15th Triennial World Congress, Barcelona, Spain

ROBUST ADAPTIVE CONTROL OF LINEARIZABLE NONLINEAR SINGLE INPUT SYSTEMS

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Abstract: In this paper a nonlinear robust adaptive control algorithm is designed and analyzed for a class of single-input nonlinear systems with unknown nonlinearities. The controller guarantees closed loop semiglobal stability and convergence of the tracking error to a small residual set. The region of attraction for semiglobal stability depends on the number of nodes and weights in the single layer neural network used to estimate the unknown plant nonlinearities. The size of the residual set depends on design parameters and can be calculated apriori. One example is used to demonstrate the performance and properties of the proposed scheme. *Copyright © 2002 IFAC*

Key words: Adaptive control; linearizable systems; nonlinear control; robustness; switching functions.

1. INTRODUCTION

The traditional way of designing feedback control system is based on the use of Linear Time Invariant (LTI) models for the plant. Off-line frequency domain techniques could be used to fit such an LTI model to experimental data and identify its parameters. In the case, where the parameters of the LTI model change with time, gain scheduling, on-line parameter identification, adaptive control, robust control techniques etc. are developed over the years to address such situations. The reliance on LTI models for control design purposes often puts limitations on the performance improvement that could be achieved for the plant under consideration. For example if the plant consists of strong nonlinearities, its approximation by an LTI model, may considerably reduce the region of attraction in the presence of disturbances and other modeling uncertainties. During the recent years, considerable research efforts have been made to deal with the design of stabilizing controllers for classes of nonlinear plants. These efforts are described in detail in a recent survey paper (Kokotovic and Arcak, 2001) where a very elegant and informative historical perspective of the evolution of nonlinear control design is presented and discussed. Most of the recent efforts (surveyed by Kokotovic and Arcak, 2001) on nonlinear control design assumed that the plant nonlinearities are known. The case where the plant nonlinearities are products of unknown constant parameters with known nonlinearities gave rise to a number of adaptive control techniques (Kosmatopoulos and Ioannou, 1999; Kristic, *et al*., 1995; Chen and Liu, 1994; Liu and Chen, 1993; Sastry and Isidori, 1989; Taylor, *et al.*, 1989).

In this paper we consider a class of single input feedback linearizable nonlinear plants with unknown nonlinearities. We assume that the plant nonlinearities are smooth functions and the nonlinear function multiplying the input satisfies a sufficient condition that guarantees that the plant is controllable. The plant nonlinear functions are estimated on-line using a single layer neural network. A nonlinear adaptive control law is designed based on these estimates to satisfy certain stability conditions derived from a selected Lyapunov-like function. The control law contains a number of robust modifications that guarantee signal boundedness even in the case where the estimated plant loses controllability at certain points in time. The proposed control scheme guarantees that for all initial conditions from a region of attraction whose size depends mainly on the number of nodes and weights of the neural network, all signals are bounded and the tracking error converges to a residual set whose size depends on certain design parameters. The size of the residual can be chosen apriori by selecting these parameters appropriately. One example of a nonlinear plant is used to demonstrate the results.

2. PROBLEM STATEMENT

Consider the single-input, single output system:

$$x^{(n)}(t) = f(\boldsymbol{x}) + b(\boldsymbol{x})u \qquad (1)$$
$$y(t) = x(t)$$

where $\boldsymbol{x} = [x(t), \dot{x}(t), \cdots, x^{(n-1)}(t)]^T \in \Re^n$, u is the scalar control input, y is the scalar system output, f, b are completely unknown smooth functions and $(\cdot)^{(n)} \overset{\text{def.}}{=} d^n(\cdot)/dt^n$. The problem is to design a control law u such that the output $y(t)$ tracks a given desired trajectory $y_d(t)$, a known smooth function of time.

Assumption 1: $b(\boldsymbol{x})$ is bounded from below by a constant $\bar{b}$, i.e., $|b(\boldsymbol{x})| \geq \bar{b}$, $\forall\, \boldsymbol{x} \in \Re^n$, and the sign of $b(\boldsymbol{x})$ is known for $\forall\, \boldsymbol{x} \in \Re^n$.

We define the scalar function $S(t)$ as the metric for describing the tracking error dynamics:

$$S(t) = (d/dt + \lambda)^{n-1} e(t) \tag{2}$$
$$e(t) = y(t) - y_d(t)$$

where λ is a positive constant defining the bandwidth of the error dynamics. The sliding surface $S(t) = 0$ represents a linear differential equation whose solution implies that $e(t)$ converges to zero with time constant $(n-1)/\lambda$ (Slotine and Li, 1991). Differentiating $S(t)$ with respect to time, we obtain:

$$\begin{aligned}\dot{S} &= e^{(n)} + \alpha_{n-1} e^{(n-1)} + \cdots + \alpha_1 \dot{e} \\ &= f(\boldsymbol{x}) + b(\boldsymbol{x})u - y_d^{(n)} + (\alpha_{n-1} e^{(n-1)} + \cdots + \alpha_1 \dot{e})\end{aligned} \tag{3}$$

where, $\alpha_{n-1}, \cdots, \alpha_1$ represent the coefficients in the Hurwitz binomial expansion of (2). Let

$$v(t) = -y_d^{(n)} + \alpha_{n-1} e^{(n-1)} + \cdots + \alpha_1 \dot{e} \tag{4}$$

Then, $\dot{S}$ can be written in the compact form:

$$\dot{S} = f(\boldsymbol{x}) + v(t) + b(\boldsymbol{x})u \tag{5}$$

If $f(\boldsymbol{x})$ and $b(\boldsymbol{x})$ were completely known functions, then the control law

$$u = \frac{1}{b(\boldsymbol{x})}\left[-f(\boldsymbol{x}) - v(t) - k_S S(t)\right] \tag{6}$$

could be used to meet the control objective provided of course that the controllability condition $b(\boldsymbol{x}) \neq 0$ for all $\boldsymbol{x}$ is satisfied (guaranteed by Assumption 1). Using (6) we obtain

$$\dot{S} = -k_S S \tag{7}$$

which implies that $S(t)$ and therefore $e^{(i)}$, i=0,1,2,…,n-1, converge to zero exponentially fast.

