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COMPLETE STABILIZATION OF DISTRIBUTED PARAMETER SYSTEMS: A CONTROLLABILITY ANALYSIS APPROACH

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Abstract: The paper studies complete stabilization of a class of distributed parameter systems described by linear evolution equations in Hilbert spaces. Based on the controllability assumption of underlying control systems, complete stabilizability conditions for linear time-varying control systems with multiple state delays as well as for a class of nonlinear control systems in Hilbert spaces are established. The feature of the obtained result is that the complete stabilizability conditions are derived from the solution of Riccati differential equation and do not involve any stability property of its evolution operator.

Keywords: Complete stabilization, controllability, abstract systems, integral quadratic constraint.

1. INTRODUCTION

Consider a distributed parameter system of the form

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad t \geq t_0, \qquad (1)$$

where $x \in X$ is the state output , $u \in U$ is the control input, X, U are infinite-dimensional spaces. The standard stabilizability question for system (1) is how the operator $K(t). : X \to U$ can be found in order to keep the closed-loop system $\dot{x}(t) = [A(t) + B(t)K(t)]x(t)$ exponentially stable in the Lyapunov sense, i.e., the evolution operator $U_K(t,s)$ generated by $A(t)+B(t)K(t)$ satisfies the condition

$$\|U_K(t,s)\| \leq Ne^{-\delta(t-s)}, \quad \forall t \geq s \geq t_0. \qquad (2)$$

The positive number δ is commonly called a Lyapunov exponent and operator $K(t)$ is called a feedback control operator. In the literature on control theory of dynamical systems, stabilizability is one of the important properties of the system and has attracted the attention of many researchers (see, e.g., Ahmed 1990; Benabdallah and Hammami, 1992; Petersen et al., 2000; Sun et al., 1998; Zak, 1990 and references therein). In practice various stability definitions have been made to extend the usefulness of the exponential stability concept. The concept of complete stabilizability is related to a strong exponential stability of its evolution operator that for every given number $\delta > 0$, there exists a feedback operator $K(t)$ such that the evolution operator $U_K(t,s)$ ratifies the condition (2). The definition of complete stabilizability originally introduced by Wonham, (see; Wonham, 1967) means that the zero-input response of the closed-loop system decays faster than $e^{\delta t}$ for any given Lyapunov exponent $\delta > 0$.

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It is obvious that if the system is completely stabilizable, then it is exponentially stabilizable, but the converse is in general not true. Some results on relationship between complete stabilizability and controllability of finite-dimensional systems can be found in (Ikeda et al., 1972; Megan, 1975; Phat and Dieu, 1992; Savkin and Petersen, 1999). For infinite-dimensional systems, the investigation of controllability and stabilizability is more complicated and requires sophisticated techniques from functional analysis theory. The difficulties increase to the same extent as passing from time-invariant to time-varying systems. Under appropriate controllability assumptions, some extensions have been given in (Curtain and Zwart, 1995) for time-invariant systems in Hilbert spaces and in (Ahmed, 1990, Ikeda et al., 1972) complete stabilizability conditions for time-varying systems were derived by using Kalman's controllability decomposition of finite-dimensional systems.

The object of this paper is to find complete stabilizability conditions for infinite-dimensional time-varying control systems with multiple delays on the states. In contrast to previous results, based on controllability assumption on the underlying control system the stabilizability conditions are obtained from the solution of the Riccati differential equation (RDE) and do not involve any stability property of the evolution operator $U(t,s)$ and hence are easy to verify and construct. An efficient algorithm to find feedback controllers via solving RDE is also proposed. The approach allows us to derive sufficient conditions for complete stabilizability of nonlinear control systems.

