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DISCRETIZATION OF CONTINUOUS TIME-DELAY SYSTEMS

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Abstract: Many works are related to the analysis and control of either continuous or else discrete time-delay systems. However, the discretization of continuous time-delay systems has not been extensively studied. In this work, sampled-data time-delay systems with internal and external point delays are described by approximate discrete time-delay systems in the discrete domain. Those approximate discrete systems allow the hybrid control of time-delay systems. Several examples complete the paper, showing the correctness of the discretization process.

Keywords: Time-delay systems, discretization, hybrid systems.

1. INTRODUCTION

In the last years, time-delay systems have been studied by many authors. Different works have been devoted to the analysis of either continuous or else discrete time-delay systems, with point or distributed delays. Special effort has been made in the study of the stability of this kind of systems. As a result, different stability analysis methods have been developed, based on Lyapunov-like algebraic approaches, LMI techniques or frequency domain methods, (Jugo, 2001; Niculescu, 2001).

Focused on control design, an important effort has been dedicated to the consecution of closed-loop finite-spectrum assignment using output feedback or state-space feedback, (Jugo, 2000; Wang *et al.*, 1999). Recent works are oriented to the robust control design, (Niculescu, 2001; Mahmoud, 2000).

Other works are devoted to the study of discrete delay systems, (Mahmoud, 2000). However, the discretization and design of hybrid controllers for continuous time-delay systems has not been extensively studied. In De la Sen and Luo (1994), discretization of continuous systems is considered using FIR filters. On the other hand, several works are focused to the design of hybrid controllers for systems with input time delay, (Shied *et al.*, 1999; Tsai *et al.*, 1999; Yen and Wu, 1994).

In this work, a discretization procedure is developed through a recursive solution of the continuous time-delay system. This process is of interest since the approximate discrete model allows the design of hybrid controllers (continuous plant+discrete controller). The proposed method gives the possibility of increasing the approximation accuracy depending on the application and, thus, the design methodology of robust controllers is of special interest and adequate in this case, (Niculescu, 2001; Mahmoud, 2000).

This paper is organized as follows: first, a solution for the state vector of continuous time-delay systems is given by a recursive equation. This solution can be expressed by a symbolic exponential equation which leads to a symbolic state transition matrix. Then, using this transition matrix, the discretization is performed yielding an approximate discrete time-delay system. Several considerations about numerical computation of the approximate system are discussed, and two different examples show the application of the methodology. Finally, same conclusions end the paper.

2. SYMBOLIC STATE TRANSITION MATRIX

Consider the next continuous time-delay system:

$$\begin{aligned}\dot{x}(t) &= A_0x(t) + A_1x(t-h) + A_2x(t-2h) \\ &\qquad + \ldots + A_nx(t-nh) \\ &\quad +B_0u(t) + B_1u(t-h') \\ &\qquad + \ldots + B_mu(t-mh') \\ y(t) &= C^Tx(t) \end{aligned} \tag{1}$$

The solution of the system (1) can be obtained by direct integration by parts. For clarity of exposition and without loss of generality, the next simpler system is considered:

$$\begin{aligned}\dot{x}(t) &= A_0x(t) + A_1x(t-h) + B_0u(t) + B_1u(t-h') \\ y(t) &= C_0^Tx(t)\end{aligned} \tag{2}$$

The transition of the state vector from t_0 to t_1 can be obtained from the following expression

$$\int_{t_0}^{t_1} \dot{x}(t)dt = x(t_1) - x(t_0) \tag{3}$$

$$= \int_{t_0}^{t_1} (A_0x(t) + A_1x(t-h) + B_0u(t) + B_1u(t-h'))dt$$

The right term of (3) can be integrated by parts. Considering the first addend of the integral, the integration process leads to the following equations

$$\begin{aligned}\int_{t_0}^{t_1} A_0x(t)dt &= A_0x(t)t|_{t_0}^{t_1} - \int_{t_0}^{t_1} A_0t\dot{x}(t)dt \\ \int_{t_0}^{t_1} A_0t\dot{x}(t)dt &= \int_{t_0}^{t_1} A_0t(A_0x(t) + A_1x(t-h) \\ &\qquad +B_0u(t) + B_1u(t-h))dt \\ \int_{t_0}^{t_1} A_0^2tx(t)dt &= A_0^2t^2x(t)|_{t_0}^{t_1} - \int_{t_0}^{t_1} A_0^2\frac{t^2}{2}\dot{x}(t)dt \\ \int_{t_0}^{t_1} A_0A_1tx(t-h)dt &= A_0A_1t^2x(t-h)|_{t_0}^{t_1} \\ &\qquad - \int_{t_0}^{t_1} A_0A_1\frac{t^2}{2}\dot{x}(t-h)dt \\ &\quad \vdots \end{aligned} \tag{4}$$

