15th Triennial World Congress, Barcelona, Spain

A CONVEX OPTIMIZATION APPROACH TO THE MODE ACCELERATION PROBLEM

S. O. Reza Moheimani * , **Dunant Halim** **

* *Senior Lecturer, School of Electrical Engineering and Computer Science and the Centre for Integrated Dynamics and Control, University of Newcastle, NSW, Australia, Fax: +61-2-49216993, Email: reza@ee.newcastle.edu.au*
** *School of Electrical Engineering and Computer Science, University of Newcastle, NSW, Australia, Email: dunanth@ee.newcastle.edu.au*

Abstract: The purpose of this paper is to introduce an alternative procedure to the mode acceleration method when the underlying structure model includes damping. We will show that the problem can be cast as a convex optimization problem that can be solved via linear matrix inequalities.

Keywords: convex optimization, mode analysis, model reduction, modelling errors, damping, simulation

1. INTRODUCTION

Strictly speaking, dynamics of a large number of systems such as structures, acoustic enclosures, etc., consist of an infinite number of modes. Dynamics of these systems are known to be governed by certain partial differential equations. These partial differential equations are often discretized using the modal analysis procedure. As a result of this discretization the partial differential equation is approximated by an infinite sum. However, it is well-known that in order to represent the dynamics of such systems, including a large number of modes in the series will suffice (Hughes, 1987).

For control design purposes, these modes can be categorized into two groups. These are *the in-bandwidth modes* (those modes that lie within the bandwidth of interest from the control point of view), and *the out-of-bandwidth modes.* In control design problems, very often the infinite series is truncated by removing the out-of-bandwidth modes and keeping those modes that lie within the bandwidth of interest. Poles of the truncated model are precisely the same as the in-bandwidth poles of the infinite dimensional system. However, zeros of the truncated model may be significantly different from those of the actual system. A controller that is designed using such a model may perform poorly when implemented on the real system as the performance of the feedback controller is largely dictated by the open loop zeros of the underlying system. It is, therefore, important to improve the in-bandwidth model of the system so that high performance controllers can be designed.

One approach to minimizing the truncation error is to add a feed-through term to the truncated model, where the feed-through term is made up of the sum of DC contents of all the truncated high-frequency modes. In the aeroelasticity literature this method is referred to as the *mode acceleration method* (Bisplinghoff and Ashley, 1962). The mode acceleration method will result in zero error at the DC. However, the error will increase as we move to higher frequencies within the bandwidth of interest. Furthermore, this method is not optimal by any measure. In reference (Moheimani, 2000), it is shown that a feed-through term can be obtained by minimizing the weighted H_2 norm of the error system and an analytic solution to the optimization problem is presented. In reference (Moheimani and Clark, 2000), the same problem is addressed by adding an out-of-bandwidth mode to the system, hence reducing the in-bandwidth error even further than that reported in (Moheimani, 2000).

All of the results reported in above references are developed for models that have zero damping associated with all the modes. This will not be a cause of concern as long as the actual damping terms are very small. This may be true for some systems, however, when the underlying structure has significant damping, the procedures reported in the literature may not perform in a satisfactory manner. This paper is aimed at developing a procedure for minimizing the in-bandwidth error when the underlying system may have significant damping associated with each mode. Our approach is to set up an optimization problem and solve it using the recently developed convex optimization techniques (Boyd *et al.*, 1994).

2. PROBLEM STATEMENT

Dynamics of many systems such as flexible beams and plates, strings, acoustic ducts and enclosures are governed by specific partial differential equations. For example, dynamics of a thin beam is governed by Bernoulli-Euler beam equation (Meirovitch, 1990) and its associated boundary conditions. These partial differential equations are often discretized using the modal analysis procedure (Meirovitch, 1986). Following this procedure, one would typically obtain a model of the form

$$G(s) = \sum_{i=1}^{\infty} \frac{\alpha_i}{s^2 + \omega_i^2}.$$

Associated with each mode, there exists a specific level of damping, which is often ignored at earlier stages of the analysis. For control design purposes the series is truncated by removing those high frequency modes that lie out of the bandwidth of interest. That is $G(s)$ is approximated by

