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ALMOST DISTURBANCE DECOUPLING FOR NONLINEAR SYSTEMS VIA CONTINUOUS FEEDBACK [1]

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Abstract: This paper addresses the problem of almost disturbance decoupling with internal stability (ADD) for inherently nonlinear systems with uncontrollable unstable linearization. Although achieving ADD in the sense of the L_2–gain is usually impossible by *smooth* state feedback, we show that there exists a *non-smooth but continuous* state feedback control law, yielding a closed-loop system which is globally strongly stable in the absence of disturbance, and in the presence of disturbance, whose L_2–gain between the disturbance input and the system output is less than or equal to an arbitrarily small number $\gamma > 0$. In contrast to the existing results in the literature, all the growth conditions imposed previously to achieve ADD via smooth state feedback are completely removed under this *continuous feedback* framework, enabling one to deal with a significantly larger class of nonlinear systems. *Copyright © 2002 IFAC*

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1. INTRODUCTION

The problem of almost disturbance decoupling with internal stability (ADD) has attracted considerable attention since it was first formulated for linear systems by Willems in 1980's. Over the past two decades, many interesting and important results have been obtained for both linear and nonlinear systems (Willems, 1981; Weiland and Willems, 1989; Isidori, 1995; Nijmeijer and van der Schaft, 1990; Marino *et al.*, 1989, 1994). In this work, we investigate the ADD problem for a class of nonlinear systems with uncontrollable unstable linearization, which cannot be dealt with by any existing *smooth* feedback method.

The disturbance decoupling problem for affine nonlinear systems was studied by a number of researchers during 1980's. With the help of the differential geometric control theory, a series of interesting results were obtained (Isidori, 1995; Nijmeijer and van der Schaft, 1990; Marino *et al.*, 1989, 1994). In the books (Isidori, 1995; Nijmeijer and van der Schaft, 1990), a solution was presented to the problem of *exact disturbance decoupling* without internal stability, and a necessary and sufficient condition was derived for the problem to be solvable by smooth static state feedback. The investigation of the so-called *almost disturbance decoupling* problem was initially carried out in (Marino *et al.*, 1989), using a singular perturbation method. The solution in (Marino *et al.*, 1989) to the ADD problem was characterized in terms of the L_∞ induced norm from the disturbance input to the system output. However, the important issue like internal stability of the closed-loop system was not addressed in (Marino *et al.*, 1989), even in the absence of disturbance. The lack of stability makes the result of (Marino *et al.*, 1989) difficult to be used in practical applications. The stability issue was addressed later in the work (Marino *et al.*, 1994), where an elegant recursive design technique known as adding a linear integrator was presented, leading to a global solution to the ADD problem with internal stability for a class of feedback linearizable or minimum-phase nonlinear systems in a lower-triangular form. The result of (Marino *et al.*, 1994) was then extended to a

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class of minimum-phase nonlinear systems whose zero-dynamics are not necessarily independent of the disturbance $w(t)$ (Isidori, 1996a). Recently, the ADD results of (Isidori, 1996a; Marino *et al.*, 1994) have been further generalized to a class of *non-minimum-phase* nonlinear systems (Isidori, 1996b). In a similar direction, *global inverse* L_2-gain design for feedback linearizable system was reported in (Isidori and Lin, 1998).

It should be observed that so far most of the existing solutions to the ADD problem have been established under the assumptions that the controlled plants are *feedback linearizable* (at least partially) and *affine* in the control input. When the system under consideration is inherently nonlinear in the sense that the linearized system is not controllable (hence it is not even locally feedback linearizable), and the system is non-affine in the control input, very few ADD results are available. In the paper (Qian and Lin, 2000), we consider the ADD problem for a class of high-order nonlinear systems of the form

$$\begin{aligned} \dot{x}_i &= x_{i+1}^{p_i} + f_i(x_1, \cdots, x_i) + \phi_i(x_1, \cdots, x_i)w, \\ y &= h(x_1), \quad x_{n+1} := u, \quad 1 \le i \le n, \end{aligned} \tag{1.1}$$

where $u \in \mathbb{R}$, $y \in \mathbb{R}$ and $w \in \mathbb{R}^s$ are the system input, output and disturbance signal, respectively, $p_i,\ i = 1, \cdots, n$, are positive *odd* integers and $h(\cdot),\ f_i(\cdot),\ \phi_i(\cdot),\ i = 1, \cdots, n$, are C^1 functions with $h(0) = 0,\ f_i(0) = 0,\ \ i = 1, \cdots, n$.

