15th Triennial World Congress, Barcelona, Spain

WHETHER THE DELTA OPERATOR MODELS ARE REALLY BETTER FOR SMALL SAMPLING PERIODS

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Abstract: It is noted that the superiority of the delta operator (DO) models over the shift operator (SO) ones, for small sampling periods, concerns only the recording of the model coefficients and some analytical calculations. It is also noted that in the simulations, making it possible to obtain the time response of the output for any input, as well as in the model identification under limited measurement accuracy, the superiority of the DO over SO models, for small sampling periods, disappears. These observations have an essential meaning since the mentioned problems of simulation and identification are important for applications.

Keywords: Digital control, discrete-time systems, delta operator models, shift operator models.

1. INTRODUCTION

In these days most control systems contain regulators implemented with microprocessors. In this case both the parameters and the signals of the regulator are recorded using the arithmetic with finite word length (FWL). In connection with this two kinds of errors are distinguished: first, the errors due to the FWL implementation of the regulator coefficients and second, the errors due to the roundoff of the regulator signals. Additional errors of the roundoff type result from the FWL realization of the analogue to digital (A/D) and digital to analogue (D/A) converters.

The FWL effects were researched mainly in connection with realization of digital filters. By comparison, the FWL effects have received much less attention in control literature. One exception is the book of Gevers and Li (1993) where in two chapters the control problems with accounting FWL effects are summarized and some new results are described. In (Gevers and Li, 1993) the attention is focused on the choice of the regulator model structure realized with FWL, for which both the above mentioned errors take the minimal values.

Performance requirements often command that reasonably small sampling period be used. It is known that in this case the use of FWL leads to poor properties of the pulse transfer function (TF) models $H(z)$ based on Z-transform (i.e. the shift operator (SO) models). In connection with this Middleton and Goodwin (1990) to improve description properties of discrete-time (DT) systems, use the delta operator (DO) models. They stress that the latter models for small sampling periods have better numerical properties than the SO models. The cause of this is explained in (Gessing, 1999) where the method for estimating the WL needed for recording the SO model parameters is proposed. The superiority of the DO models for small sampling period in identification problems is described in (Goodwin *et al*, 1992).

In the present paper the common view about superiority of the DO over SO models for small sampling periods is questioned. It is noted that the superiority concerns only the recording of the model coefficients and of some analytical calculations. The latter are less important from the application point of view. In the simulation making it possible to determine the output of the model for any input, as well as in the model identification under limited measurement accuracy, this superiority disappears. It seems that the mentioned simulation and identification problems are more important to practice.

The contribution of the paper is in showing that the superiority of the DO over SO models, for small sampling periods, disappears in model simulation making it possible to determine the output for any input, as well in model identification under limited measurement accuracy.

2. SHIFT AND DELTA OPERATOR MODELS

Consider the system composed of sampler, zero order hold, and a linear continuous-time (CT) plant G in series shown in Fig. 1. The system G has a rational strictly proper transfer function (TF)

$$G(s) = \frac{Y(s)}{U^*(s)} = \frac{B(s)}{A(s)} = \tag{1}$$
$$= \frac{b_0 s^m + b_1 s^{m-1} + ... + b_m}{s^n + a_1 s^{n-1} + ... + a_n}$$

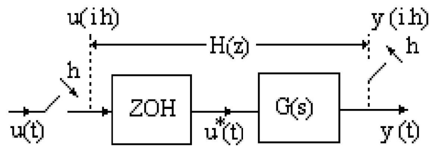

Fig. 1. Discrete-time system.

where $U^*(s) = \mathcal{L}[u^*(t)]$, $Y(s) = \mathcal{L}[y(t)]$, $\mathcal{L}$ is the symbol of Laplace transform; u^* and y are the input and output of the plant G; $B(s)$ and $A(s)$ are polynomials of degrees m and n, respectively, $m < n$. The considered discrete-time (DT) system may be described by a shift operator (SO) model

$$H(z) = \frac{\bar{Y}(z)}{\bar{U}(z)} = \frac{\bar{B}(z)}{\bar{A}(z)} = \tag{2}$$
$$= \frac{\bar{b}_1 z^{n-1} + \bar{b}_2 z^{n-2} + ... + \bar{b}_n}{z^n + \bar{a}_1 z^{n-1} + ... + \bar{a}_n}$$

where $\bar{U}(z) = \mathcal{Z}[u(ih)]$, $\bar{Y}(z) = \mathcal{Z}[y(ih)]$, $\mathcal{Z}$ is the symbol of Z-transform; u and y are the input and output of the system; $i = 0, 1, 2, ...$ is the discrete time and h is the sampling period; $\bar{B}(z)$ and $\bar{A}(z)$ are the polynomials determined in (2).