$$G_N(s) = \sum_{i=1}^{N} \frac{\alpha_i}{s^2 + \omega_i^2}.$$

It can be observed that poles of $G_N(s)$ are similar to the first N poles of $G(s)$. However, as a result of the truncation, zeros of $G_N(s)$ may be different from the in-bandwidth zeros of $G(s)$. The reason for this is that each truncated mode does contain a DC term. Removing these high frequency modes generates an error that might be significant at low frequencies. The problem is more severe if the actuator and sensor are colocated as noted in (Clark, 1997). This problem can be addressed by adding a feed-through term to $G_N(s)$. That is,

$$\hat{G}_N(s) = \sum_{i=1}^{N} \frac{\alpha_i}{s^2 + \omega_i^2} + \sum_{i=N+1}^{\infty} \frac{\alpha_i}{\omega_i^2}.$$

This technique is referred to as the *mode acceleration method* (see page 350 of (Bisplinghoff and Ashley, 1962)). The feed-through term added to $G_N(s)$ is the sum of DC contents of all the truncated modes. This reduces the error at $\omega = 0$ to zero. However, the error will increase as we move to higher frequencies within the bandwidth of interest. In references (Moheimani and Clark, 2000) and (Moheimani, 2000) it is suggested that an optimization problem can be set up to reduce the in-bandwidth error. The solutions given in these references are optimal in the H_2 sense. However, it is assumed that the effect of damping on all the modes can be ignored. In this paper, we allow for each mode to include a specific amount of damping and we develop a convex optimization based solution to the problem. Furthermore, we allow for multi-variable models in our analysis.

3. OPTIMIZATION

Consider the multi-variable input-output model of a structure obtained via modal analysis procedure

$$\mathbf{G_M}(s) = \sum_{i=1}^{M} \frac{\mathbf{\Psi_i}}{s^2 + 2\zeta_i\omega_i s + \omega_i^2} \quad (1)$$

where M may be a large number and $\mathbf{\Psi_i} \in \mathbf{R}^{m\times n}$ for $i = 1, 2, \ldots, M$.

This model is truncated by keeping the first N modes, i.e.,

$$\mathbf{G_N}(s) = \sum_{i=1}^{N} \frac{\mathbf{\Psi_i}}{s^2 + 2\zeta_i\omega_i s + \omega_i^2}. \quad (2)$$

A feed-through term is then added to (2)

$$\hat{\mathbf{G}}_\mathbf{N}(s) = \sum_{i=1}^{N} \frac{\mathbf{\Psi_i}}{s^2 + 2\zeta_i\omega_i s + \omega_i^2} + \mathbf{K} \quad (3)$$

where the optimal $\mathbf{K} \in \mathbf{R}^{m\times n}$ is to be determined such that:

$$\mathbf{K}^* = \arg \min_{\mathbf{K}\in\mathbf{R}^{m\times n}} \|\mathbf{W}(s)(\mathbf{G_M}(s) - \hat{\mathbf{G}}_\mathbf{N}(s))\|_2^2. \quad (4)$$

Here $\mathbf{W(s)}$ is a low-pass weighting function whose purpose is to emphasize the in-bandwidth error. The cut-off frequency of this filter is typically chosen to lie within the range $\omega_N \le \omega \le \omega_{N+1}$.