As illustrated by the counter-example in (Qian and Lin, 2000), the standard L_2–gain characterization between the system output and disturbance is not a well-posed problem for the high-order nonlinear system (1.1). Therfore, the ADD problem for the system (1.1) was reformulated in terms of L_2–L_{2p_1} gain (Qian and Lin, 2000). It was proved that the ADD problem in the L_2–L_{2p_1} sense is still solvable by *smooth* static state feedback under the following conditions which can be viewed as a *high-order* version of the feedback linearization conditions (Qian and Lin, 2000):

A*1.1.* For $i = 1, \cdots, n$,

$$|f_i(x_1, \cdots, x_i)| \le (|x_1|^{p_i} + \cdots + |x_i|^{p_i})\rho_i(x_1, \cdots, x_i),$$

where $\rho_i(\cdot) \ge 0$ is a smooth function.

A*1.2.* $p_1 \ge p_2 \ge \cdots \ge p_n \ge 1$ are *odd* integers.

In the case when **A1.1** and **A1.2** do not hold, the ADD problem for system (1.1) remains unsolved. As a matter of fact, it becomes quite challenging and difficult because without the two growth conditions, system (1.1) may contain uncontrollable modes associated with eigenvalues on the open right-half plane. Consequently, it is impossible to achieve ADD with internal stability by using existing methods which are all based on *smooth* feedback design. To overcome the topological obstruction caused by uncontrollable unstable linearization, a *non-smooth* feedback design method that goes beyond conventional smooth feedback designs must be developed. In the recent work (Qian and Lin, 2001a,b), a *continuous* feedback framework has been developed to solve the global stabilization problem of system (1.1) with $w = 0$, without imposing any growth condition such as **A1.1** and **A1.2**. In this paper, we shall show that the *non-smooth but continuous* feedback design approach proposed in (Qian and Lin, 2001a,b) can be further exploited to solve the ADD problem for a class of nonlinear systems (e.g. (1.1)) with uncontrollable unstable linearization.

The main objectives of this paper are: i) to prove that the ADD problem can be solved by *continuous* state feedback for the triangular system (1.1) without imposing any growth condition, although it is not solvable by any *smooth* feedback; ii) to illustrate how the continuous feedback design approach of (Qian and Lin, 2001a,b) can be successfully used to study the ADD problem for a larger class of nonlinear systems beyond (1.1), such as cascade systems and non-strict-triangular systems. iii) to achieve global disturbance attenuation in the L_2–gain sense, which has been proved to be impossible within the smooth feedback framework (Qian and Lin, 2000).

2. A PARADIGM: ADD FOR SYSTEMS (1.1)

The purpose of this section is to show how the ADD problem can be solved for the lower-triangular system (1.1) with uncontrollable unstable linearization, and how a *continuous static* state feedback control law can be explicitly constructed by using the tool called *adding a power integrator* (Qian and Lin, 2001a,b). This design technique was proposed recently in (Qian and Lin, 2001a,b), resulting in a solution to the problem of global stabilization for a number of nonlinear systems including (1.1) in the absence of $w(t)$, without imposing any growth condition.

It has been known that global stabilization of nonlinear systems with uncontrollable unstable linearization can only be achieved by *continuous* (rather than smooth) state feedback (Qian and Lin, 2001a,b). This is also true for the ADD problem because feedback stabilization can be regarded as a special case of the ADD problem. Within the continuous framework, the solution of the resulting closed-loop system is usually not unique due to the use of *non-Lipschitz continuous* state feedback. Therefore, a new notion of stability, i.e. global strong stability (GSS) in the sense of Kurzweil (Kurzweil, 1956; Qian and Lin, 2001b), must be used. The control objective is to find a *continuous static state* feedback control law, such that the resulting closed-loop system is globally strongly stable at the origin when $w = 0$, and the influence of the disturbance $w(t)$ on the output $y(t)$ of the system is arbitrarily small in the presence of $w(t)$. To be precise, the following problem known as almost disturbance decoupling with internal stability will be studied in the paper.