In the case where, f, b are unknown, (6) can no longer be used. As in the linear case, we can use the Certainty Equivalence (CE) principle (Ioannou and Sun, 1996) to come up with an initial guess of a control law, which we can then modify to meet the stability and control objective.

Let us therefore start with the CE control law

$$u = \frac{1}{\hat{b}(\boldsymbol{x},t)}\left[-\hat{f}(\boldsymbol{x},t) - v(t) - k_S S(t)\right] \tag{8}$$

where the unknown functions f, b are replaced by their estimates $\hat{f}$, $\hat{b}$ to be generated on-line. In the following sections we show how to generate $\hat{f}$, $\hat{b}$ and modify the CE control law in order to guarantee stability and satisfy the control objective.

3. APPROXIMATION AND ON-LINE ESTIMATION OF THE UNKNOWN NONLINEAR FUNCTIONS

Since f and b are assumed to be smooth functions, they can be approximated using, for example, a single layer neural network as:

$$f^a(\boldsymbol{x}) = \sum_{i=1}^{l_f} \theta_i^f g_i^f(\boldsymbol{x}) \tag{9}$$

$$b^a(\boldsymbol{x}) = \sum_{i=1}^{l_b} \theta_i^b g_i^b(\boldsymbol{x}) \tag{10}$$

where f^a and b^a are the approximation functions for f and b respectively, read as "$f(\boldsymbol{x})$ and $b(\boldsymbol{x})$ approximation", $g_i^f(\boldsymbol{x})$ and $g_i^b(\boldsymbol{x})$ are chosen basis functions, l_f, l_b are the number of the nodes, and θ_i^f, θ_i^b are the output weights for f and b respectively. The respective approximation errors are denoted by:

$$d_f(\boldsymbol{x}) = f(\boldsymbol{x}) - f^a(\boldsymbol{x}) \tag{11}$$

$$d_b(\boldsymbol{x}) = b(\boldsymbol{x}) - b^a(\boldsymbol{x}) \tag{12}$$

Here, it is assumed that there exist a set of output weights θ_i^f, θ_i^b and number of nodes l_f, l_b, such that the smooth functions f and b can be approximated with any desired accuracy $\varepsilon_f > 0$ and $\varepsilon_b > 0$ over a compact set $\Omega \in \Re^n$ so that:

$$\left|d_f(\boldsymbol{x})\right|_{\max} = \left|f^a(\boldsymbol{x}) - f(\boldsymbol{x})\right|_{\max} \leq \varepsilon_f \tag{13}$$

$$\left|d_b(\boldsymbol{x})\right|_{\max} = \left|b^a(\boldsymbol{x}) - b(\boldsymbol{x})\right|_{\max} \leq \varepsilon_b, \ \forall\, \boldsymbol{x} \in \Omega \tag{14}$$

As shown in (Park and Sandberg, 1993; Sanner and Slotine, 1992) and the references therein, a wide class of basis functions and neural networks exist to satisfy the above universal approximation conditions, (13), (14). In (9) and (10) we assumed that the designer fixes the number of nodes l_f, l_b. The weight parameters θ_i^f, θ_i^b are to be estimated on line. Let $\hat{\theta}_i^f(t)$, $\hat{\theta}_i^b(t)$ be the estimates of θ_i^f, θ_i^b respectively at time t. Then the estimates of the approximation functions f^a, b^a at time t are formed as

$$\hat{f}^a(\boldsymbol{x},t) = \sum_{i=1}^{l_f} \hat{\theta}_i^f(t) g_i^f(\boldsymbol{x}) \tag{15}$$

$$\hat{b}^a(\boldsymbol{x},t) = \sum_{i=1}^{l_b} \hat{\theta}_i^b(t) g_i^b(\boldsymbol{x}) \tag{16}$$

The difference between the estimated and actual parameter values results in the estimation errors

$$\tilde{f}^a(\boldsymbol{x},t) = \hat{f}^a(\boldsymbol{x},t) - f^a(\boldsymbol{x}) = \sum_{i=1}^{l_f} \tilde{\theta}_i^f(t) g_i^f(\boldsymbol{x}) \tag{17}$$

$$\tilde{b}^a(\boldsymbol{x},t) = \hat{b}^a(\boldsymbol{x},t) - b^a(\boldsymbol{x}) = \sum_{i=1}^{l_b} \tilde{\theta}_i^b(t) g_i^b(\boldsymbol{x}) \tag{18}$$

where

$$\tilde{\theta}_i^f(t) = \hat{\theta}_i^f(t) - \theta_i^f, \ \ \tilde{\theta}_i^b(t) = \hat{\theta}_i^b(t) - \theta_i^b \tag{19}$$

are the parameter errors.

