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# SOME LOWER BOUNDS ON THE IMPULSE RESPONSE OF SISO SYSTEMS

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Abstract: This paper establishes some lower bounds on the impulse response of SISO(single-input, single-output) stable systems imposed by zeros located in the convergence region of the system. In nonminimum phase systems, the other bounds are derived, which are more severe than that of minimum phase systems owing to RHP(right half plane) zeros. The lower bounds proposed in this paper show the trade-off relations between the maximum magnitude of the impulse response and the achievable settling time. *Copyright © 2002 IFAC*

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## 1. INTRODUCTION

Any control system always has performance limitations in the frequency-domain and the time-domain imposed by the inherent characteristics of the physical system. Many works have been done to investigate these limitations in linear SISO(single-input, single-output) systems (Davison *et al.*, 1999; Freudenberg and Looze, 1985; Goodwin *et al.*, 1999; Kwon, 2002; Middleton, 1991; McWilliams and Sain, 1989), and have been extended to MIMO(multi-input, multi-output) systems (Chen, 2000; Qiu and Davison, 1993) as well as nonlinear systems (Seron *et al.*, 1999). The fundamental limitations in linear filtering designs are also investigated as the counterparts to control theories (Goodwin *et al.*, 1995). As a result, it has been realized that nonminimum phase systems (Chen, 2000; Davison *et al.*, 1999; Freudenberg and Looze, 1985; Middleton, 1991; Qiu and Davison, 1993; Seron *et al.*, 1999) or systems with $j\omega$-axis zeros (Goodwin *et al.*, 1995), compared with minimum phase systems, have more various fundamental limitations associated with the achievable closed-loop transfer function, closed-loop gain margin, loop transfer recovery, sensitivity or complementary sensitivity function, etc. In many cases, these limitations on the achievable performances are utilized to adjust trade-off relations between design specifications.

In spite of many works on the performance limitations, however, it has not been presented for the fundamental limitations on the impulse response, especially on the maximum magnitude of impulse response, in the high-order systems. It is noted that the impulse response of second-order systems is analytically well-studied in Kwon *et al.* (2000).

In this paper, for a linear SISO stable system relaxed at time $0$, the effects of zeros located in the convergence region of the system, including imaginary axis zeros and RHP(right half plane) zeros are investigated. It has also derived the time-domain integral equalities, which have to be satisfied by the impulse response of the system. Based on these integral equalities, it will be shown that the system has some lower bounds between the maximum magnitude of its impulse response and the achievable settling time, and shown that RHP zeros or $j\omega$-axis zeros necessarily imply more severe lower bounds on the impulse response.

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The layout of this paper is organized as follows: In Section 2, some integral equalities, which are used to establish time-domain performance limitations of the system, are described in the time-domain representation of the impulse response. In Section 3, we are to arrive at some lower bounds on the maximum magnitude of the impulse response with respect to the approximate values of settling time that can be practically achieved. The concluding remarks are given in Section 4.

## 2. INTEGRAL EQUALITIES OF IMPULSE RESPONSE

This section is to investigate some integral equalities of the impulse response based on its transfer function. Let us consider a SISO stable transfer function $G(s)$, which is the proper and minimal system with zero initial value. It is assumed that all poles of $G(s)$ have real part less than $\alpha < 0$.

The following lemma states new time-domain integral equalities, which are satisfied by the impulse response of $G(s)$.

*Lemma 1.* Let $g(t)$ be the impulse response of $G(s)$. Then, for $\text{Re}\,[\chi] > \alpha$, the impulse response has to satisfy

$$\int_0^\infty e^{-\sigma t}\cos(\omega t)g(t)dt = \text{Re}\,[G(\chi)]\,, \qquad (1)$$

and

$$\int_0^\infty e^{-\sigma t}\sin(\omega t)g(t)dt = -\text{Im}\,[G(\chi)]\,, \qquad (2)$$

where $\chi = \sigma + j\omega$.

