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MIXED SENSITIVITY DESIGN

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Abstract: Mixed sensitivity design of a linear multivariable control system amounts to shaping its sensitivity functions to achieve the design targets of closed-loop system performance and robustness. Both H_∞ and H_2 optimization may be used to this end. Various tools are available, in particular low and high frequency shaping, and partial pole assignment. The paper describes a technique to help placing the dominant closed-loop poles called disturbance modeling. *Copyright © 2002 IFAC*

Keywords: Control system design, mixed sensitivity, sensitivity, complementary sensitivity, *H*-infinity optimization, *H*-2 optimization, frequency shaping

1. INTRODUCTION

The mixed sensitivity approach to the design of linear multivariable control systems was first introduced in the context of H_∞ optimization (Verma and Jonckheere, 1984; Kwakernaak, 1983). It allows shaping the sensitivity function and the complementary sensitivity function of the closed-loop system to achieve design targets compatible with good performance and robustness. The present paper reviews the various features of the H_∞ design approach, in particular, low and high frequency shaping, and partial pole placement. It is also explained how disturbance modeling may be used to place the dominant closed-loop poles in a systematic way.

The best-known instance of the H_2 version of the mixed sensitivity problem is the LQG problem. This has of course been very thoroughly investigated and is documented in many textbooks. LTR, or Loop Transfer Recovery, is a specialized application of LQG that aims at recovering the favorable features of full state feedback. We explore in the present paper how the low and high frequency shaping techniques developed for the H_∞ mixed sensitivity approach may also be used in the H_2 case. Also disturbance modeling for dominant pole placement may be applied in an LTR-like context.

The H_∞ and H_2 mixed sensitivity problems that we discuss are all special cases of the standard H_∞ and H_2 problems. The actual algorithm that is used to solve the problems is more or less incidental. The famous two Riccati equation algorithm for the standard H_∞ problem (Glover *et al.,* 1989) is widely implemented but suffers from some limitations: certain stabilizability and detectability conditions need to be satisfied, the algorithm cannot handle mixed sensitivity problems with non-proper weighting functions, and only sub-optimal as opposed to optimal solutions are obtained. Algorithms based on polynomial matrix fraction representations (Kwakernaak, 1996) or descriptor representations of the generalized plant (Kwakernaak, 1998) can deal with more general problems but no fully reliable implementations are available at this time. For this reason, the paper describes how block diagram substitution may be used to circumvent the limitations of the two Riccati equation solution.

In an earlier paper (Kwakernaak, 2000) polynomial matrix and descriptor algorithms for the solution of a quite general version of the standard H_2 problem are developed. By block diagram substitution many H_2 optimization problems may be reduced to standard LQG problems, however.

2. FREQUENCY DOMAIN DESIGN TARGETS

We discuss the design of multivariable feedback systems as in the block diagram of Fig. 1. The plant is represented by its transfer matrix *P,* and the compensator by its transfer matrix *K.* The signal *v* represents the disturbances, and *w* the measurement noise. Throughout, *P* is assumed to be square, rational and invertible.

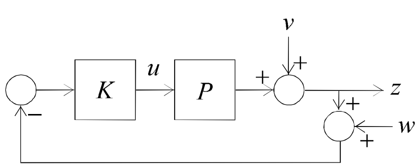

Fig. 1 Two-degree-of-freedom feedback system

Performance and robustness are characterized by various well-known closed-loop functions, in particular the sensitivity function *S,* the complementary sensitivity function T, and the input sensitivity function U, successively defined by

$$S = (I + PK)^{-1}, \quad T = PK(I + PK)^{-1}, \\ U = K(I + PK)^{-1} \tag{1}$$

S is the transfer matrix from the disturbance input v to the control system output z, T is its complement $T = I - S$, and $-U$ is the transfer matrix from the disturbance v to the plant input u.

