15th Triennial World Congress, Barcelona, Spain

A DILATION OF TRANSFER MATRICES WITH INFINITE AND FINITE IMAGINARY AXIS ZEROS

Xin XIN

Department of Communication Engineering
Faculty of Computer Science and System Engineering
Okayama Prefectural University
111 Kuboki, Soja, Okayama 719-1197, JAPAN
Email: xxin@c.oka-pu.ac.jp

Abstract: For a non-square transfer matrix with zeros on the extended imaginary axis including infinity, this paper discusses how to dilate (augment) such transfer matrix to a square one without adding extra zeros on the extended imaginary axis. The state-space construction for the dilation is proposed by using the finite and infinite eigenstructures of the transfer matrix. *Copyright © 2002 IFAC*

Keywords: Zero, Eigenvalue, Singular system, Infinite zeros, Imaginary axis zeros, Singular pencil, Generalized eigenvalue problems, Dilation, Annihilator.

1. INTRODUCTION

For multivariable control systems, we usually encounter the cases where the number of the inputs is not equal to that of outputs. The analysis and design of non-square systems, for example, the factorization or the inversion of non-square transfer systems, are complicated than those of square ones. The dilation (augmentation) of a non-square matrix to a square one preserving some properties is often used. For example, the dilation of the 4-block H_∞ control problem to 2-block one or 1-block one has been adopted in (Glover *et al.*, 1991), (Green and Limebeer, 1995), (Kimura, 1995), (Kimura, 1996).

In the above papers, the dilation is constructed under the assumption that the non-square transfer matrices do not have any zeros on the extended imaginary axis which includes the infinity (denoted as Ω_e here). For a non-square transfer matrix with Ω_e zeros, this paper tries to discuss a special dilation of it without adding extra Ω_e zeros. The reason for such dilation is that many control criteria require the assumptions of non-existence of Ω_e zeros. The offending Ω_e zeros make the analysis and design method complicated, for example, the H_∞ control problem with $j\omega$-axis zeros (Hara *et al.*, 1992), (Stoorvogel, 1991), (Scherer, 1992), (Xin *et al.*, 2000).

Let a stabilizable and detectable realization of a transfer matrix $G(s)$ with full normal rank be

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right], \quad A \in \mathbf{R}^{n\times n},\ D \in \mathbf{R}^{m\times r}, \quad (1)$$

where D is not full rank and/or $G(s)$ has finite $j\omega$-axis zeros.

We consider the case $m > r$ in this paper. The dual case $m < r$ can be treated similarly by considering $G^T(s)$.

Consider the following dilation of $G(s)$

$$\bar{G}(s) = \left[\, G(s)\ G'(s) \,\right] = \left[\begin{array}{c|cc} A & B & B' \\ \hline C & D & D' \end{array}\right], \quad (2)$$

where

$$\bar{B} := \left[\, B\ B' \,\right] \in \mathbf{R}^{n\times m},$$

$$\bar{D} := \left[\, D\ D' \,\right] \in \mathbf{R}^{m\times m}.$$

It is obvious that all the Ω_e zeros of $G(s)$ are those of $\bar{G}(s)$. The reverse statement is not always true. The *dilation problem* in this paper is to find an appropriate dilation of $G(s)$ as given in (2), if it exists, such that $G(s)$ and $\bar{G}(s)$ have the same Ω_e zeros.

On the other hand, it often needs to find a *proper* annihilator with full normal rank $m-r$ for $G(s)$ in (1) (Kimura *et al.*, 1991), i.e. find $G^{\perp}(s)$ of size $(m-r)\times m$ with full normal rank $m-r$ such that $G^{\perp}(s)G(s)=0$. However, if D is not full column rank, $G^{\perp}(s)$ is usually non-proper and is expressed by descriptor form. As an application of the solution to the dilation problem in this paper, we shall construct a unitary annihilator for $G(s)$, i.e. construct $G^{\perp}(s)$ such that $G^{\perp}(s)$ is proper and satisfies $G^{\perp}(s)(G^{\perp}(-s))^T=I_{m-r}$. Note that such unitary annihilator plays an important role in working out solutions of H_∞ control problems (Kimura *et al.*, 1991), and the problem of construction of such annihilator for $G(s)$ with infinite and finite imaginary axis zeros has not been solved yet.