**Proof.** Since $G(s)$ and $g(t)$ are the Laplace transform pair, the Laplace transform of $e^{-\chi t}g(t)$ can be written by

$$\begin{aligned} G(s+\chi) &= \int_0^\infty e^{-st}e^{-\chi t}g(t)dt \\ &= \int_0^\infty e^{-(s+\sigma)t}e^{-j\omega t}g(t)dt. \end{aligned} \qquad (3)$$

For $\text{Re}\,[\chi] > \alpha$, it is clear that the Laplace transform integral representation of $G(s+\chi)$ has the closed RHP as the region of convergence since all poles of $G(s+\chi)$ have the real part less than $\alpha$. Hence, the value of $s = 0$ is obviously in the convergence region of Eq. (3). Consequently, after evaluating Eq. (3) at $s = 0$, the result follows by using the fact that

$$e^{-j\omega t} = \cos(\omega t) - j\sin(\omega t), \qquad (4)$$

which completes the proof. □

Equations (1) and (2) in Lemma 1 have to be satisfied by the impulse response of $G(s)$ for all complex value $\chi$ in the convergence region of the system regardless of the minimum or nonminimum phase system. It is noted that the convergence region is the right side of the dominant pole. Lemma 1 implies some important results with respect to the particular values of $\chi$. In the first place, if the value of $\chi$ is taken as $0$ in Lemma 1, it gives the well-known result as follows:

*Corollary 1.* The integral of impulse response $g(t)$ is equal to the DC gain of the system $G(s)$, *i.e.*,

$$\int_0^\infty g(t)dt = \kappa, \qquad (5)$$

where $\kappa$ is the DC gain of $G(s)$.

**Proof.** Since $G(s)$ is asymptotically stable, it is clear that the Laplace transform integral representation of $G(s)$ has the closed RHP as the region of convergence. Hence, the origin is located in the region of convergence. Let us take $\chi = 0$ in Lemma 1, and the result directly follows from Eq. (1), which completes the proof. □

Note that Eq. (2) is trivial when the value of $\chi$ is taken as real values since both the right and left parts of the equation are identically $0$.

Moreover, for the system with zeros located in the right side of the dominant pole, another integral equality can be also obtained from the relationship between those zeros and the impulse response of the system.

*Lemma 2.* Let $G(s)$ have real zeros at $s = z_i$ for $i = 1, 2, \cdots, r_1$, and complex conjugate zeros at $s = a_k \pm jb_k$ for $k = 1, 2, \cdots, r_2$, which have all real parts larger than $\alpha$. Then, its impulse response $g(t)$ meets the integral equality as follows:

$$\int_0^\infty \left[E_r(t) + E_c(t)\Gamma_c(t)\right]g(t)dt = 0, \qquad (6)$$

where $E_r(t)$ is a linear combination of $e^{-z_i t}$, $E_c(t)$ is a linear combination of $e^{-a_k t}$, and $\Gamma_c(t)$ is a linear combination of $\cos(b_k t)$ and/or $\sin(b_k t)$.

**Proof.** Since $G(s)$ has the right side of the dominant pole as the region of convergence, the zeros at $s = z_i$ and $s = a_k \pm jb_k$ on the complex plane are located in the region of convergence. Let us take $\chi = z_i$ in Eq. (1), and it can be rewritten by

$$\int_0^\infty E_r(t)g(t)dt = 0, \qquad (7)$$

since $G(z_i) = 0$. Similarly, let us take $\chi = a_k \pm jb_k$ in Eqs. (1) and (2), then they can be reformulated as

$$\int_0^\infty E_c(t)\Gamma_c(t)g(t)dt = 0. \qquad (8)$$

Hence, the result comes from Eqs. (7) and (8), which completes the proof. □

Lemma 2 gives some information about the performance limitations on the impulse response of the system with those zeros. For example, if the system with a real zero at $s = z$ larger than $\alpha$, its impulse response $g(t)$ has to satisfy

$$\int_0^\infty e^{-zt} g(t) dt = 0, \tag{9}$$

which states that $g(t)$ must have sign changes at some time instant since $e^{-zt} \geq 0$ for all time $t \geq 0$. In other words, the step response of the system must have extrema such as the undershoot or the overshoot (Kwon *et al.*, 2001; León de la Barra, 1994; Vidyasagar, 1986). As a matter of fact, it is well-known that real zeros located between the dominant pole and the imaginary axis necessarily contribute to the overshoot and RHP real zeros must exhibit the initial undershoot in the step response (Middleton, 1991). It is noted that similar results related to the step response are given in Kwon and Kwon (2002).