Control system design amounts to shaping S, T and U (Kwakernaak, 1995). These are the requirements for performance:

For good disturbance attenuation the sensitivity S should be small over a suitable low frequency band. This is achieved by making the loop gain $L = PK$ large in this frequency band. We say that L is large at a frequency ω if $\|L(j\omega)\| \gg 1$ for a suitable norm. Typically the 2-norm is used, that is, the largest singular value.

Zero frequency disturbance rejection and, correspondingly, very good low frequency disturbance attenuation is obtained by integral action, that is, setting $S(0) = 0$ by letting $L^{-1}(0) = 0$.

To prevent overly large inputs and to reduce the effect of measurement noise the complementary sensitivity T should drop off to 0 as quickly as possible outside the active frequency band. This is accomplished by letting the loop gain L drop off as quickly as possible.

In the cross-over region, that is, the frequency region where the loop gain L changes over from being large to being small, both S and T should be prevented from peaking. Peaking implies a poorly damped time response.

These requirements are also needed for good robustness:

Small values of S in a low frequency band resulting from large values of the gain L imply good robustness with respect to low frequency plant perturbations such as caused by parameter uncertainty.

Small values of T at frequencies outside the active band (corresponding to small values of the loop gain L) imply robustness against high frequency perturbations caused by modeling errors and parasitic effects.

If neither S nor T exhibits excessive peaking in the crossover region then adequate stability robustness is guaranteed.

Fig. 2 shows "ideal" shapes for the norms of the sensitivity functions. The specifications of what an acceptable shape is involve notions such as bandwidth, peaking, roll-on and roll-off.

In this paper we explore how H_∞ and H_2 mixed sensitivity design may be used to achieve these design targets.

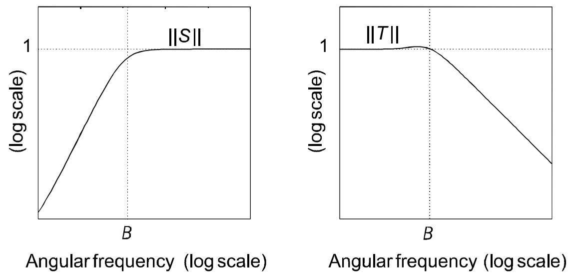

Fig. 2 Ideal shapes for the sensitivity functions

3. EXPLORATORY ANALYSIS

It goes almost without saying that the design targets need to be realistic. To establish whether they are, *exploratory analysis* of the plant is called for.

In particular, there are some essential design limitations related to the locations of the open-loop plant poles and zeros.

If the plant has right-half plane open-loop zeros then the magnitude of the smallest right-half plane zero is an upper bound for the closed-loop bandwidth. This is because good performance basically involves inversion of the plant. Since a right-half plane zero implies instability of the inverse plant this inversion may only be accomplished for low frequencies that are smaller than the magnitude of the smallest right-half plane open-loop zero. This provides an upper bound to the width of the band over which S may be made small.

If the plant has right-half plane poles then the gain may only be allowed to drop off for frequencies greater than the magnitude of the largest right-half plane pole. Otherwise the plant cannot be stabilized. This constitutes a lower bound for the frequency band outside which T drops off to small values.

If the frequency bound that follows from by (a) is less than that follows from (b) then the crossover region necessarily occupies the entire intervening interval. This does not bode well for robustness.

The bandwidth specification obviously needs to respect the limitations imposed by right-half plane poles and zeros. If it does, it needs to be checked whether the required bandwidth matches the physical capacity of the plant, that is, its ability to absorb sufficiently large inputs. This may be investigated with the help of the input sensitivity U. If the loop gain L is large then $U = C(I + PC)^{-1} \approx P^{-1}$. Thus, prior to choosing the compensator C the behavior of U over the target low frequency band may be assessed by checking the behavior of P^{-1} over this band.