The above transfer functions can be represented in state space form as follows:

$$\mathbf{G_N}(s) \stackrel{s}{=} \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{0} \end{array}\right]$$
$$\mathbf{G_M}(s) \stackrel{s}{=} \left[\begin{array}{cc|c} \mathbf{A} & \mathbf{0} & \mathbf{B} \\ \mathbf{0} & \mathbf{A_2} & \mathbf{B_2} \\ \hline \mathbf{C} & \mathbf{C_2} & \mathbf{0} \end{array}\right]$$
$$\mathbf{W}(s) \stackrel{s}{=} \left[\begin{array}{c|c} \mathbf{A_w} & \mathbf{B_w} \\ \hline \mathbf{C_w} & \mathbf{0} \end{array}\right]$$

with appropriate values for $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{A_2}$, $\mathbf{B_2}$ and $\mathbf{C_2}$. Using the above notation, an expression for the error system can be obtained as follows:

$$\begin{aligned}\mathbf{E}(s) &\stackrel{s}{=} \mathbf{W}(s)\left(\mathbf{G_M}(s) - \hat{\mathbf{G}}_\mathbf{N}(s)\right) \\ &\stackrel{s}{=} \left[\begin{array}{c|c} \mathbf{A_w} & \mathbf{B_w} \\ \hline \mathbf{C_w} & \mathbf{0} \end{array}\right] \times \left(\left[\begin{array}{cc|c} \mathbf{A} & \mathbf{0} & \mathbf{B} \\ \mathbf{0} & \mathbf{A_2} & \mathbf{B_2} \\ \hline \mathbf{C} & \mathbf{C_2} & \mathbf{0} \end{array}\right] - \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{K} \end{array}\right]\right) \\ &\stackrel{s}{=} \left[\begin{array}{c|c} \mathbf{A_w} & \mathbf{B_w} \\ \hline \mathbf{C_w} & \mathbf{0} \end{array}\right] \times \left[\begin{array}{c|c} \mathbf{A_2} & \mathbf{B_2} \\ \hline \mathbf{C_2} & -\mathbf{K} \end{array}\right] \\ &\stackrel{s}{=} \left[\begin{array}{cc|c} \mathbf{A_w} & \mathbf{B_w C_2} & -\mathbf{B_w K} \\ \mathbf{0} & \mathbf{A_2} & \mathbf{B_2} \\ \hline \mathbf{C_w} & \mathbf{0} & \mathbf{0} \end{array}\right] \\ &\stackrel{s}{=} \left[\begin{array}{c|c} \bar{\mathbf{A}} & \bar{\mathbf{B}}_1\mathbf{K} + \bar{\mathbf{B}}_2 \\ \hline \bar{\mathbf{C}} & \mathbf{0} \end{array}\right]\end{aligned}$$

where

$$\bar{\mathbf{A}} = \begin{bmatrix} \mathbf{A_w} & \mathbf{B_w C_2} \\ \mathbf{0} & \mathbf{A_2} \end{bmatrix}$$
$$\bar{\mathbf{B}}_1 = \begin{bmatrix} -\mathbf{B_w} \\ \mathbf{0} \end{bmatrix}$$
$$\bar{\mathbf{B}}_2 = \begin{bmatrix} \mathbf{0} \\ \mathbf{B_2} \end{bmatrix}$$
$$\bar{\mathbf{C}} = [\mathbf{C_w} \ \mathbf{0}]$$

Now, it is observed that H_2 norm of the error system, $\mathbf{E}(s)$ can be expressed as (Boyd *et al.*, 1994)

$$||\mathbf{E}(s)||_2^2 = \operatorname{tr}\left\{\bar{\mathbf{C}}\mathbf{P}\bar{\mathbf{C}}'\right\} \tag{5}$$

where $\operatorname{tr}(\mathbf{Q})$ represents the trace of matrix $\mathbf{Q}$ and $\mathbf{P} = \mathbf{P}' > \mathbf{0}$ is the solution to the following Lyapunov inequality

$$\bar{\mathbf{A}}\mathbf{P} + \mathbf{P}\bar{\mathbf{A}}' + (\bar{\mathbf{B}}_1\mathbf{K} + \bar{\mathbf{B}}_2)(\bar{\mathbf{B}}_1\mathbf{K} + \bar{\mathbf{B}}_2)' < 0. \tag{6}$$