For the second example, let us consider a system with two zeros on the imaginary axis at $s = \pm jb$. In this case, Eq. (6) can be simply rewritten by

$$\int_0^\infty \cos(bt) g(t) dt = 0. \tag{10}$$

Let $\tau$ be the settling time such that $g(t) = 0$ for all time $t \geq \tau$. Assume that $b\tau \ll \pi/2$. Then, using the Taylor series expansion for $\cos(bt)$, Eq. (10) yields

$$\int_0^\infty g(t) dt \simeq 0, \tag{11}$$

which contradicts Eq. (5) in Corollary 1 for the system with nonzero DC gain. It means that the complex conjugate zeros on the imaginary axis necessarily imply a lower bound on the achievable settling time of the system. This result coincides with the work of Goodwin *et al.* (1999), which has shown that fundamental limitations on the achievable settling time exist if the system has zeros on or near the imaginary axis.

## 3. LOWER BOUNDS ON THE IMPULSE RESPONSE

Based on the results of the previous section, the lower bounds on the maximum magnitude of the impulse response are to be established in this section. It is noted that the impulse response $g(t)$ of the system $G(s)$ having a strictly proper stable rational transfer function can be estimated by $C/t$, where the constant $C$ is explicitly determined by the spectral energy of $g(t)$ and the real parts of the poles and zeros of $G(s)$ (MacCluer, 1991).

Let us consider a situation in which the impulse response is identically zero after a finite time period, which is previously used in Goodwin *et al.* (1999). Although this assumption of the exact settling time would be unrealistic, corresponding results presented in this paper can be extended so that similar bounds hold under the less restrictive set of assumptions.

*Definition 1.* Let us define the exact settling time of the system as follows:

$$t_s = \inf \left\{ \tau \ : \ g(t) = 0, \ ^\forall t \geq \tau \right\}, \tag{12}$$

where $g(t)$ is the impulse response of the system.

Also, let us define the maximum magnitude of the impulse response using the $L_\infty$ norm as follows:

*Definition 2.* Define the $L_\infty$ norm by

$$\|g\|_\infty = \text{ess sup}\, |g(t)|, \ ^\forall t \geq 0, \tag{13}$$

where $g(t)$ is the impulse response of the system.

It is denoted in this chapter that $\mathbb{R}$ is the set of real numbers and $\mathbb{R}_\alpha^+$ is the set of real numbers larger than the real value $\alpha$ which is given by the real part of the dominant pole. Under the definitions, the lower bounds of $\|g\|_\infty$ can be derived as follows:

*Theorem 1.* If $\omega t_s \leq \pi/2$, the maximum magnitude of the impulse response $g(t)$ has the lower bound as follows:

$$\|g\|_\infty \geq \max_{\sigma \in \mathbb{R}_\alpha^+, \omega \in \mathbb{R}} \left[ \frac{|\text{Re}\,[G(\chi)]|}{\mathcal{M}(\sigma,\omega)} \right] \tag{14}$$

with

$$\begin{aligned} \mathcal{M}(\sigma,\omega) \triangleq \ & \frac{\sigma}{\sigma^2+\omega^2} \\ & - \frac{e^{-\sigma t_s}}{\sigma^2+\omega^2} \left[ \sigma \cos(\omega t_s) - \omega \sin(\omega t_s) \right], \end{aligned} \tag{15}$$

where $t_s$ is the exact settling time and $\chi = \sigma + j\omega$. Moreover, if $\omega t_s \leq \pi$, the maximum magnitude of $g(t)$ has the lower bound as follows:

