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QUASI-PERIODIC PHENOMENA AND PHASE-LOCKED ORBITS IN DC-DC BOOST SWITCHING REGULATORS

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Abstract: Quasi-periodic route to chaos is a common feature in DC-DC switching converters like the Boost and the Buck-Boost. A periodic orbit bifurcates into a T^2 torus through a Neimark-Sacker bifurcation, and then, quasi-periodic behavior is obtained. Further changes in the parameters may breakdown such torus resulting in chaotic behavior. Analytical expressions are deduced for the stability character of the orbits, although numerical simulations and experimental measurements are also provided to reinforce the theory. *Copyright ©2002 IFAC*

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1. INTRODUCTION

Elementary switching converters such as buck, boost and buck-boost are second order systems since they have two energy storage elements. Therefore, for any given switching condition, two first order differential equations are required to describe the total behavior of the system. The output voltage regulation is achieved by a control circuit which forces the system to switch between two basic linear configurations. These regulators are piecewise linear and therefore, between switches, we can obtain exact closed-form expression of the system trajectories. The switching instants are to be found numerically as they are solutions of transcendental equations. Possible non-periodic switching between two different configurations makes the system time dependent. Thus, the system is *non autonomous*, which as we know, requires one more state space dimension. For some feedback systems, such as PWM dc-dc converters, the switching time depends nonlinearly on the history of the state variables themselves. Therefore, we have effectively a nonlinear system. Hence this kind of piecewise linear model, at least in principle, satisfies the requirements for chaos. Chaotic behavior and nonlinear phenomena in PWM dc-dc basic power electronic regulators have been extensively studied in the last years. Several kinds of bifurcational behaviors have been found in the elementary converters with different control schemes. Period doubling route chaos is investigated for the buck converter working in the continuous conduction mode both experimentally and numerically (Hamill, *et al.*, 1992). Mathematical analysis of 1-periodic, 2-periodic and high

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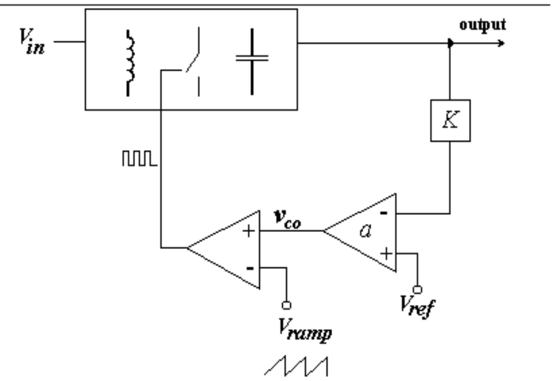

Fig. 1. Block diagram of a Dc-Dc switching regulator

periodic orbits for the same converter is presented in (Fossas and Olivar, 1996) and (di Bernardo, *et al.*, 1997). Flip bifurcations and period doubling route to chaos were also found in the buck and the boost converters working in the discontinuous conduction mode (Tse, 1994a; Tse, 1994b). More recently, Neimark-Sacker bifurcation and quasi-periodicity route to chaos were found to occur in the PWM boost and buck-boost converters (El Aroudi, *et al.*, 1999; El Aroudi, *et al.*, 2000) and border collision bifurcations were reported in buck and boost converters with different control strategies (Yuan, *et al.*, 1998; Banerjee, *et al.*, 2000). The aim of this paper is to investigate in the parameter space the mechanisms of loss of stability of the 1-periodic orbit when it experiences a Neimark-Sacker bifurcation that can explain the quasi-periodic route to chaos in a PWM voltage controlled dc-dc boost converter. In order to reduce the parameter space, dimensionless formulation will be used. A stabilization strategy will be applied in order to force the system to return to its nominal behaviour (1-periodic) after lossing stability.