The DT system shown in Fig. 1 may be also described by the delta operator (DO) model determined by (Middleton and Goodwin, 1990).

$$\hat{H}(\gamma) = H(z)_{|z=1+h\gamma} \tag{3}$$

Accounting that $\gamma = (z-1)/h$, the model (3) may be also called the forward delta operator (FDO) model. Let

$$\hat{H}(\gamma) = \frac{\hat{b}_1 \gamma^{n-1} + \hat{b}_2 \gamma^{n-2} + ... + \hat{b}_n}{\gamma^n + \hat{a}_1 \gamma^{n-1} + ... + \hat{a}_n} \tag{4}$$

Another possibility is the backward delta operator (BDO) model defined below. Denote

$$\bar{H}(z^{-1}) = \frac{z^{-n}\bar{B}(z)}{z^{-n}\bar{A}(z)} \tag{5}$$

where the polynomials $\bar{A}(z)$ and $\bar{B}(z)$ are determined in (2). Using the BDO $\bar{\gamma} = (1 - z^{-1})/h$, the following determination of the BDO model is here proposed

$$\check{H}(\bar{\gamma}) = \bar{H}(z^{-1})_{|z^{-1}=1-h\bar{\gamma}} = \tag{6}$$
$$= \frac{\check{b}_0 \bar{\gamma}^n + \check{b}_1 \bar{\gamma}^{n-1} + ... + \check{b}_n}{\bar{\gamma}^n + \check{a}_1 \bar{\gamma}^{n-1} + ... + \check{a}_n}$$

Both the delta operator models (4) and (6) have similar properties, namely their coefficients tend to the coefficients of the CT TF (1), when $h \to 0$.

3. WORD LENGTH NEEDED FOR RECORDING AND CALCULATIONS

The DO models: FDO (4) and BDO (6) have similar properties therefore in the following if we will use the acronym DO we will understand the FDO model.

It is known that for small sampling periods h a very long digital word is needed for recording the coefficients of the SO models (Gevers and Li, 1993; Gessing, 1999). It is also known that in the case of DO models (4) and also (6) a significantly shorter word length (WL) is needed for recording the coefficients of these models. This is one of the superiority of the DO models over the SO models. The WL needed for recording the DO model may be determined similarly as in (Gessing, 1999).

In relation to this, for calculation of the frequency responses of the DO models for small h a significantly smaller WL is needed than in the case of SO models.

The same remark concerns the calculation of the time responses y of the model, however only then when the input u is described by a mathematical formula for which there exists an analytical solution of the difference equation (7) corresponding to the DO model (4). In applications this is a rather seldom case.

4. MODEL SIMULATION

When the input signal u is not described by a mathematical formula the time response y of the system is usually obtained from simulations. Simulation is an important tool used in every engineering design.

The same case appears in digital control implementation in which the output of the digital controller must be calculated for any current input.

When the SO model is used then the convenient formula for simulations results from the difference equation (7) corresponding to (2):

$$y(ih+nh)+\bar{a}_1 y(ih+nh-1)+...+\bar{a}_n = \tag{7}$$
$$=\bar{b}_1 u(ih+nh-1)+...+\bar{b}_n u(ih)$$

The equation (7) determines the recurrence formula making it possible to calculate the successive y when the previous y and the sequence of u are known. In simulations the sequence of u is not given in the form of a mathematical function but usually results from current calculations. Therefore the time response y can't be determined by an analytical solution of (7).

The SO model (7) for small h needs a large WL for recording the coefficients $\bar{a}_j$, $\bar{b}_j$ and for calculation of the time response y. This is the known property of this model. The WL is increasing when $h \to 0$.