$$\|g\|_\infty \geq \max_{\sigma \in \mathbb{R}_\alpha^+, \omega \in \mathbb{R}} \left[ \frac{|\text{Im}\,[G(\chi)]|}{\mathcal{N}(\sigma,\omega)} \right] \tag{16}$$

with

$$\begin{aligned} \mathcal{N}(\sigma,\omega) \triangleq \ & \frac{\omega}{\sigma^2+\omega^2} \\ & - \frac{e^{-\sigma t_s}}{\sigma^2+\omega^2} \left[ \sigma \sin(\omega t_s) + \omega \cos(\omega t_s) \right]. \end{aligned} \tag{17}$$

**Proof.** If $\omega t_s \leq \pi/2$, then Eq. (1) in Lemma 1 can be rewritten by

$$\begin{aligned} |\text{Re}\,[G(\chi)]| &= \left| \int_0^\infty e^{-\sigma t} \cos(\omega t) g(t) dt \right| \\ &= \left| \int_0^{t_s} e^{-\sigma t} \cos(\omega t) g(t) dt \right| \\ &\leq \|g\|_\infty \int_0^{t_s} e^{-\sigma t} \cos(\omega t) dt \end{aligned} \tag{18}$$

which follows Eq. (14). Similarly, Eq. (16) can be also derived from Eq. (2) in Lemma 1 when $\omega t_s \leq \pi$, which completes the proof. □

It seems difficult to compute the lower bounds presented in Theorem 1 by analytic methods. They can be however easily changed to simple form for the specified value $\chi$. For example, if we take the imaginary part of $\chi$ for 0, Eq. (14) implies another lower bound without any relation with the imaginary part $\omega$ as follows:

$$\|g\|_\infty \geq \max_{\sigma\in\mathbb{R}_\alpha^+}\left[\frac{\sigma\,|G(\sigma)|}{1-e^{\sigma t_s}}\right]. \qquad (19)$$

As the exact settling time $t_s$ goes to $\infty$, Eq. (19) can be rewritten by

$$\|g\|_\infty \geq \max_{\sigma\in\mathbb{R}_\alpha^+}\left[\sigma|G(\sigma)|\right]. \qquad (20)$$

When $\sigma$ approaches $\infty$ in Eq. (19), it is equal to $0$ for the system with the relative degree over $2$, or equal to the leading coefficient, *i.e.*, the coefficient of the highest power, of the numerator for the system with the relative degree $1$, which is the same as the value of the impulse response at time $t = 0$. Moreover, when the complex value $\chi$ is equal to zero, the lower bound of Eq. (14) is given by

$$\|g\|_\infty \geq \frac{|\kappa|}{t_s}, \qquad (21)$$

which is practically comes from Eq. (18) with $\chi = 0$. All these lower bounds have to be satisfied by the impulse response of SISO stable systems relaxed at time $0$.

For nonminimum phase systems, we can derive more severe lower bounds than those on the impulse response of minimum phase systems.

*Theorem 2.* Let $G(s)$ be a SISO stable system with RHP complex conjugate zeros at $s = a \pm jb$ on the complex plane. Then, the maximum magnitude of $g(t)$ has a lower bound as follows:

$$\|g\|_\infty \geq \max\left[\mathcal{A}(a,b),\mathcal{B}(a,b)\right] \qquad (22)$$

with

$$\mathcal{A}(a,b) \triangleq \frac{(a^2+b^2)\,|\kappa|}{(a^2+b^2)t_s + e^{-at_s}\left[a\cos(bt_s) - b\sin(bt_s)\right] - a}, \qquad (23)$$

$$\mathcal{B}(a,b) \triangleq \frac{(a^2+b^2)\,|\kappa|}{(a^2+b^2)t_s + e^{-at_s}\left[a\sin(bt_s) + b\cos(bt_s)\right] - b}, \qquad (24)$$

where $t_s$ and $\kappa$ is the exact settling time and the DC gain of $G(s)$, respectively.