This paper is organized as follows: some preliminary results about the infinite and finite eigenstructure of $G(s)$ are given in Section 2. The dilation is designed in Section 3. The unitary annihilator of $G(s)$ is presented in Section 4. A numerical example is given in Section 5. Conclusion is made in Section 6.

Notations: In this paper, the complex plane and open left half complex plane are denoted by $\mathbf{C}$ and $\mathbf{C}_-$, respectively. Ω denotes the finite imaginary axis ($j\omega$-axis). The set of all $m\times r$ constant real matrices is denoted by $\mathbf{R}^{m\times r}$. I_r denotes the identity matrix of size $r\times r$. $0_{m\times r}$ denotes the zero matrix of size $m\times r$. The subscripts is dropped, if these dimensions are clear from the the context. A^T means the transpose of A. $|A|$ denotes the determinant of A. $\operatorname{Im} A$ and $\operatorname{Ker} A$ denote the image space and null space of A, respectively. The generalized eigenspace of $-sE+A$ corresponding to the eigenvalues in domain D is denoted by $\nu\{-sE+A;D\}$. We denote

$$C(sE-A)^{-1}B+D := \left[\begin{array}{c|c} -sE+A & B \\ \hline C & D \end{array}\right],\quad E\neq I.$$

If $E=I$, the term $-sE+A$ in the above notation is replaced by A for simplicity. For invertible matrices M and N, the identity

$$\left[\begin{array}{c|c} -sE+A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} -sMEN+MAN & MB \\ \hline CN & D \end{array}\right] \quad (3)$$

is termed as *restricted equivalent transformation* (Verghese *et al.*, 1981) under (M,N). If $N=M^{-1}$, the corresponding equivalent transformation is called as similarity transformation under M.

2. PRELIMINARIES

This section serves to recall some background material drawn largely from (Lewis, 1986), (Xin and Mita, 1998), and it defines some key quantities used subsequently through introduction of several matrices and one special coordinate base.

Consider the system matrix pencil of $G(s)$ being defined as:

$$P(s) := -sP_E + P_A, \quad (4)$$

where

$$P_E := \begin{bmatrix} I & 0 \\ 0 & 0_{m\times r} \end{bmatrix},\quad P_A := \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \quad (5)$$

Since $\dim(\operatorname{Ker} P_E) = r$, there exist linear independent infinite eigenvectors and extended infinite eigenvectors such that

$$P_E v_j^1 = 0,\quad j=1,\cdots,r, \quad (6)$$

and the extended infinite eigenvectors are defined by

$$P_E v_j^{k+1} = P_A v_j^k,\quad k=1,\cdots,k_j-1, \quad (7)$$

where $v_j^{k_j}$ is the last (highest) vector of each infinite eigenvector chain $(v_j^1,\cdots,v_j^{k_j})$ satisfying

$$P_A v_j^{k_j} \notin \operatorname{Im} P_E. \quad (8)$$

According to Lemma 4 in (Copeland and Safonov, 1992), $z_j=\infty$ $(j=1,\cdots,r)$ are zeros of $P(s)$ of orders k_j-1, and $G(s)$ has infinite zeros $z_j=\infty$ of order k_j-1. Therefore, if $k_j=1$ holds for certain j, then $z_j=\infty$ is not an infinite zero of $G(s)$.