In the case of DO model the other difference equation corresponding to (4) takes the form

$$\hat{\Delta}^n y(ih)+\hat{a}_1\hat{\Delta}^{n-1}y(ih)+...+\hat{a}_n y(ih) = \tag{8}$$
$$=\hat{b}_1\hat{\Delta}^{n-1}+\hat{b}_2\hat{\Delta}^{n-2}+...+\hat{b}_n u(ih)$$

To calculate the time response y for given u we must use in the appropriate manner the formula (8) together with the formulas

$$\hat{\Delta}^{j+1}y(ih)=[\hat{\Delta}^j y(ih+h)-\hat{\Delta}^j y(ih)]\frac{1}{h}, \tag{9}$$
$$\hat{\Delta}^{j+1}u(ih)=[\hat{\Delta}^j u(ih+h)-\hat{\Delta}^j u(ih)]\frac{1}{h},$$
$$j=0,1,...,n-1$$

Namely from (8) we calculate $\hat{\Delta}^n y(ih)$ and from (9) $\hat{\Delta}^j y(ih+h)$, $j=0,1,,n-1$. Note that in the latter calculations a large WL is needed since in the formula

$$\hat{\Delta}^j y(ih+h)=\hat{\Delta}^j y(ih)+h\hat{\Delta}^{j+1}y(ih)$$

resulting from (9), a very small value $h\hat{\Delta}^{j+1}y(ih)$ is added (for small h) to the value $\hat{\Delta}^j y(ih)$ which may be relatively large and this addition is repeated many times.

Of course the calculation of the time response y by using the formulas (8) and (9) is significantly more complicated than by using the formula (7). But one can notice that the equation (7) results from substituting the formulas (9) into the equation (8). Therefore the WL needed for calculations of time response y with a prescribed accuracy by using the formulas (8) and (9) is the same as that by using the formula (7). Thus from the point of view of the needed WL there is no superiority of the DO models over the SO ones. On the contrary from the point of view of the simplicity of calculations there is superiority of the SO models over the DO ones.

5. MODEL IDENTIFICATION

There is a common conviction that for small h, the DO models are better than the SO ones also for identification (in accordance with my knowledge the BDO models are not used at all). This conviction is justified if to short digital word is used in information processing (the numbers have to short mantissa), which however is not exactly stated in (Goodwin *et al*, 1992). From the application point of view, a more realistic is the assumption that the main source of errors comes from nonaccurateness of measurements and the errors resulting from information processing are negligible. The following simulations of the identification experiment will be performed under this assumption.

In the simulations the limited accuracy of measurements of $u(ih)$, $y(ih)$, $x(ih)$ was obtained by recording them with digital word having N digits in floating point arithmetic. The N-digit mantissa was created by accounting the first N, most significant digits of the MATLAB mantissa. In simulations $N=3$ was used, which corresponds to the relative accuracy ranging from $0.1\%-1\%$.

The identification experiment was simulated using MATLAB-SIMULINK for the system shown in Fig. 1 in which

$$G(s)=\frac{k}{s^2+3s+2}, \qquad k=1 \tag{10}$$

The estimates of DT TF $H(z)$ were calculated using RLS algorithm with forgetting factor λ and measurements with limited accuracy. The system was excited by input u obtained from a white noise generator; the same input u was used in all the simulations described below.

In Fig. 2 a) the estimates of $\bar{b}_1$, $\bar{b}_2$, $\bar{a}_1$, $\bar{a}_2$ of DT TF $H(z)$ for $h = 0.1$, $N = 3$, $\lambda = 0.96$ are plotted as functions of time. From these plots, in the from of horizontal lines, it is not possible to determine the estimates accuracy. Really, for small sampling periods some long digital words are needed for recording the coefficients of $H(z)$ and the information about these coefficients are contained in further, less meaning digits of these words (Gessing, 1999); therefore even inaccurate estimates of coefficients of $H(z)$ may have the shape of horizontal lines. To obtained a visual presentation of the accuracy, the estimates of b_0, b_2, a_1, a_2 of CT plant $G(s)$ are calculated with double precision accuracy, in each period h, using $\bar{b}_1$, $\bar{b}_2$, $\bar{a}_1$, $\bar{a}_2$ and the MATLAB functions *tf2ss*, *d2c*, *ss2tf*. The estimates of b_0, b_1, a_1, a_2, corresponding to those from Fig. 2 a, are shown in Fig. 2 b; they are almost accurate, only a_2 has some slight fluctuations.

