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ON THE POSITIVE REALIZATION OF CONTROLLABLE BEHAVIORS

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Abstract: Positive linear systems, traditionally investigated within the state-space framework, have been recently analyzed within the behavioral setting, by focusing the attention on the autonomous case. Also, the positive realization problem has been fully explored in the special case of autonomous behaviors.
In this contribution, we focus our attention on controllable behaviors. We first address the general realization problem by means of driving variable state-space representations and later analyze the possibility of realizing a controllable behavior by means of a positive driving variable representation. Several necessary and sufficient conditions for problem solvability are presented.

Keywords: Controllable behavior, state-space realization, positive realization, system matrix, minimal (left/right) annihilators.

1. INTRODUCTION

In recent years, research interests aiming at developing a general theory of positive linear system within the behavioral framework (Polderman and Willems, 1998) resulted in a few contributions which laid on firm foundations the concepts of positive behavior and of positive realizable autonomous behavior. The first original ideas and definitions appeared in a very nice paper (Nieuwenhuis, 1998) by Nieuwenhuis, where the notion of positive discrete behavior (whose trajectories are defined on the time axis $\mathbb{Z}_+$) and some preliminary results, mostly concerned with behaviors which are one dimensional (namely with trajectories in $(\mathbb{R})^{\mathbb{Z}_+}$) and autonomous, or two-dimensional (with trajectories in $(\mathbb{R}^2)^{\mathbb{Z}_+}$) and controllable, were presented. More recently, these definitions and results stimulated a special interest in autonomous behaviors, thus leading first to a complete characterization of positive autonomous behaviors (Valcher, 2000), and later to a deep analysis of the positive realization problem (undoubtedly the most challenging issue in positive system theory (Anderson *et al.*, 1996; Farina, 1996; Maeda and Kodama, 1981)) again for the special case of autonomous behaviors (Valcher, 2001). In this contribution, we aim to further extend our analysis of the positive realization problem by focusing our interest on controllable behaviors.

The realization problem for controllable behaviors proves to be a much more involved topic with respect to the analogous one for autonomous behaviors. As a first step, indeed, one has to decide which type of realization it is appropriate to refer to. In fact, for autonomous behaviors the choice is unique, meanwhile for controllable behaviors there are several possibilities. Indeed, a whole book (Kuijper, 1994) has been devoted to the analysis of some of the first order representations available for complete and hence, specifically, for controllable behaviors. Our choice has been that of focusing on *driving variable state representations* (Valcher, 2001; Willems, 1986). This choice seems to be the most natural extension of the autonomous case: indeed, the set of behavior trajectories coincides, as in the autonomous case, with the set of output trajectories of the state-space model, and there is no need to provide an arbitrary input/output partition of the system variables which would naturally lead to different results depending on the specific partition. Moreover, this latter choice would contradict, in our opinion, the spirit of the behavioral approach.

As a second problem, while autonomous behaviors are finite-dimensional sets of trajectories, controllable behaviors are not. As a consequence, all nice characterizations obtained for autonomous behaviors, and expressed in terms of proper polyhedral cones lying in $\mathbb{R}^n$, the vector space of initial conditions $\mathbf{x}(0)$, do not find any obvious extension to the case of controllable behaviors.

The search for a characterization of positive realizable controllable behaviors has preliminarily required to investigate (Valcher, 2001) the properties of the driving variable representations of controllable behaviors and, in particular, of those among them which are "minimal" (in this setting, minimality refers both to the state dimension and to the number of inputs (Willems, 1986)). These results have led to the analysis of the positive realization problem, thus resulting in a set of equivalent characterizations. As a main fact, once we refer to certain image descriptions of controllable behaviors, endowed with special features which make them "minimal" (as they correspond to moving average models of least complexity), the existence of a positive realization proves to be equivalent to the possibility of obtaining from such image representations proper rational matrices of full column rank, with positive Markov coefficients.

Before proceeding, we introduce some notation. Given any polynomial $r(z) \in \mathbb{R}[z]$, we denote by λ_R the greatest (if any) nonnegative real zero of r, namely

$$\lambda_R := \max\{\lambda \in \mathbb{R}_+ : r(\lambda) = 0\}.$$

When λ_R exists, we say that it is *dominant*, if for any other zero λ of $r(z)$ (if any), we have $|\lambda| \leq \lambda_R$ and the multiplicity of λ is not greater than the multiplicity of λ_R as zeros of $r(z)$.