Proceeding in this manner, an infinite series appears which can be expressed as follows:

$$\begin{aligned} & x(t)|_{t_0}^{t_1} - A_0x(t)t|_{t_0}^{t_1} + A_0^2t^2x(t)|_{t_0}^{t_1} \\ & \qquad +A_0A_1t^2x(t-h)|_{t_0}^{t_1} + \ldots \\ &= \int_{t_0}^{t_1} B_0u(t) + B_1u(t-h'))dt \\ &- \int_{t_0}^{t_1} A_0t(B_0u(t) + B_1u(t-h'))dt \\ &+ \int_{t_0}^{t_1} A_0A_1t(B_0u(t-h) \\ &\qquad +B_1u(t-(h+h')))dt + \ldots \end{aligned} \tag{5}$$

This expression seems to be a matrix exponential series which includes the delay operator. Using the operator $x(t-h) = \mu(h)x(t)$ (note that $x(t-(h+h')) = \mu(h+h')x(t) = \mu(h)\mu(h')x(t)$), the solution of the system (2) can be denoted:

$$\begin{aligned} x(t_1) &= e^{A_0(t_1-t_0)+A_1\mu(h)(t_1-t_0)}x(t_0) \\ &+ \int_{t_0}^{t_1} e^{A_0(t_1-t)+A_1\mu(h)(t_1-t)} \left(B_0u(t) + B_1\mu(h')u(t)\right) dt \end{aligned} \tag{6}$$

Here, the function $e^{A_0(t_1-t_0)+A_1\mu(h)(t_1-t_0)}$ can be considered as a symbolic transition matrix which is only valid for representation purposes and not for direct calculations. Observe that this representation is only used to simplify the notation. However, the use of the exponential notation makes easier the manipulation and calculation of the expression (5), by considerations similar to those of the Paynter technique. Note that this expression can be obtained using the fundamental solution of a Time delay system, obtaining the previous equations starting in this point.

In conclusion, the infinite series described by the exponential can be approximated by a finite series

$$\begin{aligned} x(t_1) &= M(t_1-t_0)x(t_0) \\ &\quad + \int_{t_0}^{t_1} M(t_1-t)\left(B_0u(t) + B_1\mu(h')u(t)\right) dt \end{aligned} \tag{7}$$

being $M(t) = \sum_{i=0}^{n_1} \frac{(A_0(t)+A_1(t)\mu(h))^i}{i!}$. In fact, the equation (8) is a recursive representation of the state vector.

The previous result can be generalized considering systems given by equations (1). In this case, the symbolic transition matrix would be:

$$e^{A_0(t_1-t_0)+A_1\mu(h)(t_1-t_0)+\ldots+A_n\mu(nh)(t_1-t_0)}$$

and the approximation is performed by a finite series

$$M(t) = \sum_{i=0}^{n_1} \left(\frac{(A_0 t + A_1 t\mu(h) + A_2 t\mu(2h)}{i!} \right. \quad (8)$$
$$\left. \frac{+ \ldots + A_n t\mu(nh))^i}{i!} \right)$$

In the next section, this expression is used to obtain the discrete equivalent for sampled-data time-delay systems.

3. DISCRETIZATION OF SYSTEMS WITH INTERNAL AND EXTERNAL POINT DELAYS

First, for the sake of simplicity, systems with an unique external delay and an unique internal delay are considered. Then, assuming a zero order hold and a sampling period T, the instants $t_1 = (k+1)T$ and $t_0 = kT$ can be considered in (8). So, denoting $x(t_1)$, $x(t_0)$ by x_{k+1} and x_k, respectively, the above expression results in:

$$\begin{aligned} x_{k+1} = x_k &+ \sum_{i=1}^{n_1} \frac{(A_0 T + A_1 T\mu(h))^i}{i!} x_k \\ &+ \int_0^T \left(\sum_{i=0}^{n_1} \frac{(A_0 t + A_1 t\mu(h))^i}{i!} \right) B_0 dt\, u_k \\ &+ \int_0^T \left(\sum_{i=0}^{n_1} \frac{(A_0 t + A_1 t\mu(h))^i}{i!} \right) B_1 \mu(h') dt\, u_k \end{aligned} \quad (9)$$