Now arranging all the above infinite eigenvectors in the following way to construct

$$V_\infty := \left[\, V_r \;\; V_h \,\right], \quad (9)$$

where $V_h \in \mathbf{R}^{(n+r)\times r}$ contains all the *last (highest)* infinite eigenvectors, i.e.

$$V_h := \left[\, v_1^{k_1} \;\cdots\; v_p^{k_r} \,\right] \quad (10)$$

and $V_r \in \mathbf{R}^{(n+r)\times n_r}$ are the rest ones in any order, where

$$n_r := n_\infty - r,\quad n_\infty := \sum_{j=1}^{p} k_j. \quad (11)$$

Hence, the complete infinite eigenstructure of $-sP_E+P_A$ is defined by

$$(-sP_E+P_A)V_\infty = P_A V_\infty(-sN+I_{n_\infty}), \quad (12)$$

where $N\in\mathbf{R}^{n_\infty\times n_\infty}$ is a nilpotent matrix.

From (7), we know that $P_A V_r \in \operatorname{Im} P_E$ which leads $\begin{bmatrix} C & D \end{bmatrix} V_r = 0$, then decompose

$$P_A V_\infty = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_r & V_h \end{bmatrix} =: \begin{bmatrix} T & \hat{B} \\ 0 & \hat{D} \end{bmatrix}, \quad (13)$$

which yields

$$T := \begin{bmatrix} A & B \end{bmatrix} V_r, \quad \hat{B} := \begin{bmatrix} A & B \end{bmatrix} V_h, \quad (14)$$

$$\hat{D} := \begin{bmatrix} C & D \end{bmatrix} V_h. \quad (15)$$

It follows from Lemma C.2 in (Copeland and Safonov, 1992) that $\hat{D}$ has full column rank.

From Lemma 3 in (Xin and Mita, 1998), we know that there exists an invertible matrix

$$A_\infty = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad A_{11} \in R^{n_r \times n_r}, \ A_{22} \in R^{r \times r}$$

such that

$$V_\infty A_\infty = \begin{bmatrix} T & 0 \\ 0 & I_r \end{bmatrix}, \quad N A_\infty = \begin{bmatrix} I_{r\times r} & 0 \\ 0 & 0 \end{bmatrix} \quad (16)$$

hold. Therefore,

$$\operatorname{Im} \begin{bmatrix} T & 0 \\ 0 & I_r \end{bmatrix} = \operatorname{Im} V_\infty = \nu\{-sP_E + P_A, \ \infty\}. \quad (17)$$

Next, we need to consider the finite $j\omega$-axis eigenstructure of $P(s)$. Let the dimension of $j\omega$-axis eigenspace of $P(s)$ be n_j and the eigenspace be spanned by $\begin{bmatrix} T_1^T & T_2^T \end{bmatrix}^T$ with $T_1 \in \mathbf{R}^{n \times n_j}$ and $T_2 \in \mathbf{R}^{r \times n_j}$, i.e.

$$\operatorname{Im} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \nu\{-sP_E + P_A, \ \Omega\}. \quad (18)$$

It follows that there exists Λ_j such that

$$(-sP_E + P_A) \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} T_1 \\ 0 \end{bmatrix} (-sI + \Lambda_j). \quad (19)$$

Note that all eigenvalues of Λ_j are invariant zeros of $G(s)$ on $j\omega$-axis.

Finally, consider the stable eigenspace of $W(s)$ which is the system matrix pencil of the *spectral density matrix* $G^\sim(s)G(s)$ denoted as

$$W(s) := -sW_E + W_A \quad (20)$$

with $W_E := \operatorname{diag}\{I, I, 0\}$ and

$$W_A := \begin{bmatrix} A & 0 & B \\ -C^T C & -A^T & -C^T D \\ D^T C & B^T & D^T D \end{bmatrix}. \quad (21)$$

Let the stable eigenspace of $W(s)$ be spanned by $\begin{bmatrix} U_1^T & U_2^T & U_3^T \end{bmatrix}^T$ with $U_1 \in \mathbf{R}^{n \times n_-}$, $U_2 \in \mathbf{R}^{n \times n_-}$ and $U_3 \in \mathbf{R}^{r \times n_-}$, i.e.