Almost Disturbance Decoupling with Internal Stability (ADD): Consider the nonlinear system (1.1). Given any real number $\gamma > 0$, find, if possible, a C^0 state feedback law

$$u = u_\gamma(x), \qquad \textit{with} \quad u_\gamma(0) = 0, \qquad (2.1)$$

such that the closed-loop system (1.1)-(2.1) satisfies the following:

(a) when $w = 0$, the closed-loop system (1.1)-(2.1) is globally strongly stable (**GSS**) at the equilibrium $x = 0$;
(b) for every disturbance $w(t) \in L_2$, the response of the closed-loop system (1.1)-(2.1) starting from the initial state $x(0) = 0$ is such that $\int_0^t |y(s)|^2 ds \leq \gamma^2 \int_0^t \|w(s)\|^2 ds, \qquad \forall t \geq 0.$

Remark 2.1. In (Qian and Lin, 2000) a counter-example was given, demonstrating that an L_2–L_{2p} (instead of the L_2) gain must be employed to characterize the disturbance decoupling level, when studying the ADD problem for the high-order system (1.1) by *smooth* state feedback. However, the problem formulation above suggests that using *non-smooth but continuous* state feedback, it is possible to study the ADD problem in the L_2–gain sense, as done in the case of feedback linearizable systems (Marino *et al.*, 1994; Isidori, 1995; van der Schaft, 1996; Isidori, 1996b) or linear systems (Willems, 1981).

To solve the ADD problem in the L_2 sense, we need to introduce three useful Lemmas that will be frequently used throughout this paper.

Lemma 2.2. For $x \in \mathbb{R}, y \in \mathbb{R}$, $p \geq 1$ is an integer, the following inequalities hold:

$$|x+y|^p \leq 2^{p-1}|x^p + y^p|, \qquad (2.2)$$

$$(|x| + |y|)^{\frac{1}{p}} \leq |x|^{\frac{1}{p}} + |y|^{\frac{1}{p}} \leq 2^{\frac{p-1}{p}}(|x| + |y|)^{\frac{1}{p}}. (2.3)$$

Lemma 2.3. For any positive real numbers c and d, and any real-valued function $\gamma(x,y) > 0$,

$$|x|^c|y|^d \leq \frac{c}{c+d}\gamma(x,y)|x|^{c+d} + \frac{d}{c+d}\gamma^{-\frac{c}{d}}(x,y)|y|^{c+d}.$$

Lemma 2.4. Let $a \geq 0,\ b \geq 0$, be real numbers and $p \geq 1, q \geq 1$, be integers. Then,

$$a^{p-1}b^q \leq a^p + b^{pq} \qquad (2.4)$$

The proofs of these lemmas can be found in (Qian and Lin, 2001a,b). With the aid of Lemma 2.2–2.4, a constructive solution to the ADD problem can be derived using the *adding a power integrator* technique proposed in (Qian and Lin, 2001a,b).

Theorem 2.5. Without imposing any condition, the **ADD** problem for the nonlinear system (1.1) is always solvable by *continuous* state feedback.

Proof. The proof is based on an inductive argument which simultaneously constructs a Lyapunov function, and a C^1 state feedback control law that solves the ADD problem. For a technical convenience, for any given $\gamma > 0$, we denote $\varepsilon = \frac{\gamma^2}{n}$.
Step 1. For the x_1-subsystem, consider $V_1(x_1) = \frac{x_1^2}{2}$. Since $h(x_1)$ and $f_1(x_1)$ are C^1 functions vanishing at $x_1 = 0$, there exist smooth functions $\varphi_1(x_1)$, $\rho_0(x_1)$ and $\rho_1(x_1)$, such that $|\phi_1(x_1)| \leq \varphi_1(x_1)$, $|h(x_1)| \leq |x_1|\rho_0(x_1)$, $|f_1(x_1)| \leq |x_1|\rho_1(x_1)$. Then, a direct calculation yields

$$\dot{V}_1 + y^2 - \varepsilon\|w\|^2 \leq x_1 x_2^{p_1} + x_1^2[\rho_0(\cdot) + \rho_1(\cdot) + \frac{\varphi_1^2(\cdot)}{4\varepsilon}]$$

Clearly, the C^0 *virtual* controller x_2^* defined by

$$x_2^{*p_1}(x_1) = -x_1\left(n + \rho_1(x_1) + \rho_0(x_1) + \frac{\varphi_1^2(x_1)}{4\varepsilon}\right)$$

leads to

$$\dot{V}_1 + y^2 - \varepsilon\|w\|^2 \leq -nx_1^2 + x_1(x_2^{p_1} - x_2^{*p_1}). (2.5)$$