Therefore, $\mathbf{K}^*$ can be determined by solving the following eigenvalue problem

$$\begin{aligned}&\text{minimize} \ \operatorname{tr}\left\{\bar{\mathbf{C}}\mathbf{P}\bar{\mathbf{C}}'\right\} \\ &\text{subject to} \ \begin{bmatrix} \bar{\mathbf{A}}\mathbf{P} + \mathbf{P}\bar{\mathbf{A}}' & \bar{\mathbf{B}}_1\mathbf{K} + \bar{\mathbf{B}}_2 \\ \mathbf{K}\bar{\mathbf{B}}_1{}' + \bar{\mathbf{B}}_2' & -\mathbf{I} \end{bmatrix} < \mathbf{0}, \ \mathbf{P} > \mathbf{0}\end{aligned}$$

Now, a different performance measure for minimizing the in-bandwidth error is considered, i.e., the H_∞ norm. The problem is then to determine $\mathbf{K}^*$, where

$$\mathbf{K}^* = \arg \min_{\mathbf{K} \in \mathbf{R}^{m \times n}} ||\mathbf{W}(s)(\mathbf{G_M}(s) - \hat{\mathbf{G}}_\mathbf{N}(s))||_\infty. \tag{7}$$

To solve this problem we will use the strict bounded real lemma:

Lemma 3.1. (Petersen *et al.*, 1991) The following two conditions are equivalent:

(i) $\mathbf{A}$ is stable and $||\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}||_\infty < \gamma$.
(ii) There exists a matrix $\mathbf{P} > \mathbf{0}$ such that

$$\mathbf{A}'\mathbf{P} + \mathbf{P}\mathbf{A} + \frac{1}{\gamma^2}\mathbf{P}\mathbf{B}\mathbf{B}'\mathbf{P} + \mathbf{C}'\mathbf{C} < \mathbf{0}. \tag{8}$$

Lemma 3.1 implies that the inequality

$$||\bar{\mathbf{C}}(s\mathbf{I} - \bar{\mathbf{A}})^{-1}(\bar{\mathbf{B}}_1\mathbf{K} + \bar{\mathbf{B}}_2)||_\infty < \gamma$$

holds if and only if there exists a matrix $\mathbf{P} > \mathbf{0}$ such that

$$\bar{\mathbf{A}}'\mathbf{P} + \mathbf{P}\bar{\mathbf{A}} + \frac{1}{\gamma^2}\mathbf{P}(\bar{\mathbf{B}}_1\mathbf{K} + \bar{\mathbf{B}}_2)(\bar{\mathbf{B}}_1\mathbf{K} + \bar{\mathbf{B}}_2)'\mathbf{P} + \bar{\mathbf{C}}'\bar{\mathbf{C}} < \mathbf{0}. \tag{9}$$

It is also noticed that (9) holds if and only if there exists a matrix $\mathbf{Q} > \mathbf{0}$ such that

$$\mathbf{Q}\bar{\mathbf{A}}' + \bar{\mathbf{A}}\mathbf{Q} + \frac{1}{\gamma^2}(\bar{\mathbf{B}}_1\mathbf{K} + \bar{\mathbf{B}}_2)(\bar{\mathbf{B}}_1\mathbf{K} + \bar{\mathbf{B}}_2) + \mathbf{Q}\bar{\mathbf{C}}'\bar{\mathbf{C}}\mathbf{Q} < \mathbf{0}. \tag{10}$$

It is now possible to transform (10) into a linear matrix inequality using the Schur complement (Boyd *et al.*, 1994). That is,

$$\begin{bmatrix} \mathbf{Q}\bar{\mathbf{A}}' + \bar{\mathbf{A}}\mathbf{Q} & \mathbf{Q}\bar{\mathbf{C}}' & \bar{\mathbf{B}}_1\mathbf{K} + \bar{\mathbf{B}}_2 \\ \bar{\mathbf{C}}\mathbf{Q} & -\mathbf{I} & \mathbf{0} \\ (\bar{\mathbf{B}}_1\mathbf{K} + \bar{\mathbf{B}}_2)' & \mathbf{0} & -\gamma^2\mathbf{I} \end{bmatrix} < \mathbf{0}. \tag{11}$$