$$\operatorname{Im} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \nu\{W(s), \ \mathbf{C}_-\}. \quad (22)$$

It follows that there exists a real stable $\Lambda \in \mathbf{R}^{n_- \times n_-}$ satisfying

$$W_A \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = W_E \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} \Lambda = \begin{bmatrix} U_1 \\ U_2 \\ 0 \end{bmatrix} \Lambda. \quad (23)$$

LEMMA 1. (Xin and Mita, 1998) With the quantities as defined in (14), (19) and (23), the Ω_e eigenspace of $P(s)$ and the stable eigenspace of $W(s)$ satisfy the following statements:

(i) S is nonsingular, where

$$S := \begin{bmatrix} U_1 & T_1 & T \end{bmatrix} \in \mathbf{R}^{n \times n}. \quad (24)$$

(ii)

$$X := \begin{bmatrix} U_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} U_1 & T_1 & T \end{bmatrix}^{-1} \geq 0 \quad (25)$$

is a solution of Riccati equation

$$XH_{11} + H_{11}^T X + XH_{12}X + H_{21} = 0, \quad (26)$$

where

$$H_{11} = A - \hat{B}(\hat{D}^T \hat{D})^{-1} \hat{D}^T C,$$

$$H_{12} = -\hat{B}(\hat{D}^T \hat{D})^{-1} \hat{B}^T,$$

$$H_{21} = C^T (I - \hat{D}(\hat{D}^T \hat{D}^T)^{-1} \hat{D}^T) C. \quad (27)$$

3. THE DILATION OF $G(S)$

We give one of main results of this paper as follows:

THEOREM 1. For the stabilizable and detectable realization of $G(s)$ with full column rank in (1), with the quantities as defined in Lemma 1, choose D' such that

$$\begin{bmatrix} \hat{D} & D' \end{bmatrix}^{-1} = \begin{bmatrix} \hat{D}^+ \\ (D')^T \end{bmatrix} \quad (28)$$

holds, and choose

$$B' := -X^+ C^T D', \quad (29)$$

where X^+ is the pseudo inverse of X. Then $\bar{G}(s)$ in (2) has the same Ω_e zeros as those of $G(s)$ in (1).

Proof. Denote the system matrix pencil of $\bar{G}(s)$ in (2) as

$$\bar{P}(s) := -s\bar{P}_E + \bar{P}_A, \quad (30)$$

where

$$\bar{P}_E := \begin{bmatrix} I & 0 \\ 0 & 0_{m\times m} \end{bmatrix}, \quad \bar{P}_A := \begin{bmatrix} A & \bar{B} \\ C & \bar{D} \end{bmatrix}. \quad (31)$$

In what follows, we shall study the infinite and finite $j\omega$-axis eigenstructures of $\bar{P}(s)$ with respect to those of $P(s)$.

First, it follows from (12) that

$$(-s\bar{P}_E + \bar{P}_A)\begin{bmatrix} V_\infty & 0 \\ 0 & I_{m-r} \end{bmatrix}$$
$$= \bar{P}_A \begin{bmatrix} V_\infty & 0 \\ 0 & I_{m-r} \end{bmatrix}\begin{bmatrix} -sN + I_{n_\infty} & 0 \\ 0 & I_{m-r} \end{bmatrix}. (32)$$

Therefore, together with (17), we obtain

$$\mathrm{Im}\begin{bmatrix} T & 0 \\ 0 & I_m \end{bmatrix} = \mathrm{Im}\begin{bmatrix} V_\infty & 0 \\ 0 & I_{m-r} \end{bmatrix} \subseteq \nu\{-s\bar{P}_E + \bar{P}_A,\ \infty\}. \quad (33)$$