4. H_∞ MIXED SENSITIVITY DESIGN

Mixed sensitivity

Mixed sensitivity design relies on optimization of a criterion that involves two or more sensitivity functions. It first arose in the context of H_∞ optimization (Kwakernaak, 1983; Verma and Jonckheere, 1984). The standard H_∞ mixed sensitivity minimization criterion is

$$\left\| \begin{bmatrix} W_1 SV \\ W_2 UV \end{bmatrix} \right\|_\infty \tag{2}$$

The less general criterion where $V = I$, which is often seen in the literature, has serious limitations. The more general criterion where the single right-hand factor V in (2) is replaced with two different factors V_1 and V_2 is a little too unstructured for analysis. Since the sensitivity functions T and U are related by $T = PU$ the sensitivity function U in the second entry may be replaced with T by suitably modifying W_2, if desired.

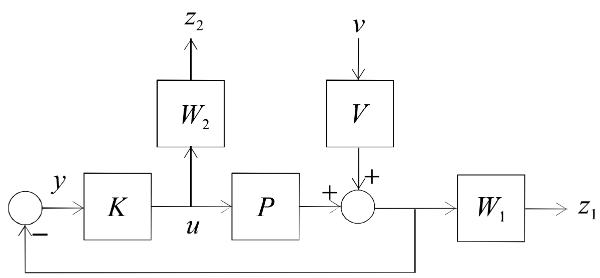

Fig. 3 Mixed sensitivity configuration

The formulation with U allows the block diagram representation of Fig. 3. In this diagram, the closed-loop transfer matrix from the disturbance driving signal v to the output

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

is

$$H = \begin{bmatrix} W_1 SV \\ -W_2 UV \end{bmatrix} \tag{3}$$

Thus, mixed sensitivity optimization amounts to minimization of the H_∞-norm of the closed-loop transfer matrix H.

Dominant pole assignment

Assume that the plant transfer matrix P has the left polynomial fraction representation $P = D^{-1}N$ with D row-reduced. Then for design purposes it is very effective to choose the rational matrix V as

$$V = D^{-1}M$$

The polynomial matrix M is so selected that V is biproper. For reasons of dimensioning and scaling it is helpful to let $V(\infty) = I$. Without loss of generality the square polynomial matrix M may be chosen to have all its roots in the left-half complex plane.

With this choice of V the solution of the H_∞ problem often (but not always) has the property that the roots of the polynomial matrix M are among the closed-loop poles. This is a consequence of the equalizing property of type B solutions of the H_∞ problem (Kwakernaak, 1996). Type B solutions generally seem to occur in the mixed sensitivity problem when the weight W_2 is small. This is exactly what is needed to make the roots of M the dominant poles.

The roots of D are the open-loop plant poles. Assuming that V is biproper, the roots of M generally are closed-loop poles of the mixed-sensitivity optimal control system. Hence, the open-loop poles are reassigned to the locations of the roots of M. This is called *partial pole assignment* — *partial* because there are other closed-loop poles besides the roots of M alone. By making sure that these pre-assigned poles are the *dominant* closed-loop poles an important degree of control over the closed-loop behavior is available.

Low and high frequency shaping

Another feature of the mixed sensitivity method consists of the possibilities it offers for low and high frequency shaping.

We first consider low frequency shaping. Suppose that V is so chosen that

$$V(s) \approx V_0 / s \quad \text{for} \quad s \to 0$$

with V_0 nonsingular finite, while also $W_1(0)$ is nonsingular finite. Then if the mixed sensitivity problem has at all a solution inspection of (2) shows that if the H_∞-norm is finite then necessarily $S(s) \approx sS_0$ for $s \to 0$, with S_o some constant matrix. This implies $S(0) = 0$ and, hence, integral control, which guarantees low sensitivity at low frequencies.