Now, the optimization problem (7) can be solved via the solution to the following eigenvalue problem:

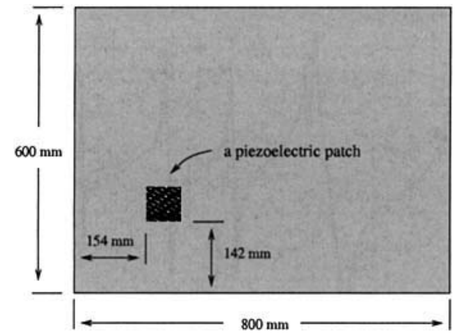

Fig. 4.1. The plate model

$$\begin{aligned} &\text{minimize} \quad \beta \\ &\text{subject to} \quad \begin{bmatrix} \mathbf{Q}\bar{\mathbf{A}}' + \bar{\mathbf{A}}\mathbf{Q} & \mathbf{Q}\bar{\mathbf{C}}' & \bar{\mathbf{B}}_1\mathbf{K} + \bar{\mathbf{B}}_2 \\ \bar{\mathbf{C}}\mathbf{Q} & -\mathbf{I} & \mathbf{0} \\ (\bar{\mathbf{B}}_1\mathbf{K} + \bar{\mathbf{B}}_2)' & \mathbf{0} & -\beta\mathbf{I} \end{bmatrix} < 0, \\ &\qquad\qquad \mathbf{Q} > 0. \end{aligned}$$

4. SIMULATION RESULTS

Simulation results are presented in this section to demonstrate the effectiveness of the proposed LMI approach. MATLAB LMI toolbox is used to perform the LMI optimizations explained in Section 3.

Here, a flexible structure system is considered: a plate with pinned boundary conditions. Two piezoelectric ceramic patches are attached symmetrically to either side of the plate, which work as an actuator and a sensor respectively. Piezoelectric actuators and sensors have been used in many vibration control applications of flexible structures (Moheimani and Ryall, 1999; Clark *et al.*, 1998; Dimitriadis *et al.*, 1991).

The structure consists of an aluminium plate of 800 mm × 600 mm × 4 mm, which is pinned all around. Two identical and colocated piezoelectric ceramic patches (72.4 mm × 72.4 mm × 0.191 mm) are used. The plate model is shown in Figure 4.1. For dimension and other physical properties of the structure, refer to (Halim and Moheimani, 2000).

A model of the structure is obtained via modal analysis technique (Meirovitch, 1986; Reismann, 1988). The transfer function from the actuator-voltage to the sensor-voltage has a similar form with (1) if the model is truncated up to M modes. In the simulation, only the first six modes are included in the truncated plate model, $G_N(s)$, i.e. $N = 6$. The feed-through term calculation is based on the higher-order model of 25 modes, $G_M(s)$, i.e. $M = 25$. A low-pass filter of 4^{th} order, with the cut-off frequency of 249.7 Hz, is used in the simulation. The cut-off frequency is chosen to be between the 6^{th} and 7^{th} resonant frequencies.