2. PRELIMINARIES

The following notations will be used in force throughout. R denotes the set of all real numbers; X denotes a Hilbert space with the norm $\|.\|$ and the inner product $\langle .,. \rangle$; $L(X)$ (respectively, $L(X,Y)$) denotes the Banach space of all linear bounded operators S mapping X into X (respectively, X into Y) endowed with the norm

$$\|S\| = \sup\{\|Sx\| : x \in X, \|x\| \leq 1\}.$$

$L_2([t,s],X)$ denotes the set of all strongly measurable 2−integrable X−valued functions on $[t,s]$; $C[t,s]$ denotes the Banach space of all continuous functions on $[t,s]$;D(A), Im(A), A* and A^{-1} denote the domain, the image, the adjoint and the inverse of the operator A, respectively; cl M denotes the closure of a set $M; I$ denotes the identity operator. Operator $Q \in L(X)$ is called non-negative definite ($Q \geq 0$) if $\langle Qx, x\rangle \geq 0$, for all $x \in X$. If $\langle Qx, x\rangle > 0$ for $x \neq 0$, then Q is called positive definite ($Q > 0$). In case if there is a number $\delta > 0$ such that

$$\langle Qx, x\rangle \geq \delta \|x\|^2, \quad \forall x \in X,$$

then Q is called strictly positive definite and denotes by $Q \gg 0$; Operator $Q \in L(X)$ is called self-adjoint if $Q = Q^*$; $BC([0,\infty), X^+)$ denotes the set of all linear bounded self-adjoint non-negative definite operators $Q(t) \in L(X)$, continuous and bounded in $t \in [0,\infty)$. Let X, U be Hilbert spaces. Consider the linear time-varying system $[A(t), B(t)]$ given by

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad t \geq t_0,$$
$$x(t_0) = x_0, \tag{3}$$

where $x(t) \in X, u(t) \in U, A(t): X \to X, B(t) \in L(U,X)$. In the sequel, we say that control $u(t)$ is admissible if $u(t) \in L_2([t_0,s],U), s \in [t_0,\infty)$. We make the following assumptions on the system (3) throughout the paper. (i) $B(t) \in L(U,X)$ and $B(.)u$ is a bounded continuous function on $[t_0,\infty)$ for all $u \in U$. (ii) The operator $A(t): D(A(t)) \subset X \to X$, cl$D(A(t)) = X$, is a bounded function in $t \in [t_0,\infty)$ and generates a strong evolution operator $U(t,\tau): \{(t,\tau): t \geq \tau \geq t_0\} \to L(X)$ (see, e.g. Benssousan et al., 1992; Curtain and Zwart, 1995) such that the system (3), for every admissible control $u(t)$, has a unique solution given by

$$x(t) = U(t,t_0)x_0 + \int_{t_0}^{t} U(t,\tau)B(\tau)u(\tau)d\tau.$$

The concept of controllability is concerned with the question of existence of admissible controls which steer any state to an another state of the system in finite time. Depending on the properties involved one defines variety of controllability concepts. Various aspects of controllability theory can be found in (Ahmed, 1990; Kalman, 1960; Phat, 1996 and references therein).

Definition 2.1. The system $[A(t), B(t)]$ is called globally null-controllable in time $T > 0$, if every state can be transfered to 0 in some time $T > 0$ by an admissible control $u(t)$, i.e., for every $x \in X$ there is an admissible control $u(t)$ such that

$$U(T,t_0)x + \int_{t_0}^{T} U(T,s)B(s)u(s)ds = 0.$$

Proposition 2.1. (Curtain and Zwart, 1995) *The system $[A(t), B(t)]$ is globally null-controllable in time T if and only if there is a number $c > 0$ such that for all $x \in X$:*

$$\int_{t_0}^{T} \|B^*(s)U^*(T,s)x\|^2 ds \geq c\|U^*(T,t_0)x\|^2.$$

Definition 2.2. The system $[A(t), B(t)]$ is called exponentially stabilizable if there exists an operator function $K(t) \in L(X,U)$ such that the condition (2) holds for some $N > 0, \delta > 0$.