Note this: If $S(s) \approx sS_0$ for $s \to 0$ then $T(s) \approx I$ so that

$$\begin{aligned} W_2(s)U(s)V(s) &= W_2(s)P^{-1}(s)T(s)V(s) \\ &\approx W_2(s)P^{-1}(s)V_0 / s \quad \text{for} \quad s \to 0 \end{aligned}$$

Therefore, if the criterion (2) is to be finite then $W_2(s)P^{-1}(s)$ needs to contain a factor s. If $P^{-1}(s)$

does not have this factor s then $W_2(s)$ needs to be chosen so that this factor is present.

Another way to design for integral control is to let the weighting function W_1 have a factor $1/s$. In this case it is not necessary to let W_2 have a factor s.

High frequency shaping is used for U. Inspection of the expressions

$$U = C(I+PC)^{-1}, \quad T = PC(I+PC)^{-1}$$

shows that if the loop gain $L = PC$ is strictly proper — which is a very sensible design requirement — then for high frequencies $U \approx C$. Hence, C inherits the high frequency roll-off characteristics of U. Since for high frequencies $T \approx PC$ the roll-off characteristics of T follow from those of P and C.

To illustrate high frequency shaping suppose that $W_2(s) \approx W_{2\infty} s^k$ for $s \to \infty$. Since by assumption V is biproper necessarily $U(s) \approx U_\infty s^{-k}$ for $s \to \infty$ if the criterion (2) is finite. Thus, high frequency roll-off may be controlled by an appropriate choice of W_2. Adequate high frequency roll-off is important to ensure high frequency robustness, to reduce the effect of measurement noise, and to eliminate unnecessary high frequency activity of the plant input.

Transformation by block diagram substitution

Including a factor $1/s$ in V or W_1 as proposed in the preceding section makes the generalized plant for the resulting H_∞ problem non-stabilizable. This is a serious handicap for the use of standard state space algorithms for the solution of the H_∞ problem.

Also, modifying the weighting function W_2 for high frequency shaping easily leads to a non-proper transfer function for W_2. Since such a transfer function cannot be represented in state form, again the standard state space algorithms for the solution of the H_∞ problem cannot be used.

Both these difficulties may be resolved by block diagram substitution (Krause, 1992). By way of example, consider the case that W_1 has a pole at 0, corresponding to a non-stabilizable mode. The configuration of Fig. 1 may be modified to that of Fig. 4. Because the uncontrollable block W_1 is now inside the loop the controllability problem has been removed. Once the compensator K_o that solves the modified mixed-sensitivity problem has been found, the compensator K for the original problem follows as $K = K_o W_1$. A disadvantage of this approach is that there usually are cancellations between the factors K_o and W_1, which is numerically unattractive.

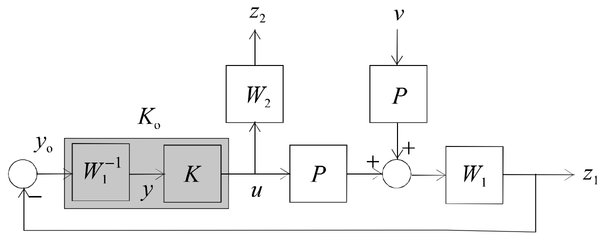

Fig. 4 Modified mixed sensitivity configuration

We see from Fig. 4 that by this substitution the plant transfer function has been modified from P to $W_1 P$. If W_1 has a factor $1/s$ then so does the modified plant. This well-known device to obtain integrating action is sometimes called the *integrator-in-the loop* method.

If W_2 is nonproper then block diagram substitution is used to pull W_2^{-1} inside the loop.

Loop shaping by disturbance modeling

In an earlier subsection we propose to design the dominant closed-loop dynamics by suitably choosing the weighting matrix V. Since in the MIMO case it is not clear at all how to choose V we outline a way do this by what we call *disturbance modeling*.