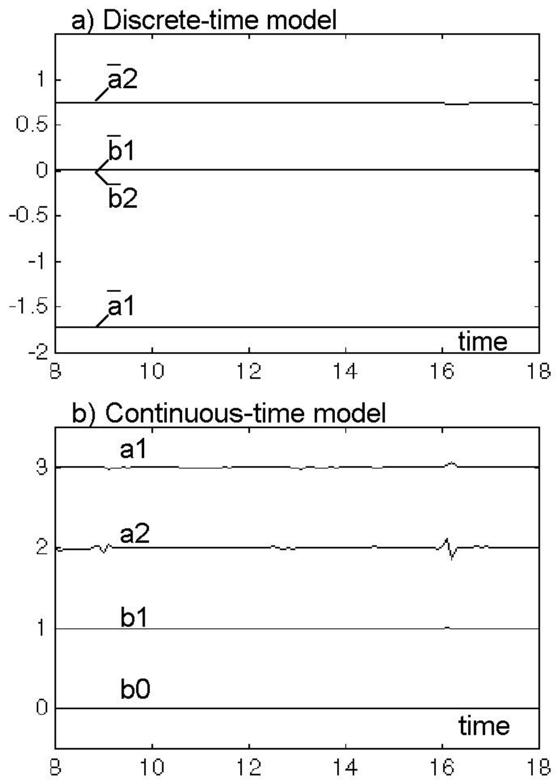

Fig. 2. Estimates for $h = 0.1$, $\lambda = 0.96$, $N = 3$.

In Fig. 3 a and 3 b the estimates of b_0, b_1, a_1, a_2 of CT plant are shown; they were calculated as previously from those of $\bar{b}_1$, $\bar{b}_2$, $\bar{a}_1$, $\bar{a}_2$, for $h = 0.05$, $N = 3$, $\lambda = 0.96$ and $\lambda = 0.9$), respectively; the estimates of $\bar{b}_1$, $\bar{b}_2$¡ $\bar{a}_1$, $\bar{a}_2$ are not shown since they, as previously, have the form of horizontal lines. It is seen that the estimates of a_1, a_2 are yet more inaccurate – they have some big fluctuations – bigger if λ is smaller. This means that the measurement accuracy determined by $N = 3$ is not sufficient for identifying the DT model with $h = 0.05$ on the basis of measurements of u and y.

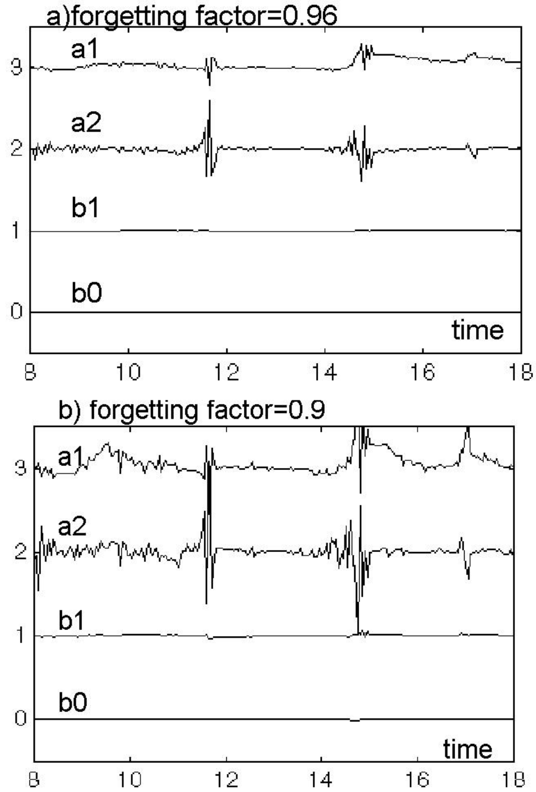

Fig. 3. Estimates of CT plant calculated from those of SO model for $h = 0.05$, $N = 3$.