The estimator and parameter errors are not available for measurement, therefore equations (17)-(18) are used only for analysis. In the following section we present the adaptive laws that generate the parameter estimates $\hat{\theta}_i^f$, $\hat{\theta}_i^b$ together with the control law.

4. ROBUST ADAPTIVE CONTROL LAW

The CE control law (8) cannot be used to stabilize the closed loop system for a number of reasons. First $\hat{f}$, $\hat{b}$ cannot be generated on-line directly, only $\hat{f}^a$, $\hat{b}^a$ can be used in the control law. Second, the estimates $\hat{f}^a$, $\hat{b}^a$ may differ considerably from the actual ones, leading to the wrong control action initially. Third, there is no guarantee that $\hat{b}^a(\boldsymbol{x},t)$ will not assume values close to zero. In such case the estimated plant is close to lose controllability leading to possible large values for u. In order to take care of these problems the CE control law (8) is modified to:

$$u = \frac{\hat{b}^a(\boldsymbol{x},t)}{(\hat{b}^a(\boldsymbol{x},t))^2 + \delta_b}\{-k_S S(t) - v(t) - \hat{f}^a(\boldsymbol{x},t) - \sigma_v |v(t)| \operatorname{sat}(S/\Phi) - \sigma_f |\hat{f}^a(\boldsymbol{x},t)| \operatorname{sat}(S/\Phi)\} \quad (20)$$

where $k_S > 0$ is a design constant, $\delta_b > 0$, $\sigma_v > 0$, $\sigma_f > 0$, $\Phi > 0$ are small design constants and

$$\operatorname{sat}(S/\Phi) = \begin{cases} 1, & \text{if } S > \Phi \\ S/\Phi, & \text{if } |S| \le \Phi \\ -1, & \text{if } S < -\Phi \end{cases} \quad (21)$$

The parameters $\hat{\theta}_i^f(t)$, $\hat{\theta}_i^b(t)$ in $\hat{f}^a(\boldsymbol{x},t)$, $\hat{b}^a(\boldsymbol{x},t)$ respectively are updated as follows:

$$\dot{\hat{\theta}}_i^f(t) = k_f S_\Delta g_i^f(\boldsymbol{x}) \quad (22)$$

$$\dot{\hat{\theta}}_i^b(t) = k_b S_\Delta u g_i^b(\boldsymbol{x}) + \rho k_b \sigma_b \operatorname{sgn}(b) |S_\Delta| (|u'| + |u|) g_i^b(\boldsymbol{x}) \quad (23)$$

where

$$u' = \frac{1}{(\hat{b}^a(\boldsymbol{x},t))^2 + \delta_b}\{-k_S S(t) - v(t) - \hat{f}^a(\boldsymbol{x},t) - \sigma_v |v(t)| \operatorname{sat}(S/\Phi) - \sigma_f |\hat{f}^a(\boldsymbol{x},t)| \operatorname{sat}(S/\Phi)\} \quad (24)$$

$$S_\Delta(t) = S(t) - \Phi \operatorname{sat}(S(t)/\Phi) \quad (25)$$

k_f, $k_b > 0$ are the adaptive gains chosen by the designer, $\sigma_b > 0$ is a small design parameter, $\operatorname{sgn}(\cdot)$ is the sign function (sgn(x)=1, if $x \ge 0$ and sgn(x)=-1 otherwise), and ρ is a continuous switching function given by:

$$\rho = \begin{cases} 0, & \text{if } |\hat{b}^a| \ge \bar{b} - \varepsilon_b \\ (\bar{b} - \varepsilon_b - |\hat{b}^a|)/\Delta, & \text{if } \bar{b} - \varepsilon_b - \Delta < |\hat{b}^a| < \bar{b} - \varepsilon_b \\ 1, & \text{if } |\hat{b}^a| \le \bar{b} - \varepsilon_b - \Delta \end{cases} \quad (26)$$

where $\Delta > 0$ is a design parameter used to avoid discontinuity in ρ. A continuous switching function ρ, instead of a discontinuous one, is used in order to guarantee the existence and uniqueness of the solution of the closed-loop system (Polycarpou and Ioannou, 1993).

By design, the control law in (20) will never become singular since $(\hat{b}^a(\boldsymbol{x},t))^2 + \delta_b > \delta_b > 0$, $\forall \boldsymbol{x}, t$. Therefore, the proposed controller overcomes the difficulty encountered in implementing some adaptive control laws where the identified model becomes uncontrollable at some points in time. It is also interesting to note that $u \to 0$ with the same speed as $\hat{b}^a \to 0$. Thus, when the estimate $\hat{b}^a$ approaches zero, the control input remains bounded and also reduces to zero. In other words in such case it is pointless to control what appears to the controller as uncontrollable plant. The control law (20)-(23) is designed using stability and Lyapunov type arguments and its properties are described by the following theorem.

Theorem: Consider the system (1), the control law (20) and the adaptive laws (22), (23). If assumption 1 holds and $\bar{b}$ satisfies the condition $\bar{b} > \sqrt{\delta_b} + 3\varepsilon_b + \Delta$, then given small positive numbers Φ and ε_f, there exist positive constants $\delta_1^* < 1$, $\delta_2^* < 1$, $\delta_3^* < 1$ such that for $k_S \Phi \ge \varepsilon_f / (1 - \delta_1^*)$, $\sigma_v \ge \delta_1^* / (1 - \delta_1^*)$, $\sigma_f \ge \delta_1^* / (1 - \delta_1^*)$, $\sigma_b \ge \max\{\delta_2^*, \delta_3^*\}$, $\tilde{\theta}_i^f(0), \tilde{\theta}_i^b(0) \in \Omega_\theta$ and $\boldsymbol{x}(0) \in \Omega_x$, where $\Omega_\theta \subset \Re^{l_f + l_b}$, $\Omega_x \subset \Omega \subset \Re^n$, all signals in the closed-loop system are bounded and the tracking error and its derivatives are bounded from above by $|e^{(i)}(t)| \le 2^i \lambda^{i-n+1} \Phi$, $i = 0, 1, \cdots, n-1$.