Suppose that the plant P, possibly modified by block diagram substitution as previously explained, is described in state space form as

$$\dot{x} = Ax + Bu, \quad y = Cx.$$

We now model the effect of the disturbances by modifying the system equations to

$$\dot{x} = Ax + Bu + Gv, \quad y = Cx + w \tag{4}$$

The disturbance has the components v and w. The assumption is that G may be sensibly chosen by design considerations. In this model, the transfer matrix from the disturbances to the plant output is

$$V(s) = \begin{bmatrix} C(sI-A)^{-1}G & I \end{bmatrix}$$

This transfer function generally is not square. Consideration of the H_∞ mixed sensitivity criterion, however, shows that V may be replaced with $\bar{V}$, which is obtained by spectral co-factorization of $V(s)V^T(-s)$, that is, $\bar{V}(s)\bar{V}^T(-s) = V(s)V^T(-s)$ such that $\bar{V}$ is square and has all its poles and zeros in the closed left-half plane (Kwakernaak, 1986).

Thus, the approach we propose is to choose the gain G such that the zeros of $\bar{V}$ – meant to be the dominant closed-loop poles – assume suitable locations. The design application in the companion paper (Kwakernaak, 2001) illustrates the procedure.

It is not really necessary to compute the spectral factor $\bar{V}$ because the standard state space H_∞ algorithm accommodates the equations (4).

5. H_2 MIXED SENSITIVITY DESIGN

LQG and LTR

The best-known H_2 optimization problem is of course the LQG problem. Fig. 5 shows the configuration. The compensator *K* is the interconnection of an optimal observer (Kalman filter) and state feedback. The system is driven by the white system noise *v* and the white measurement noise *w*.

A simple computation (see further on) shows that the LQG problem amounts to an H_2 problem. That is, solving the H_2 problem is equivalent to minimizing the H_2-norm of a transfer function that involves various sensitivity functions. It is not exactly the H_2-norm of an expression such as (3) because the configuration of Fig. 5 involves measurement noise, which the configuration of Fig. 3 does not.

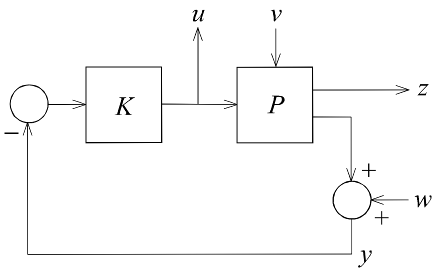

Fig. 5 LQG configuration

The LQG problem has been very extensively investigated, which facilitates its application. A special case is the situation where Loop Transfer Recovery (LTR) may be applied. We briefly review the main results.

In the LGQ problem the plant, including system and measurement noise, are represented in state form as

$$\dot{x} = Ax + Bu + Gv, \quad z = Dx, \quad y = Cx + w \qquad (5)$$

The white system noise *v* has intensity *V* and the white measurement noise *w* has intensity *W*. The optimization criterion is

$$\lim_{t\to\infty} E\left(z^T(t)Qz(t) + u^T(t)Ru(t)\right) \qquad (6)$$

The solution relies on solving two algebraic Riccati equations. It is documented in many textbooks, including Kwakernaak and Sivan (1972).

LTR (Saberi *et al.,* 1993) relies on the assumption that $P(s) = C(sI - A)^{-1}B$ is square and has no right half plane zeros. There are two approaches. The first approach is to assume that the system noise *v* is additive to the input, that is, $G = B$. Then if the intensity *W* of the measurement noise approaches the zero matrix. the loop gain of the closed-loop system approaches the loop gain of the system under state feedback (assuming that in Fig. 5 the loop is opened at the plant input.) The loop gain under state feedback has various favorable properties. In particular it has guaranteed robustness margins.

The dual approach is to assume non-inferential control, that is, $D = C$, and to let the weighting matrix *R* approach the zero matrix. Again robustness margins may be guaranteed.

The LTR approach is subject to criticism but it is transparent. Also, it accommodates the design targets small sensitivity at low frequencies, small complementary sensitivity at high frequencies, and little peaking at crossover frequencies.