In Fig. 4 a and 4 b the estimates b_0, b_1, a_1, a_2 of CT plant $G(s)$ are shown; they were calculated similarly as previously from the estimates of $\hat{b}_1$, $\hat{b}_2$, $\hat{a}_1$, $\hat{a}_2$ of DO model $\hat{H}(\gamma)$, for $h = 0.05$, $N = 3$, $\lambda = 0.96$ and $\lambda = 0.9$, respectively. Comparing the plots of Fig. 3 and Fig. 4 we see that they are similar. This and other performed simulations show that the accuracy of estimation of SO and DO models, under assumptions made, is comparable. This observation is in contradiction to the view that for small sampling periods the DO models may be identified more accurately than the SO models (Goodwin *et al*, 1992). Note that the latter statement is also true but under different assumption: that the main source of errors results from too short mantissa used for calculations (information processing). I would like to stress that the assumptions made in the present paper say that the main source of errors is the limited measurements accuracy and the information processing is relatively accurate; these assumptions are well justified from the application point of view.

6. CONCLUSIONS

It is known that in the case of small sampling period a large WL is needed for recording the SO

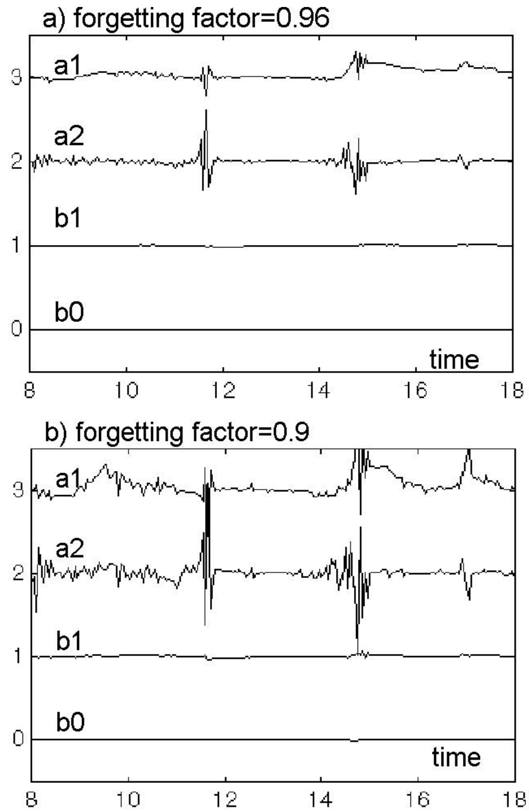

Fig. 4. Estimates of CT plant calculated from those of DO model for $h = 0.05$, $N = 3$.

model coefficients. Therefore the large WL is also needed for calculation of the frequency and time responses when the SO model is used.

Somewhat different situation is in the case of the DO model which need for recording their coefficients a significantly smaller WL. The same remark concerns the analytical calculations of the frequency and time responses. However for calculations of the latter this statement is true only for such inputs which are described by some mathematical functions and for which there exists an analytical solution of the corresponding difference equation. This is a rather seldom case in applications.

The superiority of the DO models over SO ones disappears in the simulation in which for any input the time response of the output is calculated. This is an important observation because in simulations usually this case appears. The same case appears in digital control implementation in which the output of the digital controller must be calculated for any current input.

The superiority of the DO models over SO ones, for small sampling periods, disappears also in the case of model identification under limited measurement accuracy and relatively accurate calculations (information processing). The made assumptions are fully justified from the practical point of view. The statement about the lack of the superiority is in contradiction with the common view based on (Goodwin *et al*, 1992) where the case of less accurate information processing was considered.

To summarize, since the simulation for any input signal and identification under a limited measurement accuracy are (from application point of view) more important, then the statement about the superiority of the DO over SO models, for small sampling periods, seems to be not fully justified.

ACKNOWLEDGEMENT

The paper was partially supported by the Science Research Committee (KBN), grant No. 8 T11A 012 19.

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