Next, it yields from (19) that

$$\begin{bmatrix} -sI + A & B & B' \\ C_1 & D & D' \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \\ 0 \end{bmatrix} = \begin{bmatrix} T_1 \\ 0 \\ 0 \end{bmatrix}(-sI + \Lambda_j), \quad (34)$$

which yields

$$\mathrm{Im}\left\{\begin{bmatrix} T_1 \\ T_2 \\ 0 \end{bmatrix}\right\} \subseteq \nu\left\{-s\bar{P}_E + \bar{P}_A,\ \Omega\right\}. \quad (35)$$

Now denote the system matrix of $\bar{G}^{\sim}(s)\bar{G}(s)$ as

$$\bar{W}(s) := \begin{bmatrix} -sI + A & 0 & \bar{B} \\ -C^TC & -sI - A^T & -C^T\bar{D} \\ \bar{D}^TC & \bar{B}^T & \bar{D}^T\bar{D} \end{bmatrix}. (36)$$

We shall explore the relationship between the stable eigenspaces of $\bar{W}(s)$ and $W(s)$.

To begin with, using (27) and (28), we have

$$H_{21} = C^TD^{'}(D')^TC \geq 0. \quad (37)$$

Pre-multiplying by $(I - XX^+)$ and post-multiplying by $(I - X^+X)$ of (26), together with (37), we have

$$(D')^TC(I - X^+X) = 0. \quad (38)$$

It follows from (29) that

$$(D')^TC + (B')^TX = 0 \quad (39)$$

holds. Post-multiplying S of (24) to the above equation yields

$$(D')^TCU_1 + (B')^TU_2 = 0. \quad (40)$$

Together with (23), we obtain

$$\begin{bmatrix} -sI + A & 0 & B & B' \\ -C^TC & -sI - A^T & -C^TD & -C^TD' \\ D^TC & B^T & D^TD & 0 \\ (D')^TC & (B')^T & 0 & I_{m-r} \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ 0 \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ 0 \end{bmatrix}(-sI + \Lambda), \quad (41)$$

which follows that

$$\mathrm{Im}\left\{\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ 0 \end{bmatrix}\right\} \subseteq \nu\left\{\bar{W}(s),\ \mathbf{C}_-\right\}. \quad (42)$$

Thus,

$$\mathrm{Im}\left\{\begin{bmatrix} U_1 \\ U_3 \\ U_2 \\ CU_1 + \begin{bmatrix} D & D' \end{bmatrix}\begin{bmatrix} U_3 \\ 0 \end{bmatrix} \end{bmatrix}\right\}$$
$$\subseteq \nu\left\{\begin{bmatrix} -sI + A & \bar{B} & 0 & 0 \\ C & \bar{D} & 0 & -I \\ 0 & 0 & -sI - A^T & -C^T \\ 0 & 0 & -\bar{B}^T & -\bar{D}^T \end{bmatrix}, \mathbf{C}_-\right\}. (43)$$

Note that the sum of the eigenspace dimensions of the left sides of (43), (35) and (32) is $n+m$ owing to nonsingularity of $\begin{bmatrix} U_1 & T_1 & T \end{bmatrix}$. Since $-s\bar{P}_E + \bar{P}_A$ is a square pencil, the sum of the eigenspace dimensions of the left sides of (43), (35) and (32) is also $n + m$. Hence, the equations in (43), (35) and (32) must hold.

Therefore, the infinite zeros and finite $j\omega$-axis zeros of $\bar{P}(s)$ are the same as those of $P(s)$, respectively. Hence, the dilation $\bar{G}(s)$ in (2) with D' in (28) and B' in (29) preserves the Ω_e zeros of $G(s)$.