**Proof.** For the RHP complex conjugate zeros at $s = a \pm jb$, Eq. (6) can be simply written by

$$\begin{aligned} 0 &= \int_0^\infty e^{-at}\cos(bt)g(t)dt \\ &= \int_0^{t_s} e^{-at}\cos(bt)g(t)dt \\ &= \int_0^{t_s}\left[e^{-at}\cos(bt)-1\right]g(t)dt + \int_0^{t_s} g(t)dt, \end{aligned} \qquad (25)$$

which yields the relations as follows:

$$\begin{aligned} \left|\int_0^{t_s} g(t)dt\right| &= \left|\int_0^{t_s}\left[1-e^{-at}\cos(bt)\right]g(t)dt\right| \\ &\leq \|g\|_\infty \int_0^{t_s}\left[1-e^{-at}\cos(bt)\right]dt. \end{aligned} \qquad (26)$$

Hence, Eq. (26) implies that

$$\|g\|_\infty \geq \mathcal{A}(a,b). \qquad (27)$$

Similar lower bound, which is given by

$$\|g\|_\infty \geq \mathcal{B}(a,b), \qquad (28)$$

also comes from

$$\int_0^\infty e^{-at}\sin(bt)g(t)dt = 0. \qquad (29)$$

Hence, the result follows from Eqs. (27) and (28), which completes the proof. □

For example, let us consider the system with the unit DC gain and RHP complex conjugate zeros at $s = 0.5 \pm 0.5j$ on the complex plane. The curves of Fig. 1 represent the bounds of Eqs. (21) and (22) with respect to the exact settling time $t_s$, respectively. It can be seen that when $t_s$ is large, the bounds of Eqs. (21) and (22) are similar to each other, but when the smaller $t_s$, the bound of Eq. (22) is more severe constraint than that of Eq. (21).

If the system has RHP real zeros, the lower bound of Eq. (22) in Theorem 2 can be represented by

$$\|g\|_\infty \geq \frac{z_1|\kappa|}{z_1 t_s + e^{-z_1 t_s} - 1}, \qquad (30)$$

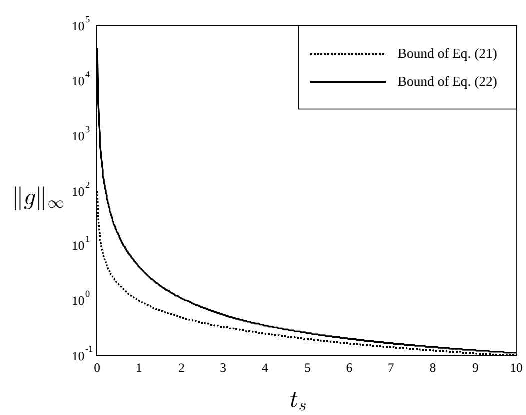


Fig. 1. Lower bounds on $\|g\|_\infty$ of the system with RHP zeros at $s = 0.5 \pm 0.5j$ and $\kappa = 1$.

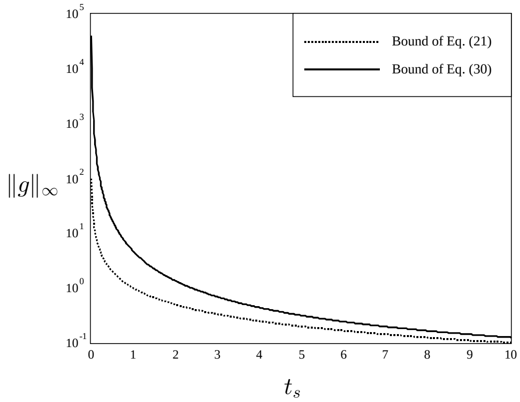


Fig. 2. Lower bounds on $\|g\|_\infty$ of the system with an RHP real zero at $s = 0.5$ and $\kappa = 1$.