2. STATE SPACE MODELLING OF PWM CONTROLLED DC-DC SWITCHING REGULATORS

The block diagram of a PWM controlled dc-dc switching regulator is shown in Fig. 1. Due to the switching action, the system passes through different configurations. We will suppose that the system switches from one configuration to another whenever the control voltage $v_{co} = a(V_{REF} - (k_v v_c + k_i i_L))$ is equal to the $T-$periodic modulating signal $v_{ramp} = V_L + (V_U - V_L)(t/T) \bmod 1$. The system switches from one phase to the other one whenever the function $v_{co} - v_{ramp}$ changes its sign in such a way that $v_{co} > v_{ramp}$ in the ON configuration (i.e., the switch is ON) and $v_{co} < v_{ramp}$ in the OFF configuration (the switch is OFF). Therefore the switching condition is $v_{co} - v_{ramp} = 0$. The system switches between the two phases at time instants nT ($n \in \mathbf{N}$) and at the switching instant, within a PWM period, for which $v_{co} = v_{ramp}$. When the inductor current becomes zero during the OFF phase, the discontinuous conduction mode (DCM) takes place. The system switches from the configuration OFF to a third configuration ($OFF^{'}$). Let us define the dimensionless variables and parameters:

- Variables

$$v(t) = \frac{v_C(t)}{V_{IN}},$$
$$i(t) = \frac{\sqrt{L/C}}{V_{IN}} i_L(t),$$
$$\tau = \frac{t}{2\pi\sqrt{LC}}$$

- Parameters

$$Q = \frac{R}{\sqrt{L/C}}, \quad Q_S = \frac{\sqrt{L/C}}{R_S},$$
$$V_R = \frac{V_{REF}}{k_v V_{IN}} - \frac{V_U + V_L}{2Ak_v V_{IN}},$$
$$V_D = \frac{V_U - V_L}{Ak_v V_{IN}},$$
$$Z = \frac{Ak_i/k_v}{\sqrt{L/C}}, \quad T_N = \frac{T}{2\pi\sqrt{LC}}$$

In the dimensionless formulation, the system switches from one phase to the other one whenever the difference function $v_{con} - v_r$ changes its sign in such a way that $v_{con} < v_r$ in the ON configuration and $v_{con} > v_r$ in the OFF configuration, where, $v_{con} = v + Zi$ and $v_r = V_R + V_D/2 - V_D(\tau/T_N) \bmod 1$ are the new (dimensionless) control voltage and sawtooth modulating signal respectively. Therefore the switching condition is $v_{con} - v_r = 0$. The DCM takes place when the dimensionless inductor current becomes zero. During each phase the dynamic of the system is described by:

$$\dot{x} = Ax + B \quad (1)$$

where $x = (v, i)^{'}$ is the vector of the dimensionless state variable and the overdot stands for derivation with respect to dimesionless time τ ($\dot{x} = dx/d\tau$). The solution during each phase interval is available and takes the following form:

$$x(\tau) = e^{A(\tau-\tau_S)}x(\tau_S) + \int_{\tau_S}^{\tau} e^{A(\tau-\sigma)}Bd\sigma \quad (2)$$

which can be further written as:

$$x(\tau) = \Phi(\tau - \tau_S)x(\tau_S) + \Psi(\tau - \tau_S) \quad (3)$$

where $\Phi(\tau) = e^{A\tau}$, $\Psi(\tau) = A^{-1}(\Phi(\tau) - I)B$, I is the identity matrix and τ_S is the instant at which the system switches from one configuration to another. Namely, the system switches between the two phases at time instants nT_N ($n \in \mathbf{N}$) and at the switching instant τ_n, within a PWM period, for which the control voltage crosses the modulating signal v_r. $x(\tau_S)$ is the state vector at the switching instant.

Although the dynamic behavior in each linear topology is easy to obtain, the closed-loop dynamics are quite complicated, sometimes including periodic, quasi-periodic and chaotic behavior, this depending on the initial conditions of the state variables and the values of the parameters.