Definition 2.3. The system $[A(t), B(t)]$ is called completely stabilizable if for every number $\delta > 0$, there exists an operator function $K(t) \in L(X, U)$ such that the condition (2) holds.

Thus, in a stronger sense, the complete stabilizability means that the closed-loop system can be made exponentially stable for any given Lyapunov exponent $\delta > 0$. Associated with system $[A(t), B(t)]$ we consider a quadratic cost functional of the form

$$J(u) = \int_{t_0}^{\infty} [\langle R(t)u, u\rangle + \langle Q(t)x, x\rangle]dt, \quad (4)$$

where $R(t) \gg 0, Q(t) \in BC([t_0, \infty), X^+)$ and consider the following abstract RDE

$$\dot{P} + A^*P + PA - PBR^{-1}B^*P + Q = 0. \quad (5)$$

Definition 2.4. An operator $P(t) \in L(X)$ is said to be a solution of RDE (5) if for all $t \geq t_0, \forall x \in D(A(t))$:

$$\langle \dot{P}x, x\rangle + \langle PAx, x\rangle + \langle Px, Ax\rangle$$
$$-\langle PBR^{-1}B^*Px, x\rangle + \langle Qx, x\rangle = 0.$$

For solving optimal quadratic control problem (3), (4) we have the following

Proposition 2.2. (Benssounsan et al., 1992) *Assume that the optimal quadratic control problem (3), (4) is solved in the sense that for every initial state x_0, there is an admissible control $u(t)$ such that the cost functional (4) exists and is finite. Then for every $Q(t) \in BC([t_0, \infty), X^+)$, the RDE (5) has a solution $P(t) \in BC([t_0, \infty), X^+)$. Moreover, the control $u(t)$ given in the feedback form*

$$u(t) = -R^{-1}(t)B^*(t)P(t)x(t), \quad t \geq t_0, \quad (6)$$

minimizes functional (4).

3. STABILIZABILITY CONDITIONS

Consider a distributed parameter system of the form

$$\dot{x}(t) = A(t)x(t) + \sum_{i=1}^{r} A_i(t)x(t - h_i) + B(t)u(t),,$$
$$x(t) = \phi(t), \quad t \in [-h_r, t_0], \quad (7)$$

where $t \geq t_0, 0 \leq h_1 \leq ... \leq h_r; r \geq 1$. We assume that the operator functions $A(t), A_1(t), B(t)$ satisfy the assumptions stated in the previous section such that for every initial condition $\phi(t) \in C[-h_r, t_0]$ and admissible control $u(t)$, the system (7) has an unique solution. We define $x_t = x(t + s), -h_r \leq s \leq t_0$ and

$$\|x_t\| = \sup_{s\in[-h_r,t_0]} \|x(t + s)\|.$$

For some $\delta > 0$ we set
$\tilde{A}(t) = A(t)+\delta I, \tilde{A}_i(t) = e^{\delta h}A_i(t), \tilde{B}(t) = e^{\delta t}B(t)$, and consider the Riccati differential equation

$$\dot{P} + \tilde{A}^*P + P\tilde{A} - P\tilde{B}\tilde{B}^*P + Q = 0, \quad t \geq t_0. (8)$$

The following theorem shows that the feedback control stabilizer for system (7) can be found from the solution of Riccati differential equation (8).

Theorem 3.1. *Assume that for every $\delta > 0$ and $Q(t) \in BC([t_0, \infty), X^+)$ the Riccati equation (8) has a unique solution $P(t) \in BC([t_0, \infty), X^+)$ such that*

$$Q(t) \geq (r + 1)I + \sum_{i=1}^{r} P(t)\tilde{A}_i(t)\tilde{A}_i^*(t)P(t). \quad (9)$$

Then the system (7) is completely stabilizable.