In the paper, all (discrete) sequences will be defined on the set $\mathbb{Z}_+$ of nonnegative integers. The left (backward) shift operator on $(\mathbb{R}^q)^{\mathbb{Z}_+}$, the set of sequences defined on $\mathbb{Z}_+$ and taking values in $\mathbb{R}^q$, is defined as

$$\begin{aligned}\sigma : (\mathbb{R}^q)^{\mathbb{Z}_+} &\to (\mathbb{R}^q)^{\mathbb{Z}_+}\\ : (\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \cdots) &\mapsto (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \cdots).\end{aligned}$$

By exploiting the usual bijective correspondence between discrete sequences and formal power series, every sequence $\mathbf{v} = \{\mathbf{v}(t)\}_{t\in\mathbb{Z}_+} \in (\mathbb{R}^q)^{\mathbb{Z}_+}$ will bijectively correspond to the power series $\hat{\mathbf{v}}(z) := \sum_{t\in\mathbb{Z}_+} \mathbf{v}(t)z^{-t}$ in $\mathbb{R}[[z^{-1}]]^q$. As a consequence, the action of the operator σ in $(\mathbb{R}^q)^{\mathbb{Z}_+}$ corresponds to the multiplication by the power z in $\mathbb{R}^q[[z^{-1}]]$. Notice, however, that upon multiplying by z we have to leave off the nonpolynomial part in z^{-1} (i.e. the positive powers of z).

To every polynomial matrix $R(z) = \sum_{i=0}^{L} R_i z^i \in \mathbb{R}[z]^{p\times q}$ we can associate the polynomial matrix operator $R(\sigma) = \sum_{i=0}^{L} R_i\sigma^i$ (from $(\mathbb{R}^q)^{\mathbb{Z}_+}$ to $(\mathbb{R}^p)^{\mathbb{Z}_+}$), mapping every sequence $\{\mathbf{w}(t)\}_{t\in\mathbb{Z}_+}$ into the sequence $\{R(\sigma)\mathbf{w}(t)\}_{t\in\mathbb{Z}_+}$, where $R(\sigma)\mathbf{w}(t) = R_0\mathbf{w}(t) + R_1\mathbf{w}(t+1) + \ldots + R_L\mathbf{w}(t+L)$, for every $t \in \mathbb{Z}_+$. $R(\sigma)$ describes an injective map if and only if R is a right prime matrix, and a surjective map if and only if R is of full row rank.

2. ELEMENTARY FACTS ABOUT CONTROLLABLE BEHAVIORS

Before proceeding, it is convenient to briefly summarize some basic definitions and results about behaviors whose trajectories have support in $\mathbb{Z}_+$. Further details can be found in (Nieuwenhuis, 1998; Valcher, 2000; Willems, 1986).

In this paper, by a *dynamic system* we mean a triple $\Sigma = (\mathbb{Z}_+, \mathbb{R}^q, \mathfrak{B})$, where $\mathbb{Z}_+$ represents the *time set*, $\mathbb{R}^q$ is the *signal alphabet*, namely the set where the system trajectories take values, and $\mathfrak{B} \subseteq (\mathbb{R}^q)^{\mathbb{Z}_+}$ is the *behavior*, namely the set of trajectories which are compatible with the system laws. A behavior $\mathfrak{B} \subseteq (\mathbb{R}^q)^{\mathbb{Z}_+}$ is said to be *linear* if it is a vector subspace (over $\mathbb{R}$) of $(\mathbb{R}^q)^{\mathbb{Z}_+}$, and *left shift-invariant* if $\sigma\mathfrak{B} \subseteq \mathfrak{B}$.

A linear left shift-invariant behavior $\mathfrak{B} \subseteq (\mathbb{R}^q)^{\mathbb{Z}_+}$ is *complete* if for every sequence $\tilde{\mathbf{w}} \in (\mathbb{R}^q)^{\mathbb{Z}_+}$, the condition $\tilde{\mathbf{w}}|_{\mathcal{S}} \in \mathfrak{B}|_{\mathcal{S}}$ for every finite set $\mathcal{S} \subset \mathbb{Z}_+$ implies $\tilde{\mathbf{w}} \in \mathfrak{B}$, where $\tilde{\mathbf{w}}|_{\mathcal{S}}$ denotes the restriction to $\mathcal{S}$ of the trajectory $\tilde{\mathbf{w}}$ and $\mathfrak{B}|_{\mathcal{S}}$ the set of all restrictions to $\mathcal{S}$ of behavior trajectories.