where $\mu(h) x_k = x(Kt - h)$. Choosing $T = h$ and, additionally, taking into account that $iT \leq h' \leq (i+1)T$ and $u(t-h') = u_{k-i}$ for some integer i, (De la Sen and Luo, 1994), the previous equation can be expressed as

$$\begin{aligned} x_{k+1} = F_0 x_k &+ F_1 x_{k-1} + \ldots + F_{n_1} x_{k-n_1} \\ &+ G_0 u_k + G_1 u_{k-i} + \ldots + G_{i+n_1} u_{k-(i+n_1)} \end{aligned} \quad (10)$$

However, depending on the value of the delay h, the election of this value as sampling period can be inadequate in the sense of the Shannon (Discretization) Theorem. Then, the sampling period can be chosen $T = \frac{h}{j}$ for some integer j, where T is a valid value.

In general, considering systems given by equations (1) and $T = \frac{h}{j}$, the approximate discrete system is given by the next state-space description:

$$\begin{aligned} x_{k+1} = x_k &+ M(T) x_k \\ &+ \left(\int_0^T M(t) B_0 dt \right) u_k \\ &+ \left(\int_0^T M(t) B_1 \mu(h') dt \right) u_k \\ &+ \left(\int_0^T M(t) B_2 \mu(2h') dt \right) u_k \\ &\vdots \\ &+ \left(\int_0^T M(t) B_m \mu(mh') dt \right) u_k \end{aligned} \quad (11)$$

Finally,

$$\begin{aligned} x_{k+1} = F_0 x_k &+ F_1 x_{k-j} + \ldots + F_{n*n_1} x_{k-j*n*n_1} \\ &+ G_0 u_k + G_1 u_{k-j} \\ &+ \ldots + G_{i_m + m*n_1} u_{k-(i_m + j*m*n_1)} \end{aligned} \quad (12)$$

Note that i_m can be not equal to $i*m$. For example if $h' = 0.4$, $T = 1$ and $m = 3$ then $i = 1$, but $i_m = 2 \neq 3$. Under this procedure, the approximate discrete model is near to an exact representation of the original system in the sampling time instants, depending on the number of terms n_1 chosen on the approximation.

In addition, after the discretization process, the resulting discrete model has a greater number of delays than the original continuous system. In any case, considering constant and known delays, the discrete-time delay system can be expressed by a standard discrete linear system of higher order.

On the other hand, selecting a lower sampling period, the number of terms that must be considered is lower, thus obtaining a simpler discrete time-delay system. Nevertheless, the order of the approximation is not lower, since a lower period leads to a higher discrete delay (more z^{-1} operators are needed).

4. NUMERICAL COMPUTATION

The numerical computation of the described approximate discrete system can be performed easily by using Matlab like software. The number of terms considered in the series can be chosen depending on the application but a good criterion can be derived from the Paynter technique.

Considering as a main objective the conservation of the gain at low frequencies under the discretization process, the maximum element p to be considered on the series can be chosen by imposing the following condition:

$$\frac{1}{p!}(nq)^p e^{nq} \leq 0.001$$

where n is the system order and $q = max|F_{ij}T|$, being F_{ij} the maximum absolute value of the elements of the matrix $F = \sum_{i=0}^{n} A_i$, (i.e., the system matrix with the delay $h = 0$) and T the sampling period.

Using this technique, some matrices (related to some delayed discrete states) could be negligible and, therefore, a post-analysis might be required. Anyway, the discrete system can be automatically computed in a relatively easy way.

5. EXAMPLES OF APPLICATION

In this section two different examples of application of the proposed approach are presented. Those examples show the validity of the methodology.