Sketch of the proof. Let $t_0 = 0, \delta > 0$, be an arbitrary number and let $y(t) = e^{\delta t}x(t)$. Then system (7) is transformed to the system

$$\dot{y}(t) = \tilde{A}(t)y(t) + \sum_{i=1}^{r} \tilde{A}_i(t)y(t - h_i) + \tilde{B}(t)u(t),$$
$$y(t) = e^{\delta t}\phi(t) := \psi(t), \quad t \in [-h_r, 0]. \quad (10)$$

We shall prove that every solution of system (10) is bounded. Let $Q(t) \in BC([0, \infty), X^+)$ and the Riccati equation (8) has a unique solution $P(t) \in BC([0, \infty), X^+)$ such that the condition (9) holds. Let

$$u(t) = -\frac{1}{2}\tilde{B}^*(t)P(t)y(t), t \geq 0, \quad (11)$$

and we consider a Lyapunov function for the system (10) of the form

$$V(t, y_t) = \langle P(t)y, y\rangle + \sum_{i=1}^{r} \int_{t-h_i}^{t} \langle y(s), y(s)\rangle ds.$$

Taking derivative of $V(t, y_t)$ along the solution $y(t)$ of the system and using the control (11), we may arrive the following estimation

$$\dot{V}(t, y_t) \leq -\langle[Q - rI - \sum_{i=1}^{r}\langle P_i^*(t)P_i(t)]y(t), y(t)\rangle.$$

Therefore, if the condition (9) holds then

$$\dot{V}(t, y_t) \leq -\|y(t)\|^2, \quad \forall t \geq 0. \quad (12)$$

Integrating both sides of (12) from 0 to t, we obtain

$$\int_0^{\infty} \|y(s)\|^2 ds \leq c,$$

for some number $c > 0$, which implies that the solution $y(t)$ is bounded. Thus, there is a number

$N > 0$ such that $\|y(t)\| \leq N\|\psi_t\|, \quad \forall t \geq 0$. Consequently, every solution $x(t)$ of system (7) satisfies

$$\|x(t)\| \leq N\|\phi_t\|e^{-\delta t}, \quad \forall t \geq t_0.$$

Theorem is proved.

Corollary 3.1. *Assume that for every $\delta > 0$ the Riccati equation*

$$\dot{P} + (A + \delta I)^* P + P(A + \delta I) -$$
$$P\big[e^{2\delta t}BB^* - \sum_{i=1}^{r} e^{2\delta h_i} A_i A_i^*\big]P + (r+1)I = 0,$$

has a unique solution $P(t) \in BC([t_0, \infty), X^+)$, then the system (7) is completely stabilizable.

We need the following lemma, the proof is based on Proposition 2.2.

Lemma 3.1. *If system $[A(t), B(t)]$ is globally null-controllable in finite time, then for every operator $Q(t) \in BC([t_0, \infty), X^+)$ the Riccati differential equation (5), where $R(t) = I$ has a solution $P(t) \in BC([t_0, \infty), X^+)$ and the feedback control (6) minimizes the cost functional (4).*

Thus, the global null-controllability is, by Lemma 3.1, a sufficient condition for existence of the solution of the Riccati equation, then we can derive the following sufficient conditions for complete stabilizability of system (7) under appropriate assumption on the operators $A_i(t), i = 1, 2, ..., r$. The proof is similar along to the proof of Theorem 3.1 using Proposition 3.1 and Lemma 3.1.

Theorem 3.2. *Let system $[A(t), B(t)]$ be globally null-controllable in finite time. Assume that*

$$rI \geq \sum_{i=1}^{r} P(t)e^{2\delta h_i} A_i(t)A_i^*(t)P(t). \qquad (13)$$

where $\delta > 0; P(t) \in BC([t_0, \infty), X^+)$ is a solution of the Riccati equation

$$\dot{P} + \tilde{A}^*P + P\tilde{A} - P\tilde{B}\tilde{B}^*P + (2r+1)I = 0 \qquad (14)$$