3. DICRETE-TIME MODELLING AND STABILITY ANALYSIS: THE POINCARÉ MAP

In order to simplify the structure of the phase space and reduce the dimension of the system, the Poincaré section is frequently used. It is constructed by viewing the intersection between the trajectory of the dynamical system and a certain cut plane. In the case of periodically driven systems as power electronic converters, a plane Σ in the cylindrical space is considered. The trajectory of the system intersects Σ every period of the driving signal. A map P which relates two successive points in the Poincaré section can be defined. Such a map is called the Poincaré map. Viewing the state of the system each driving period is similar to the action of a stroboscope flashing with the same period of the driving signal. For this reason, this map is called also the *stroboscopic* map. Under some operating conditions, it is more suitable to use another kind of discrete-time modelling for PWM power electronic converters (Di Bernardo and Vasca, 2000). In this paper we will use only the stroboscopic map.

$$\begin{aligned} P : \Sigma &\mapsto \Sigma \\ x_n &\mapsto x_{n+1} \quad n = 1, 2, ... \end{aligned}$$

where $x_n = (v(nT_N), i(nT_N))'$ is the value of the state vector at the instants nT_N.

Using the solutions of the state equations of each linear phase and linking them at the switching instants this map can written as:

$$\begin{aligned} x_{n+1} &\overset{def}{=} f(x_n, \tau_n) \\ &= \Phi_2(T_N - \tau_n)[\Phi_1(\tau_n)x_n + \Psi_1(\tau_n)] \\ &+ \Psi_2(T_N - \tau_n) \end{aligned} \quad (4)$$

The fixed points x^* of this map can be obtained by enforcing periodicity: $x_{n+1} = x_n = x^*$.

$$\begin{aligned} x^* &= [I - \Phi_2(T_N - \tau^*)\Phi_1(\tau^*)]^{-1} \\ &\cdot[\Phi_2(T_N - \tau^*)\Psi_1(\tau^*) + \Psi_2(T_N - \tau^*)] \end{aligned} \quad (5)$$

where τ^* is the switching instant corresponding to the fixed point x^* which is given by the following equation

$$\begin{aligned} g(x^*, \tau^*) &\overset{def}{=} \gamma[\Phi_1(\tau^*)x^* + \Psi_1(\tau^*)] - v_r(\tau^*) \\ &= 0 \end{aligned} \quad (6)$$

where $\gamma = (1, Z)$ is the dimensionless gain vector corresponding to k.

3.1 Floquet (characteristic) multipliers for stability analysis of periodic solutions

For periodic solutions, which correspond to fixed points of the Poincaré map P, the stability is determined by their Floquet (or characteristic) multipliers m_i which are the eigenvalues of the linearized map DP at the fixed point x^*. Near this fixed point, the local dynamics are governed by:

$$\delta x_{n+1} = DP(x^*)\delta x_n$$

For PWM switching regulators, τ_n is given by the switching equation.

$$g(x_n, \tau_n) = 0$$

The stroboscopic map P is nonlinear in x_n and its corresponding Jacobian matrix DP is given by the following equation

$$\begin{aligned} DP &= \frac{\partial f}{\partial x_n} - \frac{\partial f}{\partial \tau_n}(\frac{\partial g}{\partial \tau_n})^{-1}\frac{\partial g}{\partial x_n}\bigg|_{(x_n,\tau_n)=(x^*,\tau^*)} \\ &= \Phi_2(T_N - \tau^*) \\ &\cdot[I - \frac{((C_1 - C_2)x^* + (D_1 - D_2))\gamma}{\gamma(C_1 x^* + D_1) + V_D/T_N}]\Phi_1(\tau^*) \end{aligned} \quad (7)$$

where τ^* is the switching time corresponding to the fixed point x^*. The eigenvalues of the DP (*Floquet or the characteristic multipliers*) govern the stability of the system. A necesary and sufficient condition for stability is that all the characteristic multipliers has a modulus smaller than one ($|m_i| < 1$). If one of them cross the unite circle, the system losses its stability.