Inductive Step. Suppose at Step k, there are a C^1 Lyapunov function $V_k : \mathbb{R}^k \to \mathbb{R}$, which is positive definite and proper, and a set of C^0 virtual controllers $x_1^*, \cdots, x_{k+1}^*$, defined by $x_1^* = 0,\ \xi_1 = x_1$ and for $l = 2, \cdots, k+1$,

$$\xi_l = x_l^{p_1\cdots p_{l-1}} - x_l^{*p_1\cdots p_{l-1}}, x_l^{*p_1\cdots p_{l-1}} = -\xi_{l-1}\beta_{l-1}(\cdot)$$

with $\beta_1(x_1) > 0, \cdots,\ \beta_k(x_1, \cdots, x_k) > 0$, being *smooth,* such that

$$\dot{V}_k(x_1, \cdots, x_k) + y^2 - k\varepsilon\|w\|^2 \leq -(n-k+1)\sum_{l=1}^{k}\xi_l^2$$
$$+\xi_k^{2-\frac{1}{p_0 p_1 \cdots p_{k-1}}}\left[x_{k+1}^{p_k} - x_{k+1}^{*p_k}\right] \quad p_0 := 1. (2.6)$$

Obviously, (2.6) reduces to inequality (2.5) when $k = 1$. Since p_0 is identical to one, in what follows we simply omit p_0 in (2.6).

We claim that (2.6) also holds at Step $k+1$. To prove this claim, we consider the Lyapunov function $V_{k+1} : \mathbb{R}^{k+1} \to \mathbb{R}$, defined by

$$V_{k+1}(x_1, \cdots, x_{k+1}) = V_k + W_{k+1}(x_1, \cdots, x_{k+1}) (2.7)$$

$$W_{k+1}(\cdot) = \int_{x_{k+1}^*}^{x_{k+1}} \left(s^{p_1\cdots p_k} - x_{k+1}^{*p_1\cdots p_k}\right)^{2-\frac{1}{p_1\cdots p_k}} ds,$$

which was introduced in (Qian and Lin, 2001b) when studying the global stabilization problem. This Lyapunov function will also be the key in designing a C^0 state feedback control law that solves the ADD problem for the nonlinear system (1.1) with uncontrollable unstable linearization.

For the sake of space, we only quote several key properties of V_{k+1} and W_{k+1} from (Qian and Lin, 2001b). The interested readers is referred to (Qian and Lin, 2001b) for detailed proofs.
Property 1 $V_{k+1}(\cdot)$ is positive definite and proper.
Property 2 $W_{k+1}(x_1, \cdots, x_{k+1})$ is C^1. Moreover $\frac{\partial W_{k+1}}{\partial x_{k+1}} = \xi_{k+1}^{2-\frac{1}{p_1\cdots p_k}}$ and for $1 \leq l \leq k$

$$\left|\frac{\partial W_{k+1}}{\partial x_l}\right| \leq \bar{c}_{k+1}|\xi_{k+1}|\left|\frac{\partial x_{k+1}^{*p_1\cdots p_k}}{\partial x_l}\right|, \ \bar{c}_{k+1} > 0. (2.8)$$

Property 3 There is a C^∞ $\rho_{k+1}(\cdot) \geq 0$ such that

$$|f_{k+1}(\cdot)| \leq (|\xi_1|^{\frac{1}{p_1\cdots p_k}} + \cdots + |\xi_k|^{\frac{1}{p_1\cdots p_k}})\rho_{k+1}(\cdot).$$

Property 4 There is a C^∞ $C_{k+1,l}(x_1,\cdots,x_{k+1}) \geq 0$ such that for $l = 1,\cdots,k$

$$\left|\frac{\partial x_{k+1}^{*p_1\cdots p_k}}{\partial x_l}(x_{l+1}^{p_l} + f_l(\cdot))\right| \leq \textstyle\sum_{j=1}^{k+1} |\xi_j| C_{k+1,l}(\cdot).$$