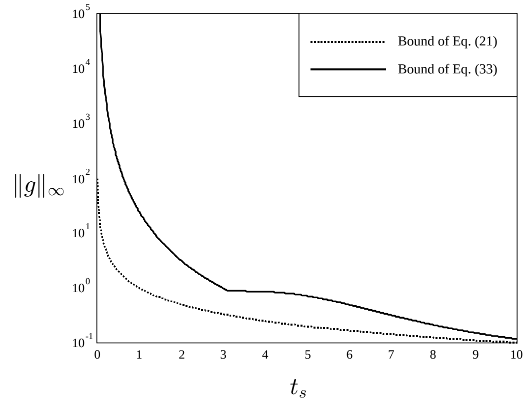


Fig. 3. Lower bounds on $\|g\|_\infty$ of the system with complex zeros at $s = \pm 0.5j$ and $\kappa = 1$.

where $z_1$ is an RHP real zero nearest to the imaginary axis. The lower bound of Eq. (30) obviously provide a more severe lower bound than that of Eq. (21), *i.e.*, the impulse response of the system with real RHP zeros has the larger lower bound than that of minimum phase system. When $\kappa = 1$ and $z_1 = 0.5$, the curves of Fig. 2 show the trends of the lower bounds (21) and (30) with respect to $t_s$. The dotted and solid curves show the loci of Eqs. (21) and (30), respectively.

For a system with complex conjugate zeros located on the imaginary axis at $s = \pm jb$, $\mathcal{A}(a, b)$ and $\mathcal{B}(a, b)$ are simply reduced by

$$\mathcal{A}(a, b) = \frac{b|\kappa|}{bt_s - \sin(bt_s)}, \tag{31}$$

and

$$\mathcal{B}(a, b) = \frac{b|\kappa|}{bt_s + \cos(bt_s) - 1}, \tag{32}$$

*i.e.*, the lower bound on the impulse response of the system is described by

$$\|g\|_\infty \geq \max\left[\frac{b|\kappa|}{bt_s - \sin(bt_s)},\ \frac{b|\kappa|}{bt_s + \cos(bt_s) - 1}\right]. \tag{33}$$

When $\kappa = 1$ and $b = 0.5$, the dotted and solid curves of Fig. 3 represent the loci of Eqs. (21) and (33) with respect to $t_s$, respectively. It can be shown that as the $t_s$ becomes smaller, the bound of Eq. (33) is more severe constraint than that of Eq. (21) similar to the case of Fig. 1 or Fig. 2.

The proposed lower bounds on the impulse response clearly show the trade-off relations with the exact settling time $t_s$; as $t_s$ becomes small, the lower bounds become arbitrarily large and *vice versa*. Hence, to make $\|g\|_\infty$ small, the desired settling time of the system has to be increased. Note that the system with RHP zeros or zeros on the imaginary axis has more severe lower bounds on the maximum magnitude of the impulse response.

In the unity-feedback control scheme, the impulse response has the other lower bounds imposed by open-loop unstable poles and RHP zeros, which is similar results to the step response in Kwon and Kwon (2002). It states that the maximum magnitude of impulse response is extremely large when the closed-loop system has unstable open-loop poles located in the left vicinity of RHP zeros (Kwon, 2002).

## 4. CONCLUSION

This paper has presented some new results on time-domain integral equalities for the impulse response of SISO stable systems in terms of the complex value $\chi$ in the convergence region of the system. It has also shown that nonminimum phase systems has another integral equality imposed by RHP zeros. These integral equalities may imply some performance limitations on the impulse response of the system.

Based on these equalities, it has shown that all systems have some lower bounds on the maximum magnitude of its impulse response with respect to the achievable settling time, which is given by Theorem 1. Moreover, it has also shown that systems with RHP zeros or complex conjugate zeros on the imaginary axis have more severe lower bounds on the impulse response. From those lower bounds, it has been shown that if we wish to make $\|g\|_\infty$ small, the desired settling time of the system has to be increased. Hence, the results presented in this paper will provide guidelines for designing feedback controller of any system since the zeros of the system is not changed in spite of the feedback.

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