Linear left shift-invariant complete behaviors are kernels of polynomial matrices in the left shift operator σ, which amounts to saying that the trajectories $\mathbf{w} = \{\mathbf{w}(t)\}_{t\in\mathbb{Z}_+}$ of $\mathfrak{B}$ can be identified with the set of solutions in $(\mathbb{R}^q)^{\mathbb{Z}_+}$ of a system of difference equations

$$R_0\mathbf{w}(t) + R_1\mathbf{w}(t+1) + \cdots + R_L\mathbf{w}(t+L) = 0,$$

$t \in \mathbb{Z}_+$, with $R_i \in \mathbb{R}^{p\times q}$, and hence described by the equation

$$R(\sigma)\mathbf{w} = 0, \qquad (1)$$

where $R(z) := \sum_{i=0}^{L} R_i z^i$ belongs to $\mathbb{R}[z]^{p\times q}$. In the sequel, a behavior $\mathfrak{B}$ described as in (1) will be denoted, for short, as $\mathfrak{B} = \ker(R(\sigma))$. Also, we will restrict our attention to linear, left shift-invariant and complete behaviors $\mathfrak{B} \subseteq (\mathbb{R}^q)^{\mathbb{Z}_+}$, and refer to them simply as *behaviors*.

One of the main properties of a behavior is controllability (Polderman and Willems, 1998; Willems, 1991).

Definition 2.1 *A behavior $\mathfrak{B} \subseteq (\mathbb{R}^q)^{\mathbb{Z}_+}$ is said to be* controllable *if there exists some nonnegative integer L such that for every $t \in \mathbb{Z}_+$ and every pair of trajectories $\mathbf{w}_1, \mathbf{w}_2 \in \mathfrak{B}$, there exists $\mathbf{w} \in \mathfrak{B}$ such that $\mathbf{w}|_{[0,t)} = \mathbf{w}_1|_{[0,t)}$ and $\mathbf{w}|_{[t+L,+\infty)} = \mathbf{w}_2|_{[t,+\infty)}$.*

Controllable behaviors are endowed with very strong properties. In particular, for a controllable behavior $\mathfrak{B}$ there exist (Wood and Zerz, 1999) an $m \in \mathbb{N}$, an $L \in \mathbb{Z}_+$, and matrices $M_i \in \mathbb{R}^{q\times m}$, for $i = 0, 1, \dots, L$, such that $\mathfrak{B}$ coincides with the set of all trajectories $\mathbf{w} \in (\mathbb{R}^q)^{\mathbb{Z}_+}$ generated by the difference equation

$$\mathbf{w}(t) = M_0\mathbf{u}(t) + \cdots + M_L\mathbf{u}(t+L), \quad (2)$$

$t \in \mathbb{Z}_+$, where $\mathbf{u} \in (\mathbb{R}^m)^{\mathbb{Z}_+}$ is an arbitrary *driving sequence* (Willems, 1986). This amounts to saying that there is a polynomial matrix $M \in \mathbb{R}[z]^{q\times m}$, $M(z) := \sum_{i=0}^{L} M_i z^i$, such that $\mathbf{w} \in \mathfrak{B}$ if and only if $\mathbf{w} = M(\sigma)\mathbf{u}$, for some $\mathbf{u} \in (\mathbb{R}^m)^{\mathbb{Z}_+}$. The set of trajectories, with support in $\mathbb{Z}_+$, thus obtained is denoted by $\mathrm{im}(M(\sigma))$.

Theorem 2.2 (Polderman and Willems, 1998; Willems, 1991) *Let $\mathfrak{B} \subseteq (\mathbb{R}^q)^{\mathbb{Z}_+}$ be a behavior. The following facts are equivalent:*

i) *$\mathfrak{B}$ is controllable;*
ii) *there exists a left prime matrix $R \in \mathbb{R}[z]^{p\times q}$ such that $\mathfrak{B} = \ker(R(\sigma))$;*
iii) *there exists a right prime matrix $M \in \mathbb{R}[z]^{q\times r}$ such that $\mathfrak{B} = \mathrm{im}(M(\sigma))$.*

It is worthwhile noticing that for a controllable behavior $\mathfrak{B}$ defined on $\mathbb{Z}_+$, the left prime kernel description $\mathfrak{B} = \ker(R(\sigma))$ and the right prime image description $\mathfrak{B} = \mathrm{im}(M(\sigma))$ are always related, as for behaviors defined on $\mathbb{Z}$, by the following property: R is a minimal left annihilator (MLA) (Rocha, 1990) of M and M is a minimal right annihilator (MRA) of R.