With the help of Properties 1-4 and Lemmas 2.2-2.3, it can be shown that

$$\dot{V}_{k+1} + y^2 - (k+1)\varepsilon\|w\|^2 \leq -(n-k)\sum_{l=1}^{k}\xi_l^2$$

$$+\xi_{k+1}^{2-\frac{1}{p_1\cdots p_k}} x_{k+2}^{p_{k+1}} + \xi_{k+1}^2\rho_{k+1}(\cdot),\ \ \rho_{k+1}(\cdot) \geq 0 \quad (2.9)$$

Clearly, the C^0 controller x_{k+2}^* defined by

$$x_{k+2}^{*p_1\cdots p_{k+1}} = -\xi_{k+1}\beta_{k+1}(x_1,\cdots,x_{k+1})$$

with $\beta_{k+1}(\cdot) := (n-k+c_{k+1}+\bar{\rho}_{k+1}(\cdot)+\tilde{\rho}_{k+1}(\cdot)+\hat{\rho}_{k+1}(\cdot))^{p_1\cdots p_k} > 0$ being smooth, renders

$$\dot{V}_{k+1}(x_1,\cdots,x_{k+1}) + y^2 - (k+1)\varepsilon\|w\|^2$$
$$\leq -(n-k)\sum_{l=1}^{k+1}\xi_l^2 + \xi_{k+1}^{2-\frac{1}{p_1\cdots p_k}}[x_{k+2}^{p_{k+1}} - x_{k+2}^{*p_{k+1}}],$$

which implies that (2.6) holds at Step $k+1$.

Using the inductive argument above, it is straightforward to show that at Step n, there exist a C^0 controller of the form

$$u = -(\xi_n\beta_n(x_1,\cdots,x_n))^{\frac{1}{p_1\cdots p_n}},\ \ \beta_n(\cdot) > 0 \quad (2.10)$$

and a positive definite and proper Lyapunov function $V_n(x_1,\cdots,x_n)$ of the form (2.7), such that

$$\dot{V}_n + y^2 - \gamma^2\|w\|^2 \leq -(\xi_1^2 + \cdots + \xi_n^2). \quad (2.11)$$

By Kurzweil's stability Theorem (Kurzweil, 1956; Qian and Lin, 2001b), system (1.1) is globally strongly stabilized by the C^0 state feedback law (2.10) when $w = 0$. Moreover, by positive definiteness of $V_n(\cdot)$ it follows from (2.11) that when $x(0) = 0$, $\int_0^t |y(s)|^2 ds \leq \gamma^2 \int_0^t \|w(s)\|^2 ds,\ \forall t \geq 0$. This completes the proof of Theorem 2.5.

Remark 2.6. Obviously, Theorem 2.5 recovers the global strong stabilization result obtained in (Qian and Lin, 2001a,b) when $w = 0$, for a chain of power integrators perturbed by a lower-triangular vector field. It has been shown (Qian and Lin, 2001a,b) that without imposing any condition (e.g. Assumptions 1.1–1.2), asymptotic stabilization of (1.1) can only be achieved by *non-smooth* state feedback, due to the presence of *uncontrollable unstable linearization.* In a series of papers (Bacciotti, 1992; Coron and Praly, 1991; Dayawansa *et al.*, 1990; Kawski, 1989), locally stabilizing C^0 controllers were designed for two or three-dimensional affine systems that are small time locally controllable, using homogeneous systems theory (Hermes, 1991; Kawski, 1989). However, *global* stabilization results were only obtained very recently, by using a novel *continuous* feedback design approach that effectively couples homogeneous systems theory and the adding a power integrator technique (Qian and Lin, 2001a,b). It turns out that this continuous feedback design method also led to a global solution to the ADD problem, as illustrated by Theorem 2.5.

Remark 2.7. Compared to the results in (Qian and Lin, 2000) where the ADD problem of system (1.1) was solved by C^∞ state feedback under **A1.1-A1.2**, Theorem 2.5 has improved the work (Qian and Lin, 2000) significantly in two aspects. First, using *non-smooth but continuous* state feedback we have solved the ADD problem for system (1.1) without imposing any growth condition, i.e. **A1.1-A1.2** were completely removed. Second, almost disturbance decoupling in the sense of L_2-gain (instead of L_2-L_{2p_1} gain) has been achieved by C^0 state feedback for nonlinear system (1.1). Notably, it was shown in (Qian and Lin, 2000) via a counter-example, that it is usually impossible to solve the ADD problem for (1.1) in the L_2-gain sense by any C^∞ state feedback. In this regard, the power of our C^0 feedback design is quite clear.