Proof: Let us consider the following Lyapunov-like function:

$$V(t) = \frac{1}{2} S_\Delta^2 + \frac{1}{2k_f} \sum_{i=1}^{l_f} (\tilde{\theta}_i^f(t))^2 + \frac{1}{2k_b} \sum_{i=1}^{l_b} (\tilde{\theta}_i^b(t))^2 \quad (27)$$

The time derivative $V(t)$ is then given by

$$\dot{V}(t) = S_\Delta \dot{S}_\Delta + \frac{1}{k_f} \sum_{i=1}^{l_f} \tilde{\theta}_i^f \dot{\hat{\theta}}_i^f(t) + \frac{1}{k_b} \sum_{i=1}^{l_b} \tilde{\theta}_i^b \dot{\hat{\theta}}_i^b(t) \quad (28)$$

where, $\dot{S}_\Delta = 0$ for $|S(t)| \le \Phi$ and $\dot{S}_\Delta = \dot{S}$ for $|S(t)| > \Phi$. In view of the adaptive laws (22) and (23), $\dot{V} = 0$ for $|S| \le \Phi$. Therefore, the remaining of this proof deals strictly with the case of $|S| > \Phi$. First, we analyze the first term in $\dot{V}$ in (28). Let us rewrite the control law in (20) as

$$u = \frac{\hat{b}^a(\boldsymbol{x},t)}{(\hat{b}^a(\boldsymbol{x},t))^2 + \delta_b} \bar{u} \quad (29)$$

where $\bar{u}$ is given by:

$$\bar{u} = -k_S S(t) - v(t) - \hat{f}^a(\boldsymbol{x},t) - \sigma_v |v(t)| \operatorname{sat}(S/\Phi) - \sigma_f |\hat{f}^a(\boldsymbol{x},t)| \operatorname{sat}(S/\Phi) \quad (30)$$

In view of equation (5) and $\dot{S}_\Delta = \dot{S}$ for $|S(t)| > \Phi$, $\dot{S}_\Delta$ can be written as:

$$\dot{S}_\Delta = f(\boldsymbol{x}) + v(t) + b(\boldsymbol{x}) u \quad (31)$$

By substituting the control input (29) and (30) into (31), one obtains:

$$\begin{aligned} \dot{S}_\Delta &= f(\boldsymbol{x}) + v(t) + \hat{b}^a(\boldsymbol{x},t) u + [b(\boldsymbol{x}) - \hat{b}^a(\boldsymbol{x},t)] u \\ &= f(\boldsymbol{x}) + v(t) + \bar{u} - \frac{\delta_b}{(\hat{b}^a(\boldsymbol{x},t))^2 + \delta_b} \bar{u} \\ &\quad + [b(\boldsymbol{x}) - \hat{b}^a(\boldsymbol{x},t)] u \\ &= -k_S S - \sigma_v |v(t)| \operatorname{sat}(S/\Phi) - \sigma_f |\hat{f}^a(\boldsymbol{x},t)| \operatorname{sat}(S/\Phi) \\ &\quad + [f(\boldsymbol{x}) - \hat{f}^a(\boldsymbol{x},t)] + [b(\boldsymbol{x}) - \hat{b}^a(\boldsymbol{x},t)] u - \delta_b u' \end{aligned} \quad (32)$$

Using the identities,

$$\begin{aligned} f(\boldsymbol{x}) - \hat{f}^a(\boldsymbol{x},t) &= (f(\boldsymbol{x}) - f^a(\boldsymbol{x})) - (\hat{f}^a(\boldsymbol{x},t) - f^a(\boldsymbol{x})) \\ &= d_f(\boldsymbol{x}) - \tilde{f}^a(\boldsymbol{x},t) \end{aligned} \quad (33)$$

$$b(\boldsymbol{x})-\hat{b}^a(\boldsymbol{x},t)=(b(\boldsymbol{x})-b^a(\boldsymbol{x},t))-(\hat{b}^a(\boldsymbol{x},t)-b^a(\boldsymbol{x})) \\ =d_b(\boldsymbol{x})-\tilde{b}^a(\boldsymbol{x},t) \tag{34}$$

$\dot{S}_\Delta$ becomes:

$$\dot{S}_\Delta=-k_S S-\sigma_v|v(t)|\operatorname{sat}(S/\Phi)-\sigma_f\left|\hat{f}^a(\boldsymbol{x},t)\right|\operatorname{sat}(S/\Phi) \\ -\tilde{f}^a(\boldsymbol{x},t)-\tilde{b}^a(\boldsymbol{x},t)u+d_f(\boldsymbol{x})+\{d_b(\boldsymbol{x})u-\delta_b u'\} \tag{35}$$

The last term in (35) represents the effect of the design constant δ_b in the control law and the approximation error $d_b(\boldsymbol{x})$.