3. STATE-SPACE REALIZATIONS FOR CONTROLLABLE BEHAVIORS

Let us now investigate the realization problem for controllable behaviors defined on $\mathbb{Z}_+$. The state-space realization we are interested in, in this paper, is the so-called *driving variable state-space representation* of a behavior (Willems, 1986).

Definition 3.1 *The state-space model*

$$\mathbf{x}(t+1) = F\mathbf{x}(t) + G\mathbf{v}(t) \quad (3)$$

$$\mathbf{w}(t) = H\mathbf{x}(t) + J\mathbf{v}(t), \qquad t \geq 0, \quad (4)$$

with $\mathbf{x}(t)$ the state vector, $\mathbf{v}(t)$ the driving input and $\mathbf{w}(t)$ the output vector, $n = \dim \mathbf{x}, m = \dim \mathbf{v}$ and $q = \dim \mathbf{w}$, is said to be a driving variable state-space representation *(for short, a (DV) representation) of the behavior $\mathfrak{B}$ if the following relationship holds:*

$$\mathfrak{B} \equiv \{\mathbf{w} \in (\mathbb{R}^q)^{\mathbb{Z}_+} : \exists\, \mathbf{x} \in (\mathbb{R}^n)^{\mathbb{Z}_+}, \mathbf{v} \in (\mathbb{R}^m)^{\mathbb{Z}_+},$$
$$\textit{such that } (\mathbf{w}, \mathbf{x}, \mathbf{v}) \textit{ satisfies } (3) \div (4)\}.$$

The state-space model (3)÷(4) will be denoted, for the sake of brevity, by $\Sigma_{DV} = (F, G, H, J)$.

The following theorem provides us with a complete characterization of the (DV) representations of a controllable behavior. As it is well-known (Willems, 1986) that controllable behaviors admit reachable state-space descriptions, we will confine our attention to this class of (DV) realizations. Somehow surprisingly, by introducing this assumption on the state-space models, it turns out that the (DV) realizations of a given controllable behavior can be completely identified by means of the "numerator matrix" appearing in any right matrix fraction description of the state-space model transfer matrix.

Theorem 3.2 (Valcher, 2001) *Let $\mathfrak{B} \subseteq (\mathbb{R}^q)^{\mathbb{Z}_+}$ be a controllable behavior and let $R(z) \in \mathbb{R}[z]^{p\times q}$ be a left prime matrix, providing a kernel description of $\mathfrak{B}$. Let $\Sigma_{DV} = (F, G, H, J)$ be a reachable state-space model of dimension n, with q outputs and m inputs, and assume that $W(z)$ is the transfer matrix of $\Sigma_{DV} = (F, G, H, J)$, i.e.,*

$$W(z) = H(zI_n - F)^{-1}G + J,$$

and $N(z)D^{-1}(z)$ is any of its right matrix fraction descriptions. Then, Σ_{DV} is a (DV) realization of $\mathfrak{B}$ if and only if $R(z)$ is an MLA of $N(z)$.

Let us now address the problem of obtaining, among all possible (DV) realizations of a given controllable behavior, a minimal one. As clarified in (Willems, 1986), the notion of minimal (DV) realization of a behavior involves two different types of minimality, namely the minimality with respect to the number of inputs and the minimality with respect to the dimension of the state-space. In fact, a *minimal (DV) realization of* $\mathfrak{B}$ is just a (DV) realization of $\mathfrak{B}$ which is minimal with respect to both quantities.