Remark 2.8. In the case of feedback linearizable systems (i.e., $p_i = 1,\ 1 \leq i \leq n$), Theorem 2.5 reduces to the early result by Marino et al. (Marino *et al.*, 1994), which provides a global solution to the ADD problem for feedback linearizable systems. Note that in that case, our Lyapunov function (2.7) reduces to the quadric Lyapunov function used in (Marino *et al.*, 1994).

Example 2.9. Consider the following system

$$\dot{x}_1 = x_2 + x_1 w,\ \ \dot{x}_2 = x_3^3 + \sin x_1 + (1+x_2)w,$$
$$\dot{x}_3 = x_4,\ \ \dot{x}_4 = u,\ \ \ y = x_1. \quad (2.12)$$

When $w = 0$, (2.12) becomes the example considered in (Rui *et al.*, 1997; Qian and Lin, 2001b), which represents a class of underactuated, weakly coupled, unstable mechanical systems that cannot be stabilized by *any smooth* state feedback. In the presence of w, system (2.12) is of the form (1.1) in which neither **A1.1** nor **A1.2** is fulfilled. Therefore, the ADD problem cannot be solved by any existing method including (Qian and Lin, 2000). However, by Theorem 2.5 the ADD problem of (2.12) is solvable by *continuous* state feedback. A C^0 static state feedback control law can be easily designed by following the constructive procedure in the proof of Theorem 2.5.

3. ADD FOR CASCADE SYSTEMS

In this section, we discuss how the solution of the ADD problem obtained so far can be extended to a class of cascade nonlinear systems of the form

$$\dot{z} = f_0(z,x_1) + \phi_0(z,x_1)w$$
$$\dot{x}_i = x_{i+1}^{p_i} + f_i(z,x_1,\cdots,x_i) + \phi_i(z,x_1,\cdots,x_i)w,$$
$$y = h(z,x_1),\ \ x_{r+1} := u,\ 1 \leq i \leq r \quad (3.1)$$

where $p_i \geq 1$, $i = 0, \cdots, r$ are odd integers, $z \in \mathbb{R}^{n-r}$, $f_i(\cdot)$, $\phi_i(\cdot)$ $i = 0, \cdots, r$, and $h(\cdot)$ are C^1 functions with $f_i(0) = 0$ and $h(0,0) = 0$.

To begin with, we present a Lemma which will be used to solve the ADD problem for system (3.1). For $i = 1, \cdots, r$ denote

$$F_i(Z_i, x_{i+1}) = \begin{bmatrix} f_0(z, x_1) \\ x_2^{p_1} + f_1(z, x_1) \\ \vdots \\ x_{i+1}^{p_i} + f_i(Z_i) \end{bmatrix}, \; \Phi_i(Z_i) = \begin{bmatrix} \phi_0(z, x_1) \\ \phi_1(z, x_1) \\ \vdots \\ \phi_i(Z_i) \end{bmatrix}$$

with $Z_i = [z, x_1, \cdots, x_i]^T$

Lemma 3.1. Suppose for an integer k, $1 \leq k \leq r-1$, there are smooth functions $\beta_i(Z_i)$, $i = 0, \cdots, k$ and a C^1 Lyapunov function $V_k : \mathbb{R}^{n-r+k} \to \mathbb{R}$, which is positive definite and proper, such that

$$\frac{\partial V_k}{\partial Z_k} F_k(Z_k, x_{k+1}^*) + \frac{1}{4\gamma^2}\left(\frac{\partial V_k}{\partial Z_k}\Phi_k(Z_k)\right)^2 + h^2(Z_1)$$
$$\leq -\omega_k(\cdot)[\|z\|^2 + x_1^2 + x_2^{2p_1} + \cdots + x_k^{2p_1\cdots p_{k-1}}] \quad (3.2)$$

for a C^∞ $\omega_k(Z) > 0$, and

$$\xi_k = x_k^{p_1\cdots p_{k-1}} + \beta_{k-1}(\cdot)[x_{k-1}^{p_1\cdots p_{k-2}} + \cdots + \beta_1(\cdot)[x_1 + \beta_0^T(z)z]]$$
$$x_{k+1}^{*p_1\cdots p_k} = -\beta_k(Z_k)\xi_k, \quad \frac{\partial V_k}{\partial x_k} = \xi_k^{2-\frac{1}{p_1\cdots p_{k-1}}},$$
$$\left\|\frac{\partial V_k}{\partial Z_k}\right\| \leq (\|z\| + |x_1| + |x_2|^{p_1} + \cdots + |x_k|^{p_1\cdots p_{k-1}})C_k(Z_k)$$

where C^∞ $C_k(\cdot) \geq 0$, and $p_1 \cdots p_{k-1} = 1$ when $k = 1$.