4. BIFURCATION PHENOMENA OBTAINED BY COMPUTER SIMULATION AND EXPERIMENTAL MEASUREMENTS

A bifurcation diagram for the boost converter is computed when the period T_N is swept in the range $(0.04, 0.34)$ with the following values of the

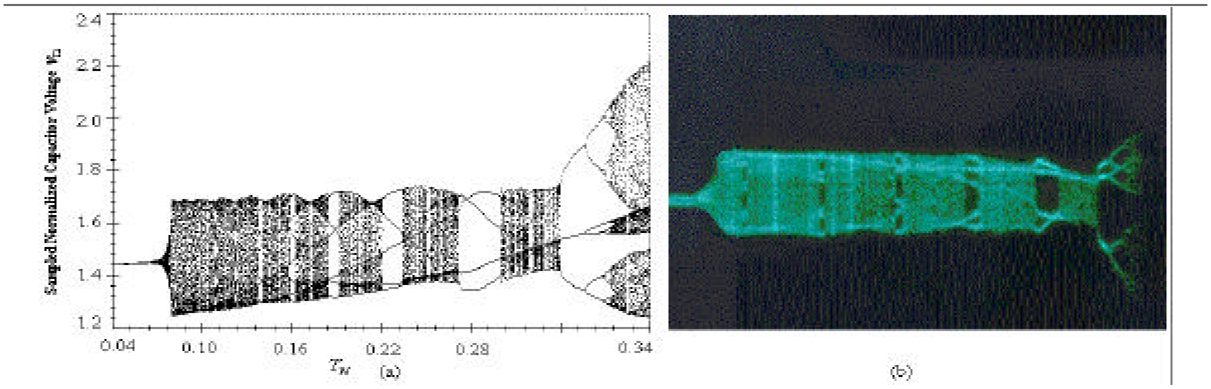

Fig. 2. Bifurcation diagram taking T_N as bifurcation parameter (a) computer simulation, (b) experimental mesurements

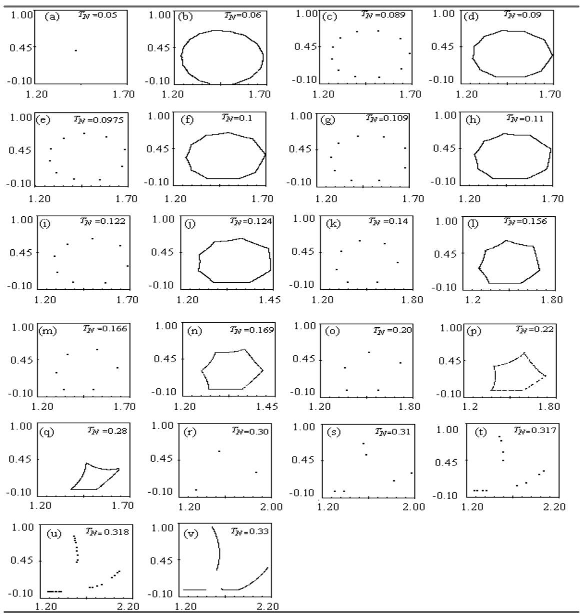

Fig. 3. Evolution of the attractor of the boost converter when T_N is taken as bifurcation parameter. Horizontal axis v, vertical axis i

remaining parameters $Q = 5.23$, $Q_S = 6.5$, $V_D = 1.4$, $V_R = 1.2$ and $Z = 0$ (voltage mode control). The result is shown in Fig. 2-a. This bifurcation diagram is also obtained experimentally and the result is represented in Fig. 2-b. Figure 3 shows the evolution of the stroboscopic map attractor when T_N is varied.

A summary of the bifurcations obtained follows: For T_N · 0.05, the attractor is a periodic orbit with small voltage ripple and the conduction mode in this case is continuous for each cycle.

Fig. 3a shows the fixed point of the Poincaré map at $T_N = 0.05$; the transient state near this fixed point indicates that it is a stable focus. This may be confirmed by the fact that the characteristic multipliers are complex conjugates with modulus smaller than one. For Fig. 3a, the characteristic multipliers are $m_\pm = 0.946 \pm j0.319$. Their modulus being $\mid m_\pm \mid$· 1.