Define:

$$\delta_1 \overset{\text{def.}}{=} \frac{\varepsilon_b}{\bar{b}-\varepsilon_b-\Delta}+\frac{\delta_b}{(\bar{b}-\varepsilon_b-\Delta)^2+\delta_b} \tag{36a}$$

$$\delta_2 \overset{\text{def.}}{=} \frac{2\delta_b}{\bar{b}-\varepsilon_b} \tag{36b}$$

$$\delta_3 \overset{\text{def.}}{=} \frac{\varepsilon_b}{\bar{b}-\varepsilon_b} \tag{36c}$$

Then, as shown in the Appendix, the absolute value of the last term in $\dot{S}_\Delta$ can be expressed as:

$$|d_b(\boldsymbol{x})u-\delta_b u'|\le\delta_1|\bar{u}|+\rho\delta_2\left|\tilde{b}^a(\boldsymbol{x},t)\right||u'|+\rho\delta_3\left|\tilde{b}^a(\boldsymbol{x},t)\right||u| \tag{37}$$

Since $|S|\le|S_\Delta|+\Phi$, one has

$$|\bar{u}|\le k_S|S|+(\sigma_v+1)|v(t)|+(\sigma_f+1)\left|\hat{f}^a(\boldsymbol{x},t)\right| \\ \le k_S|S_\Delta|+k_S\Phi+(\sigma_v+1)|v(t)|+(\sigma_f+1)\left|\hat{f}^a(\boldsymbol{x},t)\right| \tag{38}$$

and (37) can be rewritten as:

$$|d_b(\boldsymbol{x})u-\delta_b u'|\le\delta_1 k_S|S_\Delta|+\delta_1 k_S\Phi+\delta_1(\sigma_v+1)|v(t)| \\ +\delta_1(\sigma_f+1)\left|\hat{f}^a(\boldsymbol{x},t)\right|+\rho\delta_2\left|\tilde{b}^a(\boldsymbol{x},t)\right||u'|+\rho\delta_3\left|\tilde{b}^a(\boldsymbol{x},t)\right||u| \tag{39}$$

Then in view of (35), and using (25) and $S_\Delta\operatorname{sat}(S/\Phi)=|S_\Delta|$, the first term in $\dot{V}$ is expressed as:

$$S_\Delta\dot{S}_\Delta=-k_S S_\Delta^2-k_S\Phi|S_\Delta|-\sigma_v|v(t)||S_\Delta| \\ -\sigma_f\left|\hat{f}^a(\boldsymbol{x},t)\right||S_\Delta|-S_\Delta\tilde{f}^a(\boldsymbol{x},t) \\ -S_\Delta\tilde{b}^a(\boldsymbol{x},t)u+S_\Delta d_f(\boldsymbol{x})+S_\Delta\{d_b(\boldsymbol{x})u-\delta_b u'\} \tag{40}$$

Using (39) in (40), in the mean while noticing (13), (14), one obtains:

$$S_\Delta\dot{S}_\Delta\le-(1-\delta_1)k_S S_\Delta^2-\{(1-\delta_1)k_S\Phi-\varepsilon_f\}|S_\Delta| \\ -\{\sigma_v-\delta_1(1+\sigma_v)\}|v(t)||S_\Delta| \\ -\{\sigma_f-\delta_1(1+\sigma_f)\}\left|\hat{f}^a(\boldsymbol{x},t)\right||S_\Delta| \\ -S_\Delta\tilde{f}^a(\boldsymbol{x},t)-S_\Delta\tilde{b}^a(\boldsymbol{x},t)u \\ +\rho\delta_2|S_\Delta|\left|\tilde{b}^a(\boldsymbol{x},t)\right||u'|+\rho\delta_3|S_\Delta|\left|\tilde{b}^a(\boldsymbol{x},t)\right||u| \tag{41}$$

In view of (22), the second term in $\dot{V}$ can be expressed as:

$$\frac{1}{k_f}\sum_{i=1}^{l_f}\tilde{\theta}_i^f(t)\dot{\hat{\theta}}_i^f(t)=\frac{1}{k_f}\sum_{i=1}^{l_f}\tilde{\theta}_i^f(t)\{k_f S_\Delta g_i^f(\boldsymbol{x})\} \\ =S_\Delta\tilde{f}^a(\boldsymbol{x},t) \tag{42}$$

Finally, using (23) the last term in $\dot{V}$ can also be expanded as:

$$\frac{1}{k_b}\sum_{i=1}^{l_b}\tilde{\theta}_i^b(t)\dot{\hat{\theta}}_i^b(t)=\frac{1}{k_b}\sum_{i=1}^{l_b}\tilde{\theta}_i^b(t)\{k_b S_\Delta u g_i^b(\boldsymbol{x}) \\ +\rho k_b\sigma_b\operatorname{sgn}(b(\boldsymbol{x}))|S_\Delta|(|u'|+|u|)g_i^b(\boldsymbol{x})\} \\ =S_\Delta\tilde{b}^a(\boldsymbol{x},t)u-\rho\sigma_b\left|\tilde{b}^a(\boldsymbol{x},t)\right||S_\Delta||u'|-\rho\sigma_b\left|\tilde{b}^a(\boldsymbol{x},t)\right||S_\Delta||u| \tag{43}$$

Here, we have used the identity $\rho\tilde{b}^a(\boldsymbol{x},t)\operatorname{sgn}(b(\boldsymbol{x}))=-\rho\left|\tilde{b}^a(\boldsymbol{x},t)\right|$. Since $\tilde{b}^a(\boldsymbol{x})=\hat{b}^a(\boldsymbol{x},t)-b^a(\boldsymbol{x})=(\hat{b}^a(\boldsymbol{x},t)+d_b(\boldsymbol{x}))-b(\boldsymbol{x})$ and $\rho\ne0$ only if $\left|\hat{b}^a(\boldsymbol{x},t)\right|\le\bar{b}-\varepsilon_b$ implies that $\left|\hat{b}^a(\boldsymbol{x},t)+d_b(\boldsymbol{x})\right|\le\bar{b}$, then, for $\rho\ne0$, the sign of $\rho\tilde{b}^a(\boldsymbol{x},t)$ is always the opposite sign of $b(\boldsymbol{x})$ $\forall\ t\ge0$.