Then the system (7) is completely stabilizable by the feedback controller

$$u(t) = -\frac{e^{-\delta t}}{2}B^*(t)P(t)x(t). \qquad (15)$$

Sketch of the proof. Let $\delta > 0$ be an arbitrary number and let $y(t) = e^{\delta t}x(t)$. As in the proof of Theorem 3.1, the system (7) is transformed to the system (10). It suffices to prove that every solution of system (10) is bounded. For this, from Definition 2.1, Proposition 2.1 it follows that the system $[\tilde{A}(t), \tilde{B}(t)]$ is globally null-controllable in T time. Therefore, in view of Lemma 3.1 the Riccati equation (14), where $R(t) = I, Q(t) = (2r+1)I$, has a solution $P(t) \in BC([t_0, \infty), X^+)$. From the assumption it follows that

$$Q = (2r+1)I \geq (r+1)I + \sum_{i=1}^{r} P\tilde{A}_i\tilde{A}_i^*P.$$

Thus, the condition (9) of Theorem 3.1 is satisfied such that the system is completely stabilizable by the feedback control (15). Theorem is proved.

Remark 3.1. The condition (13) can be replaced by the following

$$\sum_{i=1}^{r} \sup_{t\in[0,\infty)} \|A_i(t)\|^2 \leq \frac{re^{-2\delta h}}{p^2}, \quad \forall t \geq t_0, \qquad (16)$$

where $p = \sup\{\|P(t)\| : t \in [t_0, \infty)\}$. Then, the following simple step-by-step procedure can be used to find the feedback controller, which completely stabilizes the system (7):

Step 1: Verify the global null-controllability of system $[A(t), B(t)]$ by Proposition 2.1.

Step 2: Giving $\delta > 0$, find solution $P(t) \in BC([t_0, \infty), X^+)$ of Riccati differential equation (14).

Step 3: Verify the condition (16).

Step 4: The stabilizing controller is then defined by (15)

Remark 3.2. Note that if $A_i(.) = 0$, i.e. the system is undelayed in states, then the condition (9), taking $Q(t) = I$, immediately holds for all $\delta > 0$ and we obtain the following corollary, which shows that all linear closed-loop systems of globally null-controllable systems can be made exponentially stabilizable for any given stability exponent. This result extends some previous results for undelayed finite-dimensional systems given in (Ahmed, 1990; Ikeda et al., 1972; Kalman, 1960) as well as for time-invariant systems in Hilbert spaces given in (Megan, 1975; Phat and Kiet, 1999; Slemrod, 1974).

Corollary 3.2. *If system $[A(t), B(t)]$ is globally null-controllable in finite time, then it is completely stabilizable.*

By the same approach we can derive complete stabilizability conditions for a nonlinear control system of the form

$$\dot{x} = A(t)x + B(t)u + f(t, x, u), x(0) = x_0, \qquad (17)$$

where $x \in X, u \in U$, and $f(t, x, u) : [0, \infty) \times X \times U \to X$ is a some given nonlinear perturbation. We recall that nonlinear control system (17) is completely stabilizable if for every $\delta > 0$, there is a feedback operator $K(t) \in L(X, U)$ such that any solution $x(t)$ of the closed-loop system

$$\dot{x} = [Ax + KB]x + f(t, x, Kx),$$

satisfies the condition

$$\|x(t)\| \leq N e^{-\delta t}\|x_0\|, \quad \forall t \geq 0.$$

It is well known (see, e.g. Benabdallah and Hammami, 2001; Sun et al., 1998; Zak, 1990) that standard stabilization conditions for nonlinear control systems have been obtained based on the stability of the evolution operator $U(t, s)$ and on the perturbation $f(t, .)$ that for some $a > 0, b > 0$:

$$\|f(t, x, u)\| \leq a\|x\| + b\|u\|, \tag{18}$$

for all $(t, x, u) \in ([0, \infty) \times X \times U)$. In contrast to these results, as a consequence of the main result, the following complete stabilizability conditions for nonlinear control system (17) can be obtained via the global null-controllability of control system $[A(t), B(t)]$. Let us denote

$$p = \sup_{t\in[0,\infty)} \{\|P(t)\|\}, \beta = \sup_{t\in[0,\infty)} \{\|B(t)\|\}.$$