As a first example, the next system is considered:

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} x(t) + \begin{pmatrix} 0 & 0 \\ -1 & 2 \end{pmatrix} x(t-0.2) \\ &\quad + \begin{pmatrix} 0 & 0 \\ 2 & -1 \end{pmatrix} x(t-0.4) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t) \\ y(t) &= \begin{pmatrix} 1 & 1 \end{pmatrix} x(t) \end{aligned} \tag{13}$$

Choosing the period $T = 0.2$, the expression (10) can be used for obtaining the approximate discrete system:

$$\begin{aligned} x_{k+1} = &\begin{pmatrix} 0.9671415 & 0.1484107 \\ -0.2968214 & 0.5219093 \end{pmatrix} x_k \\ &+ \begin{pmatrix} -0.0202793 & 0.25727 \\ -0.1998647 & 0.1993912 \end{pmatrix} x_{k-5} \\ &+ \begin{pmatrix} 0.0323453 & -0.0081820 \\ 0.2874585 & -0.0400655 \end{pmatrix} x_{k-10} \\ &+ \begin{pmatrix} 0.0048 & 0.25727 \\ -0.1998647 & 0.1993912 \end{pmatrix} x_{k-15} \\ &+ \begin{pmatrix} -0.0015911 & 0.0004104 \\ -0.0193944 & 0.0031502 \end{pmatrix} x_{k-20} \\ &+ \begin{pmatrix} 0.0164293 \\ 0.1484107 \end{pmatrix} u_k + \begin{pmatrix} 0.0019249 \\ 0.0257272 \end{pmatrix} u_{k-5} \\ &+ \begin{pmatrix} -0.0007056 \\ -0.0081829 \end{pmatrix} u_{k-10} \\ &+ \begin{pmatrix} -0.0001529 \\ -0.0026176 \end{pmatrix} u_{k-15} \end{aligned} \tag{14}$$

The open-loop step responses of the continuous system and approximate discrete system are depicted in figure 1. On the other hand, closed-loop simulation has been performed following the scheme presented in the figure 2. Results are presented in figure 3.

In conclusion, the discrete equivalent (14) is a valid representation of the system (13)in the discrete domain.

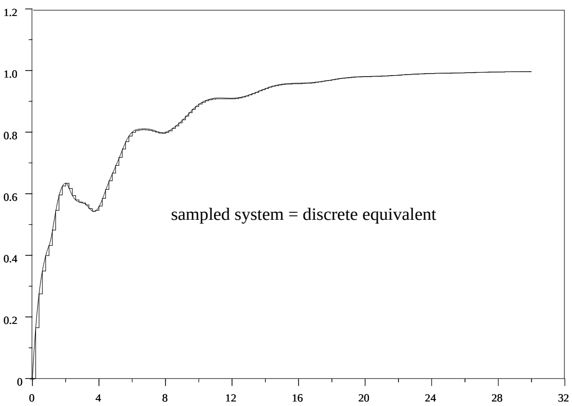

Fig. 1. Example 1: Open-loop step responses for the continuous system and the approximate discrete system.

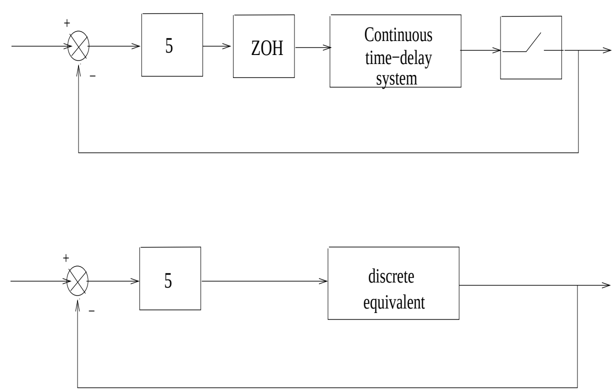

Fig. 2. Closed-loop scheme for the continuous system and the approximate discrete system.

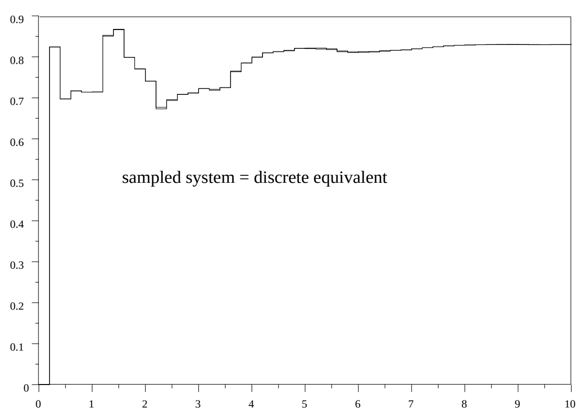

Fig. 3. Example 1: Closed-loop step response of continuous system and approximate discrete system.