Notice, first, that by resorting to the previous theorem we can obtain one specific (DV) representation. In fact, let $M^*(z) \in \mathbb{R}[z]^{q\times(q-p)}$ be a right prime and column reduced polynomial matrix such that $\mathfrak{B} = \mathrm{im}(M^*(\sigma))$ and let $\nu_1, \nu_2, \dots, \nu_{q-p}$ be its column degrees. Set

$$W^*(z) := M^*(z) \operatorname{diag}\{z^{\nu_1}, z^{\nu_2}, \ldots, z^{\nu_{q-p}}\}^{-1} \quad (5)$$

and let Σ^*_{DV} be a reachable and observable state-space realization of the (proper rational) matrix $W^*(z)$. Notice that

- by the previous theorem, Σ^*_{DV} is a (DV) representation of $\mathfrak{B}$.
- Σ^*_{DV} has $m^* :=$ number of columns of $W^* =$ number of columns of $M^* = q - p$ inputs;
- the representation of W^* given in (5) is a right coprime matrix fraction description (since M^* is right prime) and hence the dimension of Σ^*_{DV} coincides with the *McMillan degree* of $W^*(z)$ (Kailath, 1980), namely with

$$n^* := \deg\det \operatorname{diag}\{z^{\nu_1}, z^{\nu_2}, \ldots, z^{\nu_{q-p}}\} = \sum_{i=1}^{q-p} \nu_i.$$

As a result, Σ^*_{DV} is a (DV) representation of $\mathfrak{B}$ with $m^* = q-p$ inputs and $n^* = \sum_i \nu_i$ dimension. We aim to prove that such a (DV) representation is a minimal one. We aim to underline that the minimal values of n and m have already been obtained in (Willems, 1986), by means of completely different tools.

Proposition 3.3 (Valcher, 2001; Willems, 1986) *Let $M^*(z) \in \mathbb{R}[z]^{q\times(q-p)}$ be a right prime and column reduced polynomial matrix with column degrees $\nu_1, \nu_2, \ldots, \nu_{q-p}$, and set $\mathfrak{B} := \operatorname{im}(M^*(\sigma))$. Let $\Sigma_{DV} = (F, G, H, J)$ be a (DV) representation of $\mathfrak{B}$ with m inputs and dimension n. Then $m \geq m^* := q - p$ and $n \geq n^* := \sum_{i=1}^{q-p} \nu_i$.*

The previous result immediately leads to the following corollary.

Corollary 3.4 *Let $M^*(z) \in \mathbb{R}[z]^{q\times(q-p)}$ be a right prime and column reduced polynomial matrix with column degrees $\nu_1, \nu_2, \ldots, \nu_{q-p}$, and set $\mathfrak{B} := \operatorname{im}(M^*(\sigma))$. Let $\Sigma_{DV} = (F, G, H, J)$ be a (DV) representation of $\mathfrak{B}$ and denote by $W(z)$ its transfer matrix. Σ_{DV} is minimal if and only if it satisfies the following conditions:*

i) it is reachable and observable;
ii) W has McMillan degree $\sum_i \nu_i$ and it can be expressed as

$$W(z) = M^*(z) D^{-1}(z)$$

for some nonsingular square polynomial matrix $D(z)$.

To conclude, we aim at focusing on another technical result which is of some interest as it provides further insights into the relationship existing between behavior trajectories and the output trajectories of a (DV) realization of the behavior. Specifically, there is a bijective correspondence between a special class of behavior trajectories and the class of trajectories generated by a (DV) realization starting from zero initial conditions.

Let $M^*(z) \in \mathbb{R}[z]^{q\times(q-p)}$ be a right prime and column reduced polynomial matrix such that $\mathfrak{B} = \operatorname{im}(M^*(\sigma))$. Consider the proper rational matrix $W^*(z)$ defined in (5). We have just seen that if $\Sigma^*_{DV} = (F, G, H, J)$ denotes a reachable and observable (and hence minimal) state-space realization of $W^*(z)$, then Σ^*_{DV} is also a minimal (DV) representation of $\mathfrak{B}$. Also, in the general case, we know (Valcher, 2001) that $\mathbf{w} = M(\sigma)\mathbf{u}$ does not mean $\hat{\mathbf{w}}(z^{-1}) = M(z)\hat{\mathbf{u}}(z^{-1})$, for some $\hat{\mathbf{u}}(z^{-1}) \in \mathbb{R}[[z^{-1}]]^{(q-p)\times 1}$, but only that

$$\hat{\mathbf{w}}(z^{-1}) \equiv M(z)\hat{\mathbf{u}}(z^{-1}) \mod z. \quad (6)$$