Theorem 3.3. *Assume that system $[A(t), B(t)]$ is globally null-controllable. Nonlinear control system (17) is completely stabilizable if condition (18) holds with*

$$b < \frac{1}{\beta p^2}, \quad a < \frac{1 - b\beta p^2}{2p},$$

where $P(t)$ is the solution of Riccati equation (8), where $Q(t) = e^{2\delta t}I$. Moreover, the feedback control is given by (15).

4. AN EXAMPLE

Let l_2 denote the Hilbert space of all sequences $x = (x_1, x_2, ...), x_i \in R$ endowed with the norm

$$\|x\| = \Big[\sum_{i=1}^{\infty} x_i^2\Big]^{\frac{1}{2}} < +\infty.$$

Consider an abstract control system in l_2 of the form

$$\dot{x}(t) = A(t)x(t) + A_1(t)x(t-h) + B(t)u(t),$$
$$x(t) = \phi(t), \quad t \in [-h, 0], \tag{19}$$

where $h > 0, x(t), u(t) \in l_2, a > 0$ and

$$A(t) : (x_1, x_2, ...) \to (sin2tx_1, -x_2, -x_3, ...),$$
$$A_1(t) : (x_1, x_2, ...) \to \frac{a}{t+1}(x_1, x_2, ...),$$
$$B(t) : (x_1, x_2, ...) \to (\frac{1}{t+1}x_1, e^{-t}x_2, e^{-t}x_3, ...).$$

Finding the evolution operator $U(t, \tau)$ by solving the operator equation

$$\frac{d}{dt}U(t, \tau) = A(t)U(t, \tau), U(t, t) = U(\tau, \tau) = I,$$

we have for all $t \geq \tau$:

$$U(t, \tau) : (x_1, x_2, ...) \to (e^{cos^2\tau - cos^2 t}x_1, e^{-(t-\tau)}x_2, e^{-(t-\tau)}x_3, ...),$$

It is obvious that the evolution operator $U(t, \tau)$ is not exponentially stable, however for every $x = (x_1, x_2, ...) \in l_2$ and $T > 0$ we have

$$\int_0^T \|B^*(\tau)U^*(T, \tau)x\|^2 d\tau \geq c\Big[e^{-2cos^2 T}x_1^2 + e^{-2T}\sum_{i=2}^{\infty} x_i^2\Big],$$

$$\|U^*(T, 0)x\|^2 = e^{-2cos^2 T}x_1^2 + e^{-2T}\sum_{i=2}^{\infty} x_i^2.$$

Therefore, for any $T \geq 1$, the condition of Proposition 2.1 holds with $c = 1/2$, and the system $[A(t), B(t)]$ is globally null-controllable in time T. Let $\delta > 0$ be given number and let $P(t) \in BC([0, \infty), l_2)$ be the solution of Riccati equation

$$\dot{P} + (A+\delta I)^*P + P(A+\delta I) - e^{2\delta t}PBB^*P + 3I = 0.$$

Verifying the condition (16): $a \leq \frac{e^{-\delta h}}{2p}$, the linear control system (19) is then completely stabilizable by the feedback control (15).

5. CONCLUSIONS

We have presented in this paper complete stabilizability conditions for infinite-dimensional time-varying control systems with multiple state delays. The conditions are obtained from the solution of Ricatti differential equations and do not involve any stability property of the evolution operator. An algorithm to find feedback controller via solving the Riccati equation was proposed. The problem of finding the solution of abstract Riccati equations is still difficult and complicated, however various efficient approaches to this problem can be found, for instance, in (Gibson, 1983; Oostveen and Curtian, 1998).