Combining (41), (42), and (43), $\dot{V}$ satisfies:

$$\dot{V}\le-(1-\delta_1)k_S S_\Delta^2-\{(1-\delta_1)k_S\Phi-\varepsilon_f\}|S_\Delta| \\ -\{\sigma_v-\delta_1(1+\sigma_v)\}|v(t)||S_\Delta|-\{\sigma_f-\delta_1(1+\sigma_f)\}\left|\hat{f}^a(\boldsymbol{x},t)\right||S_\Delta| \\ -\rho(\sigma_b-\delta_2)|S_\Delta|\left|\tilde{b}^a(\boldsymbol{x},t)\right||u'|-\rho(\sigma_b-\delta_3)|S_\Delta|\left|\tilde{b}^a(\boldsymbol{x},t)\right||u| \tag{44}$$

By choosing $k_S,\Phi,\sigma_v,\sigma_f,\sigma_b$ such that the following conditions are satisfied,

$$\delta_1<1 \tag{45a}$$

$$k_S\Phi\ge\frac{\varepsilon_f}{1-\delta_1} \tag{45b}$$

$$\sigma_v\ge\delta_1(1+\sigma_v)\Rightarrow\sigma_v\ge\frac{\delta_1}{1-\delta_1} \tag{45c}$$

$$\sigma_f\ge\delta_1(1+\sigma_f)\Rightarrow\sigma_f\ge\frac{\delta_1}{1-\delta_1} \tag{45d}$$

$$\sigma_b\ge\max\{\delta_2,\delta_3\}=\max\left(\frac{2\delta_b}{\bar{b}-\varepsilon_b},\frac{\varepsilon_b}{\bar{b}-\varepsilon_b}\right) \tag{45e}$$

one obtains

$$\dot{V}=0,\ \text{for } |S|\le\Phi \tag{46a}$$

$$\dot{V}\le-(1-\delta_1)k_S S_\Delta^2\le0,\ \text{for } |S|>\Phi \tag{46b}$$

The results (46a-b) are valid provided (13)-(14) hold. Since (13)-(14) hold on a compact set, i.e., $\boldsymbol{x}\in\Omega$, all states need to remain in this compact set for all $t\ge0$ in order for the results to be valid. Consider the set

$$M(\boldsymbol{x},\tilde{\theta}_i^f,\tilde{\theta}_i^b)=\{\boldsymbol{x},\tilde{\theta}_i^f,\tilde{\theta}_i^b\mid V\le V_0\} \tag{47}$$

where $V_0>V(0)$ and $V_0>\Phi$ is chosen as the largest constant for which $M=\Omega_x\times\Omega_0$, where $\Omega_x\subset\Omega$. Then for $\forall\,\boldsymbol{x}(0)\in\Omega_x$ and $\tilde{\theta}_i^f(0),\tilde{\theta}_i^b(0)\in\Omega_0$ it follows from (27), (46a-b) that *V*(*t*) is bounded from above by V_0 for all $t\ge0$ which implies that $\boldsymbol{x}\in\Omega_x\subset\Omega$, $\forall\ t\ge0$.

This implies that S_Δ and $\tilde{\theta}_i^f,\tilde{\theta}_i^f$ are bounded for all $t>0$. Since *V*(*t*) is bounded from below and is nonincreasing with time, it has a limit, i.e., $\lim_{t\to\infty} V(t) = V_\infty$. Using (46b) and the fact that $S_\Delta = 0$ for $|S| \le \Phi$, we have $\lim_{t\to\infty} \int_0^t k_S S_\Delta^2(\tau)d\tau = k_S \int_0^\infty S_\Delta^2(t)dt \le \frac{V(0)-V_\infty}{1-\delta_1} < \infty$ which implies that $S_\Delta \in L_2$. From $S_\Delta, \tilde{\theta}_i^f, \tilde{\theta}_i^b \in L_\infty$, it follows that all signals are bounded which implies that $\dot{S}_\Delta \in L_\infty$. From $\dot{S}_\Delta \in L_\infty$ and $S_\Delta \in L_2$ we have $S_\Delta(t) \to 0$ as $t \to \infty$ (Ioannou and Sun, 1996). This implies that $S(t)$ converges to the region $|S| \le \Phi$ which in turn implies that the tracking error converges to a small residual set whose size is characterized by the size of the design parameter Φ. We can also establish that the tracking error and its derivatives are bounded from above by $\left|e^{(i)}(t)\right| \le 2^i \lambda^{i-n+1}\Phi$, $i = 0,1,\cdots,n-1$ (Slotine and Li, 1991) ■.