On the other hand, once we try to represent any behavior trajectory $\mathbf{w} \in \mathfrak{B}$ in terms of the (DV) representation Σ^*_{DV}, we get a free/forced decomposition:

$$\mathbf{w}(t) = HF^t\mathbf{x}_0 + \sum_{i=0}^{t-1} HF^{t-1-i}G\mathbf{v}(i) + J\mathbf{v}(t), \quad (7)$$

for a suitable initial condition $\mathbf{x}_0$ and a suitable driving input $\mathbf{v}$. Notice that W^* is an F.I.R. (finite impulse response) filter and it is well-known that the system matrix in a minimal realization of an F.I.R. filter is necessarily nilpotent. So, in particular, the "free component" $HF^t\mathbf{x}_0$ is nonzero only for sufficiently small values of t. We aim to show, now, that there is a strict relationship between the free component appearing in (7) and the fact that (6) expresses, in general, only a congruence relation and not an identity.

Proposition 3.5 *Let $\mathfrak{B} = \operatorname{im}(M^*(\sigma))$ be a controllable behavior, and suppose that $M^*(z) \in \mathbb{R}[z]^{q\times(q-p)}$ is a right prime column reduced polynomial matrix, with column degrees $\nu_1 \geq \nu_2 \geq \ldots \geq \nu_{q-p}$. Let $W^*(z)$ be the proper rational matrix defined in (5), and let $\Sigma^*_{DV} = (F, G, H, J)$ be a minimal realization of $W^*(z)$, and hence a minimal (DV) representation of $\mathfrak{B}$.*

If $\mathbf{w}$ is a trajectory in $\mathfrak{B}$ and $\hat{\mathbf{w}}(z^{-1})$ represents the corresponding power series, then $\hat{\mathbf{w}}(z^{-1}) = M^*(z)\hat{\mathbf{u}}(z^{-1})$, for some $\hat{\mathbf{u}}(z^{-1}) \in \mathbb{R}[[z^{-1}]]^{(q-p)\times 1}$, if and only if $\mathbf{w}$ is generated by Σ^*_{DV} in forced evolution (i.e., by assuming $\mathbf{x}(0) = 0$).

4. POSITIVE REALIZABLE CONTROLLABLE BEHAVIORS

We are, now, in a position to focus on the main issue of the paper, namely the positive realization problem for controllable behaviors. To this end, we first state the definition of positive realizable (controllable) behavior.

Definition 4.1 *A controllable behavior $\mathfrak{B} \subseteq (\mathbb{R}^q)^{\mathbb{Z}_+}$ is said to be* positive realizable *if there exists a (DV) realization of $\mathfrak{B}$, $\Sigma_{DV} = (F_+, G_+, H_+,$

J_+), *with* F_+, G_+, H_+ *and* J_+ *nonnegative matrices of suitable sizes.*

Before addressing the realization problem in the specific context of controllable behaviors, we aim at recalling a few facts about the positive realization problem in its more traditional setting, namely that of input/output models or, equivalently, rational transfer matrices (Anderson *et al.*, 1996; Farina, 1996; Maeda and Kodama, 1981).

The first result is a rather elementary one, and hence we omit here the proof.

Proposition 4.2 *Let* $W(z) \in \mathbb{R}(z)^{q\times m}$ *be a proper rational transfer matrix, and let* $w_{i\ell}(z)$ *be its* (i,ℓ)*th entry.* $W(z)$ *is positive realizable, by this meaning that there exists a positive state-space model* (F_+, G_+, H_+, J_+) *having* $W(z)$ *as its transfer matrix if and only if, for each* i *and* ℓ, $w_{i\ell}(z)$ *is positive realizable.*

Theorem 4.3 (Anderson *et al.*, 1996) *Let* $w(z)$ *be a proper rational function with nonnegative Markov coefficients. If* $w(z)$ *has one single pole of maximum modulus, which is positive real, (i.e., there exists* $\rho \in \mathbb{R}_+$ *such that* $\rho > |\lambda|$ *for every other pole* λ *of* $w(z)$*, and* ρ *is simple) then* $w(z)$ *is positive realizable.*

Before introducing a set of characterizations of the positive realizability property, we state and prove the following useful technical result.