Then, there is a C^1 Lyapunov function $V_{k+1} : \mathbb{R}^{n-r+k+1} \to \mathbb{R}$, which is proper and positive definite, such that
$\frac{\partial V_{k+1}}{\partial Z_{k+1}} F_{k+1}(Z_{k+1}, x_{k+2}^*) + \frac{1}{4\gamma^2}\left[\frac{\partial V_{k+1}}{\partial Z_{k+1}}\Phi_{k+1}(Z_{k+1})\right]^2 +$
$h^2(\cdot) \leq -\omega_{k+1}(Z_{k+1})(\|z\|^2 + x_1^2 + x_2^{2p_1} + \cdots + x_{k+1}^{2p_1\cdots p_k})$,
where C^∞ $\omega_{k+1}(Z_{k+1}) > 0$, and

$$x_{k+2}^{*p_1\cdots p_{k+1}} = -\beta_{k+1}(\cdot)\xi_{k+1}, \quad \xi_{k+1} = x_{k+1}^{p_1\cdots p_k} - x_{k+1}^{*p_1\cdots p_k},$$
$$\left\|\frac{\partial V_{k+1}}{\partial Z_{k+1}}\right\| \leq (\|z\| + |x_1| + |x_2^{p_1}| + \cdots + |x_{k+1}^{p_1\cdots p_k}|)C_{k+1}(\cdot),$$
$$\frac{\partial V_{k+1}}{\partial x_{k+1}} = \xi_{k+1}^{2-\frac{1}{p_1\cdots p_k}}, \quad C^\infty \;\; \beta_{k+1}(\cdot) > 0, \; C_{k+1}(\cdot) \geq 0$$

Using Lemma 3.1, a result analogous to Theorem 2.5 can be derived, which gives a solution to the ADD problem for the cascade system (3.1).

Theorem 3.2. Suppose there exists a C^2 Lyapunov function $V(z)$, which is positive definite and proper, such that

$$\frac{\partial V}{\partial z} f_0(z, v^*) + \frac{1}{4\gamma^2}\left[\frac{\partial V}{\partial z}\phi_0(z, v^*)\right]^2 + h^2(z, v^*)$$
$$\leq -\|z\|^2\omega(z) \quad (3.3)$$

where $v^*(z)$ is a C^∞ real-valued function with $v^*(0) = 0$, $\omega(z) > 0$ is a C^∞ function. Then, the ADD problem for (3.1) is solvable by C^0 static state feedback (i.e. $u = u_\gamma(z, x)$).

Proof. The key point of the proof is to show that the Hamilton-Jacobi-Isaacs (HJI) partial differential inequality (3.3) can be propagated by repeatedly using Lemma 3.1. At last step, there are C^1 positive definite and proper Lyapunov function $V_r(Z_r)$ and a C^0 $u(Z_r)$ such that

$$\frac{\partial V_r}{\partial Z_r} F_r(Z_r, u) + \frac{1}{4\gamma^2}\left(\frac{\partial V_r}{\partial Z_r}\Phi_r(Z_r)\right)^2 + h^2(Z_1)$$
$$\leq -\omega_r(Z_r)(\|z\|^2 + x_1^2 + \cdots + x_r^{2p_1\cdots p_{r-1}}),$$

where C^∞ $\omega_r(\cdot) > 0$. With this in mind, one can deduce the following dissipation inequality:

$$\dot{V}_r + y^2 - \gamma\|w\|^2 \leq -\omega_r(\cdot)(\|z\|^2 + x_1^2 + \cdots + x_r^{2p_1\cdots p_{r-1}})$$

from which the ADD result follows immediately.

