in the observable canonical form is, by construction, observable for each t. Moreover, observability can be performed in n steps at most. As for the reachability properties of the system, the following result can be proven in a totally analogous way of Theorem 4.1.

Theorem 4.2. The system in observable canonical form is reachable for each t if and only if the two polynomials $(d(\sigma,t), n(\sigma,t))$ are strongly left coprime.

5. CONCLUSION

This paper is composed by two parts. The first one deals with purely algebraic concepts relative to square matrices with periodically time-varying coefficients. Precisely, we discuss when a generic matrix of this family is algebraically equivalent to a periodic matrix in companion form. This possibility is of notable interest since matrices in companion forms are characterized by a relatively small number of free parameters and as such lead to parsimonious representations. The equivalence is provided in terms of the key notion of cyclic periodic matrix, introduced in this paper for the first time. Then, we pass to the realm of discrete-time dynamic systems. For single-input single-output periodic systems in state-space description, we define a canonical model based on the reachable canonical form and a canonical model based on the observable canonical form, both relying on dynamical matrices in periodic companion forms. From these canonical models the passage to polynomial input-output descriptions is straightforward. We discuss the reachability and observability properties of the canonical forms in terms of strong coprimeness of the periodic polynomials of the input-output description. The results so obtained clarify a number of important issues. In particular, they lead to a better comprehension of the emerging concept of transfer function in periodic control theory. The extension to multi-input multi-output periodic systems is not a trivial subject, and is currently underway.

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