Lemma 4.4 *Let* $W(z) \in \mathbb{R}(z)^{q\times m}$ *be a proper rational transfer matrix, with nonnegative Markov coefficients. Then, for every positive real number* ρ*, the strictly proper rational transfer matrix* $W_1(z) := W(z)/(z-\rho)$ *has nonnegative Markov coefficients, too.*

Proof Suppose that $W(z) = \sum_{t\geq 0} W_t\ z^{-t}$, with $W_t \geq 0$ for every index t. Since $\frac{1}{z-\rho} = \sum_{t\geq 1}\rho^{t-1}z^{-t}$, we have that $W_1(z) = \frac{1}{z-\rho}\cdot W(z) = \left(\sum_{t\geq 1}\rho^{t-1}z^{-t}\right)\cdot\left(\sum_{t\geq 0} W_t\ z^{-t}\right) = \sum_{t\geq 1}\left(\sum_{i=0}^{t-1} W_i\ \rho^{t-1-i}\right)z^{-t}$, and hence $W_1(z)$ has nonnegative Markov coefficients

$$W_{1,t} := \sum_{i=0}^{t-1} W_i\ \rho^{t-1-i}.$$

■

We are, now, in a position to provide a complete characterization of positive realizability for a controllable behavior.

Theorem 4.5 *Let* $\mathfrak{B} \subseteq (\mathbb{R}^q)^{\mathbb{Z}_+}$ *be a controllable behavior, and let* $M \in \mathbb{R}[z^{-1}]^{q\times(q-p)}$ *be a right prime polynomial matrix which provides an image description of* $\mathfrak{B}$*. The following facts are equivalent:*

i) $\mathfrak{B}$ *is positive realizable;*

ii) there exists a full row rank rational transfer matrix $Q(z)$ *such that* $W(z) := M(z)Q(z)$ *is a proper rational and positive realizable transfer matrix;*

iii) there exists a nonsingular square rational transfer matrix $\bar{Q}(z)$ *such that* $\bar{W}(z) := M(z)\bar{Q}(z)$ *is a proper rational and positive realizable transfer matrix;*

iv) there exists a nonsingular square rational transfer matrix $\bar{Q}(z)$ *such that* $\bar{W}(z) := M(z)\bar{Q}(z)$ *is a rational transfer matrix with nonnegative Markov coefficients.*

Proof i) $\Leftrightarrow$ ii) If $\mathfrak{B}$ is positive realizable and (F_+, G_+, H_+, J_+) is a positive (DV) realization of $\mathfrak{B}$, then, by Theorem 3.2 and following Remark, the transfer matrix of the state-space model (F_+, G_+, H_+, J_+), say $W(z)$, can be expressed as $W(z) = M(z)Q(z)$ for some full row-rank rational matrix. Of course, $W(z)$, being the transfer matrix of a positive state-space model is both proper rational and positive realizable.

Conversely, if there exists a full row rank rational matrix $Q(z)$ such that $W(z) = M(z)Q(z)$ is proper rational and positive realizable, then any positive realization of $W(z)$, say (F_+, G_+, H_+, J_+), provides, due to Theorem 3.2, a positive (DV) realization of $\mathfrak{B}$.

ii) $\Rightarrow$ iii) Since Q is of full row rank, let S be a selection matrix such that $\bar{Q}(z) := Q(z)S$ is nonsingular square. Of course, $\bar{W}(z) := M(z)\bar{Q}(z) = W(z)S$ is proper rational. Moreover, if (F_+, G_+, H_+, J_+) is a positive realization of $W(z)$, then (F_+, G_+S, H_+, J_+S) is a positive realization of $\bar{W}(z)$, which is, therefore, positive realizable, too.

iii) $\Rightarrow$ iv) Obvious: positive realizable transfer matrices have nonnegative Markov coefficients.

iv) $\Rightarrow$ ii) Let $\bar{Q}(z)$ be a nonsingular square rational transfer matrix such that $\bar{W}(z) := M(z)\bar{Q}(z)$ is a rational transfer matrix with nonnegative Markov coefficients. Of course, for every sufficiently large K, we have that $\bar{W}(z)\frac{1}{z^K}$ is also proper, so it entails no loss of generality assuming that $\bar{W}$ is already proper. Let ρ be a positive real number such that $\rho > \max\{|\lambda| : \lambda$ a pole of $\bar{W}(z)\}$. Set $Q(z) := \bar{Q}(z)(z-\rho)^{-1}$, so that

$$W(z) := M(z)Q(z) = \frac{1}{z-\rho}\cdot\bar{W}(z).$$

Of course, W is proper rational, as $\bar{W}$ was. By Lemma 4.4, it has nonnegative Markov coefficients, since $\bar{W}$ has and ρ is positive real. Finally, since W has nonnegative Markov coefficients and a single pole of maximum modulus, which is positive real, each of its entries $w_{i\ell}(z)$ is endowed with these two properties. So, by Theorem 4.3, $w_{i\ell}(z)$

is positive realizable for each (i,ℓ). Therefore, by Proposition 4.2, W is positive realizable. ■