LQG as a mixed sensitivity problem

We show that LQG optimization is a mixed sensitivity problem. Under the simplifying assumptions $G = B$, $D = C$ we have the block diagram of Fig. 5 $y = z = P(u+v)$. It is easily found that in the closed loop $z = SPv - Tw, \quad u = -UPv - Uw$. The sensitivity functions *S, T,* and *U* are given by (1). It follows that the LQG criterion (6) may be rendered as

$$\mathrm{tr}\int_{-\infty}^{\infty}\left(SPVP^{\sim}S^{\sim} + TWT^{\sim} + UPVP^{\sim}U^{\sim} + UWU^{\sim}\right) df$$

The argument of the integrand is $2\pi f$ throughout, and we denote $S^{\sim}(s) = S^T(-s)$, etc. Inspection of this expression shows that its minimization definitely deserves to be called a mixed sensitivity problem.

We may generalize this mixed sensitivity problem by introducing weighting and shaping filters as in Fig. 6. Correspondingly, the generalized criterion is

$$\mathrm{tr}\int_{-\infty}^{\infty}(W_1SPV_1V_1^{\sim}P^{\sim}S^{\sim}W_1^{\sim} + W_1TV_2V_2^{\sim}T^{\sim}W_1^{\sim} + W_2UPV_1V_1^{\sim}P^{\sim}U^{\sim}W_2^{\sim} + W_2UV_2V_2^{\sim}U^{\sim}W_2^{\sim})\, df \qquad (7)$$

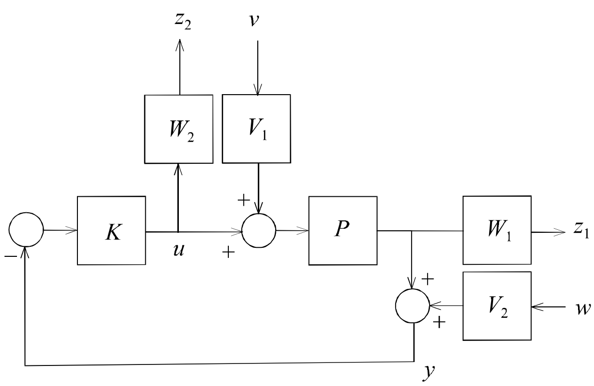

Fig. 6. Block diagram for the H_2 mixed sensitivity problem

Low and high frequency shaping

The various weighting functions that have been introduced may be used for low and high frequency shaping. The techniques introduced in the H_∞ section to design for integrating action also work here.

For high frequency shaping it is important to recognize that if the H_2 mixed sensitivity problem has a solution then each of the terms in the integrand of (7) is strictly proper. One consequence of this is that in the standard LQG problem the compensator is always strictly proper and, hence, also U and T. If for high frequency robustness T needs to have more high frequency roll-off than 1 decade/decade then the weighting function W_2 may be selected nonproper to achieve this.

Dominant pole assignment by disturbance modeling

The dominant pole assignment technique proposed for the H_∞ mixed sensitivity problem may also be applied to the H_2 mixed sensitivity approach. Again it relies on disturbance modeling. In this method, the gain matrix G for the noise in the state equation (5) is so chosen that only those modes are excited that correspond to poles that need to be moved. The amount of excitation determines the extent to which the poles are relocated.

The approach relies on making the observer poles the dominant closed-loop. The regulator poles are rendered non-dominant by choosing the weighting matrix R sufficiently small. Hence, the approach amounts to dual LTR

The design application in the companion paper (Kwakernaak, 2001) illustrates the method.

6. CONCLUSIONS

The low and high frequency shaping and disturbance modeling tools of the H_∞ and H_2 mixed sensitivity methods presented in this paper aim at shaping the closed-loop sensitivity functions to achieve generic performance and robustness qualities. The dominant pole placement feature moreover allows control over the time domain response properties.

In the companion paper (Kwakernaak, 2001) the two methods are applied to a concrete multivariable design problem.

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