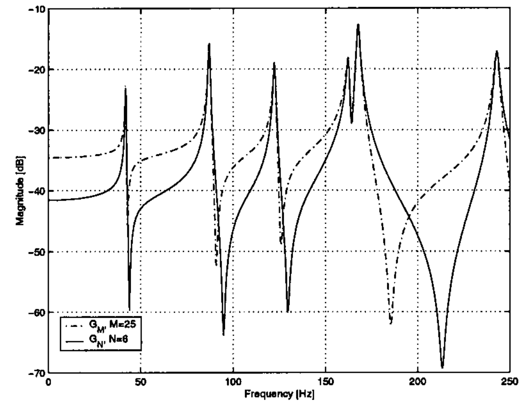

Fig. 4.2. Comparison of frequency responses (magnitude) of $G_N(s)$ and $G_M(s)$

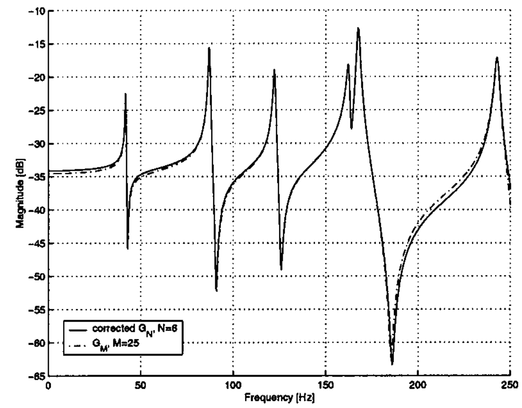

Fig. 4.3. Comparison of frequency responses (magnitude) of $\hat{G}_N(s)$ and $G_M(s)$: H_2 norm approach

Figure 4.2 shows the comparison of the frequency response (magnitude) of those two models. It can be observed that the zeros of the truncated model, $G_N(s)$, are significantly different from $G_M(s)$ since the effect of out-of-bandwidth modes are ignored. Furthermore, there are also gain differences between the two models, especially at low frequencies.

An H_2 norm approach for obtaining the feed-trough term is considered. The LMI optimization searches for the feed-through term that minimizes the H_2 norm of the error system described in (4). Figure 4.3 shows the corrected truncated model, $\hat{G}_N(s)$, in comparison with the higher-order model, $G_M(s)$. The frequency responses with frequency up to cut-off frequency are plotted since the model is only intended to be corrected up to that frequency. The zeros of the corrected model are now closer to the zeros of higher-order model. The gain differences of the two models are also smaller due to an additional gain contributed by the feed-through term of the corrected model.

Similarly, the H_∞ norm approach is used to obtain the feed-through term that minimizes the H_∞ norm of the error system described in (7). Fig-

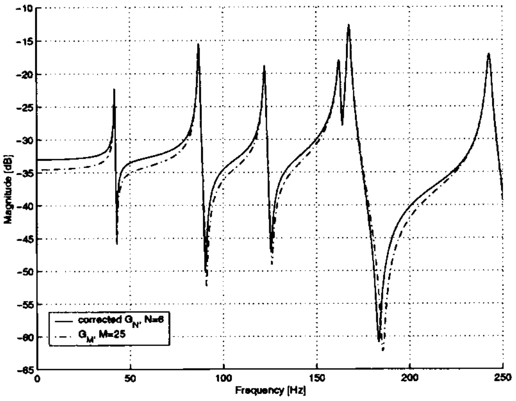

Fig. 4.4. Comparison of frequency responses (magnitude) of $\hat{G}_N(s)$ and $G_M(s)$: H_∞ norm approach

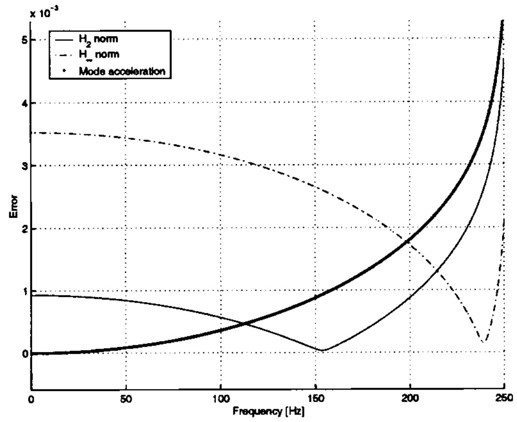

Fig. 4.5. Comparison of error frequency responses (magnitude)

ure 4.4 compares the corrected truncated model, $\hat{G}_N(s)$, and the higher-order model, $G_M(s)$. Compared to Figure 4.2, the zeros and the gain of the corrected model are closer to those of higher-order model. However, the result for the H_∞ norm approach, at frequencies lower than 215 Hz, is worse than that of the H_2 norm approach (compare with Figure 4.3). To analyze this behaviour, the error frequency response for both approaches need to be plotted.