By increasing T_N, a Neimark-Sacker bifurcation of the stroboscopic map occurs at a certain value of T_N between 0.05 and 0.06. The attractor becomes a torus T^2, and the system operation begins to fluctuate quasi-periodically between continuous and discontinuous conduction mode. If we sample stroboscopically (once per ramp cycle) the trajectory in the phase plane (v, i), we obtain an infinite set of points that, when the steady state is reached, belongs to an invariant closed curve (Fig. 3b). This ring-like structure is a typical signature of the incommensurability between the two frequencies acting on the dynamics and, therefore, of the quasi-periodic behavior. For $T_N = 0.06$ (Fig. 3b), the fixed point is a spiral source (unstable focus) and its characteristic multipliers are $0.925 \pm 0.380j$. With further variation of T_N, the Poincaré section is a discrete set of p points ($p \in \mathbf{N}$) and this behavior corresponds to the phase-locking regime (Fig. 3c). The period p of this frequency-locking behavior decreases when T_N increases (Fig. 3). For relatively large values of T_N, the attractor appears to be a torus whose Poincaré section showing a pseudo-symmetry of order equal to the period of the phase-locking periodic orbit that existed previously (Fig. 3). This scenario is repeated until the breakdown of the alternation between the quasi-periodicity and the phase-locking ($T_N \simeq 0.304$). Then, another scenario appears: a period doubling cascade route to chaos begins with a periodic orbit of period 3, and finishes by culminating in a 3-piece chaotic attractor.

5. TWO-DIMENSIONAL BIFURCATION DIAGRAM AND ARNOLD TONGUES

In order to show the bifurcation structure when varying at the same time both T_N and V_R, we

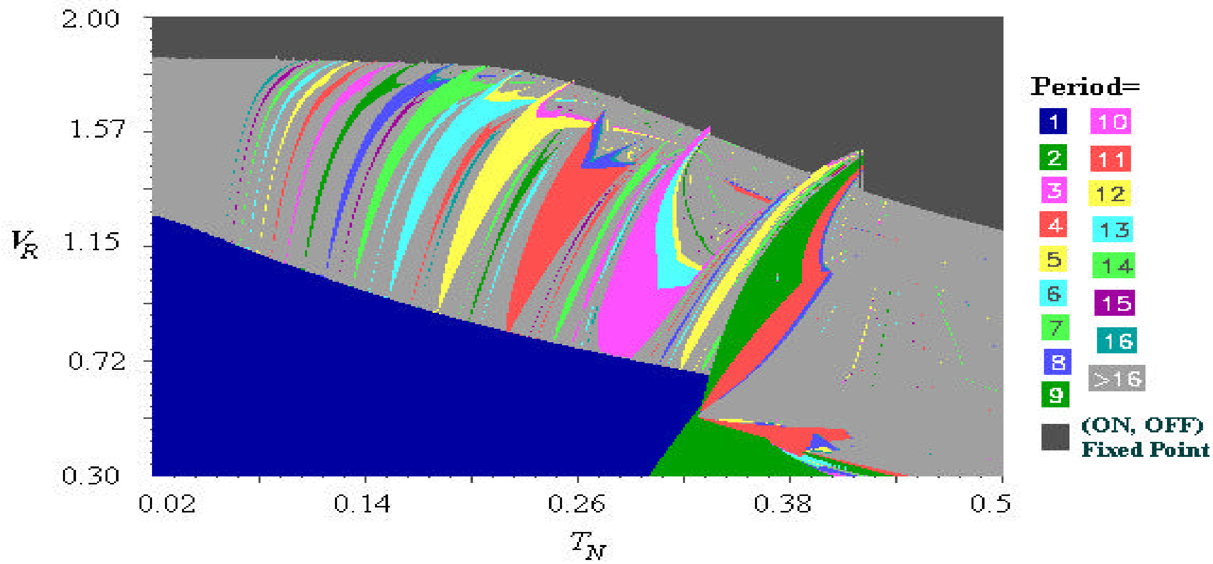

Fig. 4. 2-D bifurcation diagram of the boost converter showing Arnold tongues

have plotted the two-dimensional bifurcation diagrams corresponding to these parameters; the result is shown in Fig. 4. The two-dimensional bifurcation diagram is color-coded depending on the periodicity of the attractor. Due to possible coexisting attractors, only one of the attractors can be identified at each point of the parameter space.