4. APPLICATION: CONSTRUCTION OF UNITARY ANNIHILATOR

As an application of the above dilation, we shall construct a unitary annihilator of $G(s)$. Decompose the inverse of $\bar{G}(s)$ as

$$\bar{G}^{-1}(s) = \begin{bmatrix} G^+(s) \\ G^\perp(s) \end{bmatrix} \quad (44)$$

in accordance with (2). Therefore,

$$G^+(s)G(s) = I_r, \quad G^\perp(s)G(s) = 0 \quad (45)$$

hold. Since $D = G(\infty)$ is not full column rank, $G^+(s)$ is a non-proper matrix. As to $G^\perp(s)$, we have

THEOREM 2. For the stabilizable and detectable realization of $G(s)$ with full column rank in (1), suppose that B' and D' in $\bar{G}(s)$ in (2) are chosen according to (29) and (28), respectively. Then an annihilator of $G(s)$ is given as

$$G^{\perp}(s) = (D')^T \left[\begin{array}{c|c} L_1(A-HC)U_1 & L_1H \\ \hline -CU_1 & I_{m-r} \end{array}\right], \quad (46)$$

where L_1 is defined as

$$\begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} := \begin{bmatrix} U_1 & T_1 & T \end{bmatrix}^{-1} = S^{-1}, \quad (47)$$

and

$$H := \begin{bmatrix} \hat{B} & B' \end{bmatrix} \begin{bmatrix} \hat{D} & D' \end{bmatrix}^{-1}. \quad (48)$$

Moreover, $G^{\perp}(s)$ in (46) is a unitary matrix, i.e.

$$G^{\perp}(s)(G^{\perp}(s))^{\sim} = I_{m-r}. \quad (49)$$

where $(G^{\perp}(s))^{\sim} := (G^{\perp}(-s))^T$.

Proof. The derivation of $G^{\perp}(s)$ in (46) is omitted due to the space limit. Here, we shall just show that $G^{\perp}(s)$ is a unitary matrix.

Denoting $A_1 := A - HC$ and $R := HH^T$, we have

$$G^{\perp}(s)(G^{\perp}(s))^{\sim} = (D')^T \left[\begin{array}{cc|c} L_1A_1U_1 & L_1RL_1^T & L_1H \\ 0 & -U_1^TA_1^TL_1^T & U_1^TC^T \\ \hline -CU_1 & H^TL_1^T & I_{m-r} \end{array}\right] D'. (50)$$

Performing the similarity transformation under

$$M = \begin{bmatrix} I & 0 \\ U_1^TU_2 & I \end{bmatrix}, \quad M^{-1} = \begin{bmatrix} I & 0 \\ -U_1^TU_2 & I \end{bmatrix},$$

and using $U_2L_1 = X$ which holds owing to (25) and (47), we have

$$G^{\perp}(s)(G^{\perp}(s))^{\sim} = (D')^T \left[\begin{array}{cc|c} L_1(A_1-RX)U_1 & L_1RL_1^T & L_1H \\ W_{21} & -U_1^T(A_1-RX)^TL_1^T & U_1^T(C^T+XH) \\ \hline -CU_1 - H^TXU_1^T & H^TL_1^T & I_{m-r} \end{array}\right] D', (51)$$

where

$$W_{21} = U_1^T(XA_1 + A_1^TX + XRX)U_1.$$

Since $R = HH^T = \hat{B}(\hat{D}^T\hat{D})^{-1}\hat{B}^T + B'B'^T$. From (26) and (29), we know

$$XA_1 + A_1^TX + XRX$$
$$= XH_{11} + H_{11}^TX + XH_{12}X + H_{21} = 0.$$

It follows from (29) and (39) that

$$(D')^T(C + H^TX) = (D')^TC + (B')^TX = 0.$$

Based on these identities,

$$G^{\perp}(s)(G^{\perp}(s))^{\sim} = I_{m-r}$$

hold. This completes the proof. ■

5. A NUMERICAL EXAMPLE

Consider the following system

$$G(s) = \frac{\begin{bmatrix} s(s^2+2s+2) & 3s^2+8s+4 \\ s^2 & s+2 \\ s & s^2+2s \end{bmatrix}}{(s-1)(s+1)(s+2)}$$

$$= \left[\begin{array}{ccc|cc} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 2 & 1 & -2 & -2 & 1 \\ \hline 2 & 2 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{array}\right]. \quad (52)$$