Figure 4.5 shows the error frequency response (magnitude) for H_2 norm and H_∞ norm approaches. From zero frequency up to frequency of 214.4 Hz, the error of the H_2 norm approach is less than that of H_∞ norm approach. This is reasonable since the H_2 norm approach minimizes the error system across the frequency bandwidth. In contrast, the H_∞ norm approach minimizes the H_∞ norm of the error system, which usually occurs at a higher frequency. This means that for a better performance at low frequencies, a higher order low-pass filter is desirable in order to reduce the magnitude of error at out-of-bandwidth frequencies. However, as a consequence, the H_∞ norm approach has a better performance at higher frequencies.

This paper essentially provides an alternative way of obtaining the feed-through term for model correction. The performances of our LMI based approaches with the mode acceleration method can now be compared. In Figure 4.5, the error due to the mode acceleration method is also plotted. As expected, the error is zero at $\omega = 0$ since the method corrects the zero-frequency gain of the truncated model. However, the error increases exponentially as frequency increases. At frequencies higher than 198.35 Hz, the error of the mode acceleration method exceeds that of our LMI based approaches as shown in Figure 4.5.

5. CONCLUSION

An alternative procedure to the mode acceleration method is introduced using a convex optimization approach. Two approaches are discussed, which are the minimizations of the H_2 and H_∞ norms of the error system respectively. The H_2 norm approach out-performs the H_∞ norm approach at lower frequencies, while the H_∞ norm approach has a better high-frequency performance. These approaches perform better at higher frequencies than the mode acceleration method.

6. REFERENCES

Bisplinghoff, R.L. and H. Ashley (1962). *Principles of Aeroelasticity.* Dover Publications Inc.

Boyd, S., L. El Ghaoui, E. Feron and V. Balakrishnan (1994). *Linear Matrix Inequalities in System and Control Theory.* SIAM. Philadelphia, PA.

Clark, R.L. (1997). Accounting for out-of-bandwidth modes in the assumed modes approach: implications on colocated output feedback control. *Transactions of the ASME, Journal of Dynamic Systems, Measurement, and Control* **119**, 390–395.

Clark, R.L, W.R. Saunders and G.P. Gibbs (1998). *Adaptive Structures: Dynamics and Control.* John Wiley & Sons. Canada.

Dimitriadis, E.K., C.R. Fuller and C.A Rogers (1991). Piezoelectric actuators for distributed vibration excitation of thin plates. *ASME Journal of Vibration and Acoustics* **113**, 100–107.

Halim, D. and S.O.R. Moheimani (2000). An optimization approach to optimal placement of collocated piezoelectric actuators and sensors on a thin plate. Technical Report EE0048. Uni. of Newcastle. NSW, Australia. also to appear in *Mechatronics.*

Hughes, P.C. (1987). Space structure vibration modes: how many exist? which ones are important?. *IEEE Control Systems Magazine* pp. 22–28.

Meirovitch, L. (1986). *Elements of Vibration Analysis*. 2 ed.. McGraw-Hill. Sydney.

Meirovitch, L. (1990). *Dynamics and control of structures*. John Wiley & Sons.

Moheimani, S.O.R. (2000). Minimizing the effect of out of bandwidth modes in truncated structure models.. *Transactions of the ASME - Journal of Dynamic Systems, Measurement, and Control* **122**(1), 237–239.

Moheimani, S.O.R. and R.L. Clark (2000). Minimizing the truncation error in assumed modes models of structures.. *Transactions of the ASME, Journal of Vibration & Acoustics* **122**(3), 332–335.

Moheimani, S.O.R. and T. Ryall (1999). Considerations in placement of piezoceramic actuators that are used in structural vibration control. In: *Proceedings of the* 38^{th} *IEEE Conference on Decision & Control*. Phoenix, Arizona, USA. pp. 1118–1123.

Petersen, I.R., B.D.O. Anderson and E.A. Jonckheere (1991). A first principles solution to the non-singular H^∞ control problem. *International Journal of Robust and Nonlinear Control* **1**(3), 171–185.

Reismann, H. (1988). *Elastic plates: theory and application*. John Wiley & Sons. Canada.