As a main result of the previous theorem, we have reduced the problem of deciding whether or not a controllable behavior $\mathfrak{B} = \mathrm{im}(M(\sigma))$, with M right prime, is positive realizable, to the problem of determining whether there exists a nonsingular square rational matrix Q such that MQ is proper with nonnegative Markov coefficients. This problem solution, however, is far from being trivial. It is well-known to people familiar with the positive realization problem for proper rational transfer matrices that this aspect of the problem is almost unexplored. Indeed, even in fundamental articles about the positive realization problem as (Anderson *et al.*, 1996; Farina, 1996) the results are stated in these terms:

"if $w(z)$ is a strictly proper rational function with nonnegative Markov coefficients, then $w(z)$ is positive realizable if and only if…".

In other words, only upon assuming that $w(z)$ has nonnegative Markov coefficients, the authors provide additional conditions which are equivalent to positive realizability of $w(z)$.

By applying Proposition 3.5, we can alternatively characterize the positive realizability of a controllable behavior.

Proposition 4.6 *Let $\mathfrak{B} = \mathrm{im}(M^*(\sigma))$ be a controllable behavior, and suppose that $M^*(z) \in \mathbb{R}[z]^{q\times(q-p)}$ is a right prime column reduced polynomial matrix, with column degrees $\nu_1 \geq \nu_2 \geq \ldots \geq \nu_{q-p}$. Let $\Sigma^*_{DV} = (F,G,H,J)$ be the usual minimal (DV) representation of $\mathfrak{B}$, with transfer matrix $W^*(z)$ (see (5)). The following facts are equivalent:*

i) *$\mathfrak{B}$ is positive realizable;*

ii) *there exists some nonsingular square proper rational matrix $V(z)$ such that $W^*(z)V(z)$ is proper rational with nonnegative Markov coefficients;*

iii) *there exist $q-p$ nonnegative trajectories $\mathbf{w}^i$, $i = 1,2,\ldots,q-p$, generated by Σ^*_{DV} in forced evolution, such that the corresponding power series $\hat{\mathbf{w}}^i(z^{-1})$ are rational and $\mathrm{rank}[\,\hat{\mathbf{w}}^1(z^{-1})\ !\ldots\ \hat{\mathbf{w}}^{q-p}(z^{-1})\,] = q-p$.*

Proof i) $\Rightarrow$ ii) By Theorem 4.5 (stated in terms of M^*), if $\mathfrak{B}$ is positive realizable then there exists a nonsingular square rational matrix $\bar{Q}(z)$ such that $\bar{W}(z) := M^*(z)\bar{Q}(z)$ is a rational transfer matrix with nonnegative Markov coefficients. Obviously, as $\bar{W}(z)$ has nonnegative Markov coefficients, then also

$$\bar{W}_1(z) := \frac{1}{z^K}\ \bar{W}(z)$$

has nonnegative Markov coefficients, for every $K \geq 0$. Moreover, $\bar{W}_1$ can be expressed as $\bar{W}_1(z) = M^*(z)\left(\bar{Q}(z)\frac{1}{z^{\nu_1}}\right) = W^*(z)\cdot$ (diag $\{z^{\nu_1}, z^{\nu_2},\ldots,z^{\nu_{q-p}}\}\ \bar{Q}(z)\ \mathrm{diag}\{z^{-K},z^{-K},\ldots,z^{-K}\}) =: W^*(z)\ V(z)$, where V is proper rational provided that K is sufficiently large. But then, as both W^* and V are proper rational matrices, also $\bar{W}_1$ is, and hence condition ii) holds true.

ii) $\Rightarrow$ i) As

$$W^*(z) = M^*(z)\ \mathrm{diag}\{z^{-\nu_1}, z^{-\nu_2},\ldots,z^{-\nu_{q-p}}\},$$

the result is an immediate consequence of Theorem 4.5.

ii) $\Leftrightarrow$ iii) Upon setting $\hat{w}^i(z^{-1}) := W^*(z)\bar{Q}(z)\mathbf{e}_i$, the result is obvious from Proposition 3.5. ■

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