From its state-space realization, we have

$$P_E = \begin{bmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{bmatrix}, \quad P_A = \begin{bmatrix} 0&1&0&0&1 \\ 0&0&1&1&0 \\ 2&1&-2&-2&1 \\ 2&2&1&1&0 \\ 0&1&0&0&0 \\ 1&0&0&0&0 \end{bmatrix}.$$

First, it yields from (6) that

$$v_1^1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T, \quad v_1^2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T.$$

From $P_Ev_2^1 = P_Av_1^1$, we obtain $v_1^2 = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \end{bmatrix}^T$. Since $P_Av_1^2 \notin \operatorname{Im} P_E$, we know that v_1^2 is the highest infinite eigenvector starting from v_1^1. Also owing to $P_Av_2^1 \notin \operatorname{Im} P_E$, we know that v_2^1 is the highest infinite eigenvector starting from v_2^1. Thus, $V_h = \begin{bmatrix} v_1^2 & v_2^1 \end{bmatrix}$, $V_r = v_1^1$, $n_\infty = 3$ and $n_r = 1$. From (14) and (15), we obtain

$$T = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & -2 \end{bmatrix}, \quad \hat{D} = \begin{bmatrix} 3 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Next, since $s = 0$ is a zero of $G(s)$ in (52) on $j\omega$-axis, it follows from

$$P_A \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = 0,$$

that $T_1 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$, $T_2 = \begin{bmatrix} -1 & 0 \end{bmatrix}^T$ hold. Therefore, $n_j = 1$.

Now we calculate (23). From

$$|W(s)| = -s^2(s-3)(s+3),$$

the stable solution of $|W(s)| = 0$ is $s = -3$, and its corresponding eigenvector of $W(s)$ is

$$U_1 = \begin{bmatrix} -6 \\ -15 \\ 28 \end{bmatrix},\ U_2 = \begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix},\ U_3 = \begin{bmatrix} 17 \\ 33 \end{bmatrix}.$$

Note that $n_- = 1$ holds. Thus, $n_- + n_j + n_r = 3$, and $\begin{bmatrix} U_1 & T_1 & T \end{bmatrix}$ is nonsingular. Therefore, from (26),

$$X = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

From (28) and (29), the matrices related to the dilation are

$$D' = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ B' = \begin{bmatrix} 0 \\ -5 \\ 0 \end{bmatrix}.$$

Using (48) and (47) yields

$$H = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -5 & -2 \\ -2 & 0 & 6 \end{bmatrix},\ L_1 = \begin{bmatrix} 0 & -\frac{1}{15} & 0 \end{bmatrix}.$$

Finally, we obtain from (46)

$$G^{\perp}(s) = \left[\begin{array}{c|ccc} 3 & -\frac{1}{15} & \frac{1}{3} & \frac{2}{15} \\ \hline 15 & 0 & 1 & 0 \end{array}\right]$$
$$= \begin{bmatrix} \frac{-1}{s-3} & \frac{s+2}{s-3} & \frac{2}{s-3} \end{bmatrix}.$$

It is easy to check that the above $G^{\perp}(s)$ is unitary and is an annihilator of $G(s)$ in (52).

6. CONCLUSION

The dilation of a non-square matrix with full column rank to a square matrix preserving the Ω_e zeros has been discussed in this paper. The state-space solution to the dilation has been proposed. As an application, a unitary annihilator of the non-square transfer matrix has been presented. The obtained result is useful for analysis and design of non-square systems with infinite and finite imaginary axis, i.e, the factorization, and the left/right inversion of these systems.

Note that though the techniques are completely different, completing the singular pencil associated with a given transfer matrix is discussed in (Cabral *et al.*, 2001). The discussion between the result developed in this paper and that in (Cabral *et al.*, 2001) will be a future subject.

7. REFERENCES

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