In the next example, one external and one internal point delays are considered:

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix} 1 & -1.5 \\ -1 & -2.5 \end{pmatrix} x(t) + \begin{pmatrix} 0.25 & -0.5 \\ 0.5 & 0.5 \end{pmatrix} x(t-0.3) \\ &\quad + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t) + \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} u(t-0.5) \\ y(t) &= \begin{pmatrix} 1 & 1 \end{pmatrix} x(t) \end{aligned} \tag{15}$$

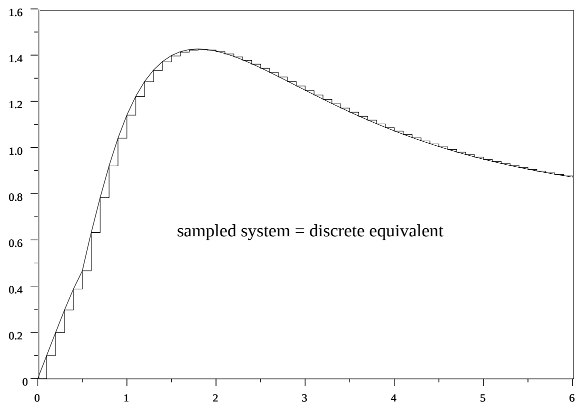

Fig. 4. Example 2: Open-loop step responses for the continuous system and the approximate discrete system.

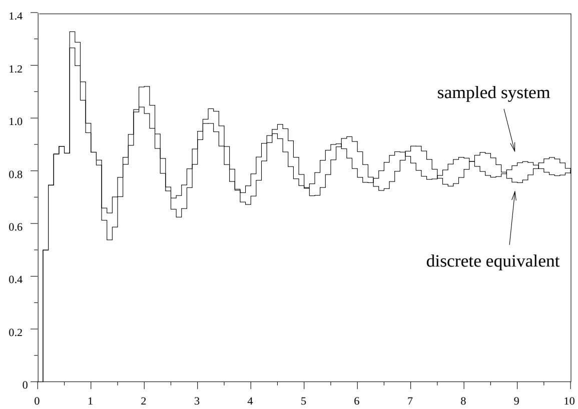

Fig. 5. Example 2: Closed-loop step response of continuous system and approximate discrete system.

In this case, the system has different internal and external delays (noncommensurate delays) and, then, combinations of those delays appear in the input of the discrete approximation. Considering $T = 0.1$ and applying the expression (10), the following discrete time-delay system is obtained:

$$\begin{aligned} x_{k+1} = &\begin{pmatrix} 1.09373 & 0.14613 \\ -0.14613 & 0.8502 \end{pmatrix} x_k \\ &+ \begin{pmatrix} -0.027252 & -0.05411 \\ -0.04331 & -0.0426 \end{pmatrix} x_{k-3} \\ &+ \begin{pmatrix} 0.00161 & 0.001966 \\ 0.001637 & 0.002234 \end{pmatrix} x_{k-6} \\ &+ \begin{pmatrix} 0.007372 \\ 0.092505 \end{pmatrix} u_k + \begin{pmatrix} -0.00264 \\ -0.022525 \end{pmatrix} u_{k-3} \\ &+ \begin{pmatrix} 0.056082 \\ 0.04257 \end{pmatrix} u_{k-5} + \begin{pmatrix} -0.0019832 \\ -0.002264 \end{pmatrix} u_{k-8} \end{aligned} \tag{16}$$

The open-loop step responses for the continuous system and the approximate discrete system are shown in figure 4. Using again the scheme presented in figure 2, the closed-loop step responses have been depicted in figure 5.

In this case, the model is insufficient for an accurate representation of the closed-loop response and a more complex approximate discrete system should be considered in order to improve the result. Precision requirements will determine the number of terms to be considered in the exponential series, depending on the application.

Anyway, simple discrete systems can be an useful tool, combined with robust control techniques.

6. CONCLUSIONS

This work presents a direct way of calculation of an approximate discrete system for continuous time-delay systems involving internal and external point delays. The method is based on a recursive solution of the state vector which can be symbolically represented by an exponential expression. The resulting infinite series can be approximated by a Paynter like technique. The discrete time-delay model is useful for the analysis, simulation and design of hybrid controllers valid for continuous time-delay systems. Several examples show the validity of the proposed discretization process, by comparing the responses of both systems.

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