Remark 3.3. When $p_i = 1$, $i = 1, \cdots, r$, system (3.1) reduces to the normal form with a triangular structure (Isidori, 1995), and the HJI partial differential inequality (3.3) becomes the one in (Isidori, 1995, 1996a), which gives a tight sufficient condition for the ADD problem to be solvable via smooth state feedback for minimum-phase nonlinear systems. In other words, in the partial feedback linearizable case, Theorem 3.2 reduces to the well-known ADD theorems proved in (Marino *et al.*, 1994; Isidori, 1995, 1996a). When $w = 0$, Theorem 3.2 recovers the global strong stabilization results in (Qian and Lin, 2001a,b), for a class of cascade systems with uncontrollable unstable linearization.

Example 3.4. Consider the cascade system

$$\dot{z} = x_1 + x_1^2 z + x_1 w, \quad \dot{x}_1 = x_2^3 + x_1 + (z+1)w,$$
$$\dot{x}_2 = u, \quad y = x_1(1 + z^2) \quad (3.4)$$

System (3.4) is not feedback linearizable, and hence cannot be handled by (Marino *et al.*, 1994; Isidori, 1995, 1996a). Neither can it be dealt with by the paper (Qian and Lin, 2000) because the Jacobian linearization is uncontrollable unstable. However, by choosing $V(z) = \frac{z^2}{2}$ and $v^*(z) = -z\beta_0(z)$ with $\beta_0(z) := \frac{1}{2(z^2 + \frac{z^2}{4\gamma^2} + (1+z^2)^2)} > 0$, it is easy to verify that (3.3) holds for $\omega(z) = \frac{\beta_0(z)}{2} > 0$. By Theorem 3.2, the ADD for (3.4) can be achieved by continuous state feedback.

All the ADD results obtained so far can only be applied to nonlinear systems with a lower-triangular form. In th remainder of this section, we briefly discuss how this structural condition can be relaxed. For simplicity, we only consider the situation where a non-triangular system involves no zero-dynamics, i.e.

$$\dot{x}_i = x_{i+1}^{p_i} + \sum_{j=0}^{p_i - 1} x_{i+1}^j f_{i,j}(x_1, \cdots, x_i)$$
$$+ \sum_{j=0}^{(p_i-1)/2} x_{i+1}^j \phi_{i,j}(x_1, \cdots, x_i) w, \quad 1 \leq i \leq n$$
$$y = h(x_1), \quad x_{n+1} := u, \quad (3.5)$$

where $f_{i,j}(\cdot)$, $\phi_{i,j}(\cdot)$ and $h(x_1)$ are C^1 functions with $f_{i,j}(0) = 0$ and $h(0) = 0$.

By combining Theorem 2.5 with the design technique in (Qian and Lin, 2001a,b) for non-triangular systems, one is able to solve the ADD problem for the nontriangular system (3.5).

Theorem 3.5. The ADD problem for system (3.5) is solvable by C^0 state feedback.

For the sake of space, a detailed proof is omitted.

Finally, it should be pointed out that in the case of $p_1 > 1$, output of the system is allowed to depend not only on x_1 but also x_2. Indeed, only the following condition is needed in order to solve the ADD problem for system (3.5).

A*3.6.* $y = h(x_1, x_2) = \sum_{j=0}^{(p_1-1)/2} x_2^j a_j(x_1)$ where $a_j(x_1)$ is a C^1 function with $a_j(0) = 0$.

Under **A3.6**, the following result can be proved.

Theorem 3.7. The ADD problem for the non-triangular system (3.5) with a generalized output satisfying **A3.6** is solvable by C^0 state feedback.

In the literature of which we are aware, all the existing results have never addressed the ADD problem for nonlinear systems whose outputs depend on the state (x_1, x_2). However, by Theorem 3.7, the ADD problem for the system

$$\dot{x}_1 = x_2^3 + x_1 + w, \ \dot{x}_2 = u, \ y = x_1 + x_1 x_2$$

is still solvable by a C^0 state feedback control law.

4. CONCLUSIONS

Within a continuous feedback framework, we have formulated the problem of almost disturbance decoupling with internal stability for a large class of nonlinear systems with uncontrollable unstable linearization. Using the adding a power integrator technique, the ADD problem was first solved by C^0 state feedback, for a chain of power integrators perturbed by a lower-triangular C^1 vector field without imposing any growth condition. This result was then extended to the ADD problem for cascade systems and non-strict-triangular systems with a more general form of the output.

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