5. SIMULATIONS

In this section, we demonstrate the properties of the proposed adaptive control law using one example.
Example: consider the following second order nonlinear system

$$\ddot{x} = -4\left(\frac{\sin(4\pi x)}{\pi x}\right)\left(\frac{\sin(\pi \dot{x})}{\pi \dot{x}}\right)^2 + \left(2 + \sin(x) + 0.01\sin(100t)\right)u \qquad (48)$$
$$y = x$$

where the nonlinear functions and parameters are unknown. The output y=x and $\dot{x}$ are assumed to be available for measurement. The output $y(t)$ is required to track a desired trajectory defined by $y_d = \sin(\pi t)$. The magnitude of the tracking error $e = y - y_d$ at steady state is required to be less than 0.05. A one hidden layer radial Gaussian network with a basis function $g_i^f(\boldsymbol{x}) = \exp[-\pi\sigma^2(\boldsymbol{x}-\xi_i)^T(\boldsymbol{x}-\xi_i)]$ is used to approximate $f(x,\dot{x})$ and $b(x)$, which in this case are the ideal bandlimited smooth functions, on a compact set $\Omega = \Omega_x \times \Omega_{\dot{x}}$, where $\Omega_x = \{x \mid x \in (-3,3)\}$, $\Omega_{\dot{x}} = \{\dot{x} \mid \dot{x} \in (-5,5)\}$,. The mean ξ_i is the center of the radial Gaussian representing the sampling grid, and σ^2 is the variance representing a measure of the width of the radial Gaussian. By choosing a sampling grid with mesh size 0.125 and variance 4π, a small uniform approximation bound of $\varepsilon_f \le 0.05$, $\varepsilon_b \le 0.02$, can be achieved (Sanner and Slotine, 1992). Furthermore $b(\boldsymbol{x}) \ge \bar{b} = 0.99$. Note that 0.01sin(100t) is considered as a disturbance term and is not estimated. The values of δ_b, Δ, are chosen to be 0.01, 0.05 respectively. Using (36a-c), $\delta_1 = 0.0334$, $\delta_2 = 0.02062$, $\delta_3 = 0.02062$. The constants $\sigma_v = 0.034$, $\sigma_f = 0.034$ and $\sigma_b = 0.03$ are chosen such that conditions (45c-e) are satisfied. By selecting $\lambda = 1$ and $\Phi = 0.05$, the requirement of tracking error less than 0.05 is achieved. Given condition (45b), the gain k_S is chosen to satisfy $k_S \ge 1.035$. Therefore with a value of $k_S = 2$, the tracking error remains bounded from above by 0.05. Figures 1 and 2 show the simulation results for the tracking error, and continuous switching function ρ.

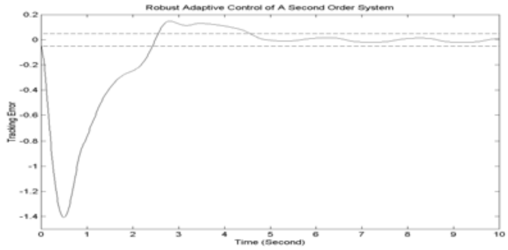

Fig. 1. Tracking error during the first 10 seconds. The dashed lines indicate the required error bound

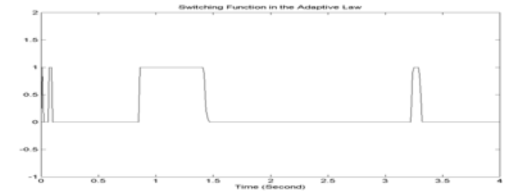

Fig. 2. The continuous switching function ρ in the adaptive law during the first 4 seconds

6. CONCLUSIONS

In this paper, we consider the control problem of a single input feedback linearizable nonlinear system with unknown nonlinearities. The nonlinearities are assumed to be smooth functions and as such can be approximated and estimated on-line using a single layer neural network. A robust adaptive controller scheme is designed that uses the estimated nonlinear functions and employs a number of robust modifications in order to compensate for uncertainties in the estimation. The control scheme guarantees semiglobal stability and convergence of the tracking error to a small residual set whose size depends on certain design parameters. Semiglobal stability is characterized by a region of attraction for stability whose size depends on the nodes of the neural network used to approximate the nonlinear functions of the plant. Our results present a methodology for choosing various design parameters so that the tracking error is guaranteed to converge and remain within desirable bounds at steady state. The extension of these results to a wider class of nonlinear system is currently under investigation.

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APPENDIX-PROOFS OF INEQUALITY (37)

In this appendix, we prove inequality (37) used in the proof of *theorem*. Let us start with the equality

$$d_b(\boldsymbol{x})u - \delta_b u' = (1-\rho)\{d_b u - \delta_b u'\} + \rho\{d_b u - \delta_b u'\} \quad (A.1)$$

From (24), (30), the function u' can be written as:

$$u' = \frac{1}{(\hat{b}^a)^2 + \delta_b}\bar{u} \quad (A.2)$$

Then,

$$\{(\hat{b}^a)^2 u' + \delta_b u' = \bar{u} \quad (A.3)$$

Since $\hat{b}^a u' = u$, (A.3) can be written as

$$\hat{b}^a u = \bar{u} - \delta_b u' \quad (A.4)$$

Substituting $\hat{b}^a = b^a + \tilde{b}^a$ into (A.4), we obtain

$$b^a u = \bar{u} - \delta_b u' - \tilde{b}^a u \quad (A.5)$$

Using (A.2), it follows that

$$\begin{aligned} b^a u &= \left\{1 - \frac{\delta_b}{(\hat{b}^a)^2 + \delta_b}\right\}\bar{u} - \tilde{b}^a u \\ &= \frac{(\hat{b}^a)^2}{(\hat{b}^a)^2 + \delta_b}\bar{u} - \tilde{b}^a u \end{aligned} \quad (A.6)$$