The stabilization conditions obtained in this paper are based on the global controllability of the underlying control system. It is well known that the constrained controllability problem has been extensively considered (see, e.g., Phat, 1996; Smirnov, 1996 and references therein), however the stabilizability question of systems with constrained controls is still far from being solved. An interesting question that remains open related to this issue is the following.

Consider control system (1), where $u(t)$ is constrained in some given subset, e.g., $u(t) \in \Omega \subset U$.

Assume that this system with constrained controls is globally null-controllable. The constrained stabilization problem is to find a feedback control operator $K(t)$ such that the control $u(t) = K(t)x(t) \in \Omega$ completely stabilizes system (1). The same question is proposed for nonlinear control system (17). Preliminary results for this problem related to finite-dimensional control systems with special constrained control sets Ω may be found in (Phat, 1996; Smirnov, 1996; Sun et al., 1998).

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REFERENCES

Ahmed N.U. (1990), *Elements of Finite-dimensional Systems and Control Theory.* Pitman SPAM,Longman Sci. Tech. Publ., vol. 37.

Bensoussan A., G.Da Prato, M.C. Delfour and S.K. Mitter (1992), *Representation and Control of Infinite-dimensional Systems.* vol. II, Birkhauser.

Benabdallah A. and M.A. Hammami (2001), On the output feedback stability for nonlinear uncertain control systems. *Int. J. Control,* vol.74, 547-551.

Curtain R.F. and H. Zwart (1995), *An Introduction to Infinite-Dimensional Linear Systems Theory.* Springer-Verlag, New York.

Gibson J.S. (1983), Infinite-dimensional Riccati equations and numerical approximations. *SIAM J. Contr. Optim.,* vol.21, 95-139.

Ikeda M., H. Maeda and S. Kodama (1972), Stabilization of linear systems. *SIAM J. on Control,* 10, 716-729.

Kalman R.E. (1960), Contribution to the theory of optimal control. *Bol. Soc. Mat. Mexicana,* vol.2, 102-119.

Megan G. (1975), On the stabilizability and controllability of linear dissipative systems in Hilbert spaces. *S.E.F., Universitate din Timisoara,* vol.32.

Oostveen J.C. and R.F. Curtain (1998), Riccati equations for strongly stabilizable bounded linear systems. *Automatica,* vol.34, 953-967.

Phat V.N., Dieu T.C. (1992), Constrained controllability of linear systems with constrained controls and states in Banach spaces. *SIAM J. Contr. Optim.,* vol.30, 311-1319.

Phat and T.T. Kiet (1999), On the stabilizability of nonlinear systems in finite-dimensional spaces. *J. of Nonl. Funct. Anal. Appl.*, vol.4, 73-85.

Phat V.N. (1996), *Constrained Control Problems of Discrete Processes.* World Scientific, Singapore.

Pertesen I.R., V.A. Ugrinovskii and A.V. Savkin (2000), *Robust Control Design Using H-infinity Methods.* Springer-Verlag, London.

Savkin A.V. and I.R. Petersen (1999), Weak robust controllability and observability of uncertain linear systems. *IEEE Trans. Aut. Contr.,* vol.44, 1037-1041.

Slemrod M. (1974), A note on complete controllability and stabilizability for linear control systems in Hilbert spaces. *SIAM J. on Control,* vol.12, 500-508.

Smirnov G.V. ()1996), Stabilization by constrained controls. *SIAM J. Contr. Optim.*, vol.34, 1616-1649.

Sun Y.J., C.H Lien and J.G. Hsieh (1998), Global exponential stabilization for a class of uncertain nonlinear systems with control constraints. *IEEE Trans. Aut. Contr.,* vol.43, 67-70.

Wonham W.M. (1967), On pole assignment in multi-input controllable linear systems. *IEEE Trans. Aut. Contr.,* vol.12, 660-665.

Zak (1990), On the stabilization and observation of nonlinear uncertain dynamic systems. *IEEE Trans. Aut. Contr.,* vol.35, 604-607