From (A.6), the control law u can be expressed as

$$u = \{\frac{(\hat{b}^a)^2}{(\hat{b}^a)^2 + \delta_b}\}\frac{\bar{u}}{b^a} - \frac{1}{b^a}\tilde{b}^a u \quad (A.7)$$

Using the fact $|b^a| = |b(\boldsymbol{x}) - d_b(\boldsymbol{x})| \geq \bar{b} - \varepsilon_b$, one has

$$\begin{aligned} |u| &\leq \frac{1}{|b^a|}|\bar{u}| + \frac{1}{|b^a|}|\tilde{b}^a||u| \\ &\leq \frac{1}{\bar{b} - \varepsilon_b}|\bar{u}| + \frac{1}{\bar{b} - \varepsilon_b}|\tilde{b}^a||u| \end{aligned} \quad (A.8)$$

Since (A.3) can also be written as

$$\{(b^a + \tilde{b}^a)^2 + \delta_b\}u' = \bar{u} \quad (A.9)$$

one has

$$\{(b^a)^2 + (\tilde{b}^a)^2 + \delta_b\}u' = \bar{u} - 2\tilde{b}^a b^a u' \quad (A.10)$$

Therefore u' can be expressed as

$$u' = \frac{1}{(b^a)^2 + (\tilde{b}^a)^2 + \delta_b}\bar{u} - \frac{2b^a}{(b^a)^2 + (\tilde{b}^a)^2 + \delta_b}\tilde{b}^a u' \quad (A.11)$$

and,

$$\begin{aligned} |u'| &\leq \frac{1}{(b^a)^2 + \delta_b}|\bar{u}| + \frac{2|b^a|}{(b^a)^2 + \delta_b}|\tilde{b}^a||u'| \\ &\leq \frac{1}{(b^a)^2 + \delta_b}|\bar{u}| + \frac{2}{|b^a|}|\tilde{b}^a||u'| \\ &\leq \frac{1}{(\bar{b} - \varepsilon_b)^2 + \delta_b}|\bar{u}| + \frac{2}{\bar{b} - \varepsilon_b}|\tilde{b}^a||u'| \end{aligned} \quad (A.12)$$

Using (A.8) and (A.12), the absolute value of the second term in (A.1) can be written in the following form:

$$\begin{aligned} &|\rho\{d_b u - \delta_b u'\}| \leq \rho|d_b||u| + \rho\delta_b|u'| \\ &\leq \rho\left\{\frac{\varepsilon_b}{\bar{b} - \varepsilon_b} + \frac{\delta_b}{(\bar{b} - \varepsilon_b)^2 + \delta_b}\right\}|\bar{u}| \\ &+ \rho\frac{2\delta_b}{\bar{b} - \varepsilon_b}|\tilde{b}^a||u'| + \rho\frac{\varepsilon_b}{\bar{b} - \varepsilon_b}|\tilde{b}^a||u| \\ &\leq \rho\left\{\frac{\varepsilon_b}{\bar{b} - \varepsilon_b - \Delta} + \frac{\delta_b}{(\bar{b} - \varepsilon_b - \Delta)^2 + \delta_b}\right\}|\bar{u}| \\ &+ \rho\frac{2\delta_b}{\bar{b} - \varepsilon_b}|\tilde{b}^a||u'| + \rho\frac{\varepsilon_b}{\bar{b} - \varepsilon_b}|\tilde{b}^a||u| \end{aligned} \quad (A.13)$$

The first term in (A.1) can be written as:

$$\begin{aligned} (1-\rho)\{d_b u - \delta_b u'\} &= (1-\rho)\frac{d_b\hat{b}^a}{(\hat{b}^a)^2 + \delta_b}\bar{u} \\ &- (1-\rho)\frac{\delta_b}{(\hat{b}^a)^2 + \delta_b}\bar{u} \end{aligned} \quad (A.14)$$

Since $(1-\rho) \neq 0$ only if $|\hat{b}^a| \geq \bar{b} - \varepsilon_b - \Delta$, one obtains:

$$\begin{aligned} &|(1-\rho)\{d_b u - \delta_b u'\}| \leq (1-\rho)\frac{|d_b|}{|\hat{b}^a|}|\bar{u}| + (1-\rho)\frac{\delta_b}{(\hat{b}^a)^2 + \delta_b}|\bar{u}| \\ &\leq (1-\rho)\left\{\frac{\varepsilon_b}{\bar{b} - \varepsilon_b - \Delta} + \frac{\delta_b}{(\bar{b} - \varepsilon_b - \Delta)^2 + \delta_b}\right\}|\bar{u}| \end{aligned} \quad (A.15)$$

From (A.13) and (A.15), (37) follows, i.e.

$$\begin{aligned} |\{d_b u - \delta_b u'| &\leq \left\{\frac{\varepsilon_b}{\bar{b} - \varepsilon_b - \Delta} + \frac{\delta_b}{(\bar{b} - \varepsilon_b - \Delta)^2 + \delta_b}\right\}|\bar{u}| \\ &+ \rho\frac{2\delta_b}{\bar{b} - \varepsilon_b}|\tilde{b}^a||u'| + \rho\frac{\varepsilon_b}{\bar{b} - \varepsilon_b}|\tilde{b}^a||u| \\ &= \delta_1|\bar{u}| + \rho\delta_2|\tilde{b}^a||u'| + \rho\delta_3|\tilde{b}^a||u| \end{aligned} \quad (A.16)$$

where δ_1 , δ_2 , δ_3 are as defined in (36a-c).