6. STABILIZATION OF PERIODIC ORBIT

When the converter is working in the quasi-periodic or in the chaotic regime, there may exist unstable periodic orbits that can be stabilized. In this section we propose a control scheme to stabilize 1-periodic orbits in the Boost converter. The method utilizes a cycle by cycle variable peak to peak ramp voltage rather than a constant one. Varying the peak to peak voltage is equivalent to varying V_U. We propose the following law to vary V_U.

$$v_{U,n} = V_U - g_v(v_n - v^*) \qquad (8)$$

where g_v is a feedback factor. Note that when the steady state is reached by the system ($v_n = v^*$), $v_{U,n} = V_U$. Therefore the nominal operation point is not alterated by the control law introduced in Eq. (8). Psim is used in order to apply this control scheme to a Boost converter. The circuit simulated is shown in Fig. 5. The values of the parameters used are: $R = 68$- , $L = 5.4mH$, $R_S = 2$- , $C = 32\mu F$, $V_{ref} = 12V$, $V_L = 2V$, $V_U = 9V$, $k_v = 1$, $k_i = 0$, $a = 1$, $T = 400\mu s$ and $V_{IN} = 5V$. With these values the fixed point of the stroboscopic map is $x^* = (v^*, i^*) = (7.7992, 0.1151)$ and its characteristic multipliers are $\lambda_\pm = 0.6106 \pm 0.8601j$ wich corresponds to an unstable focus, and the dynamics of the system is a quasiperiodic regime. When the control scheme introduced in Eq. (8) is activated the fixed point becomes stable and the the periodic orbit is reached after a few cycles of transitory (Fig. 6).

7. CONCLUSIONS

In this paper we have shown that quasi-periodic route to chaos is a common feature in DC-DC power electronic circuits, specifically, in Boost switching regulators. Periodic orbits bifurcates to a quasiperiodic attractor (torus). Such a torus may breakdown to give chaotic behavior. One-dimensional bifurcation diagrams show that there are alternating windows of periodic orbits (phase-locked). Two dimensional bifurcation diagrams show that these phase-locked produce in the parameter space the so called *Arnold tongues*. Different numerical tools were used to study the complicated phenomena in the system. A control strategy is proposed to stabilize a periodic orbit after losing stability by Neimark-Sacker bifurcation, and Psim simulations show that it can work succesfully. Experimental realization of these controllers could be interesting for power electronic regulation and will be investigated in further works.

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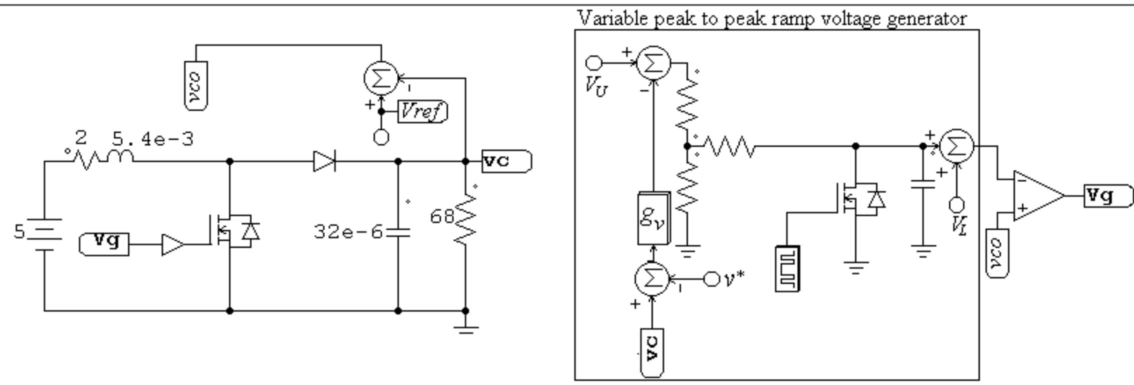

Fig. 5. Psim simulated Boost converter under PWM control with variable peak to peak ramp voltage

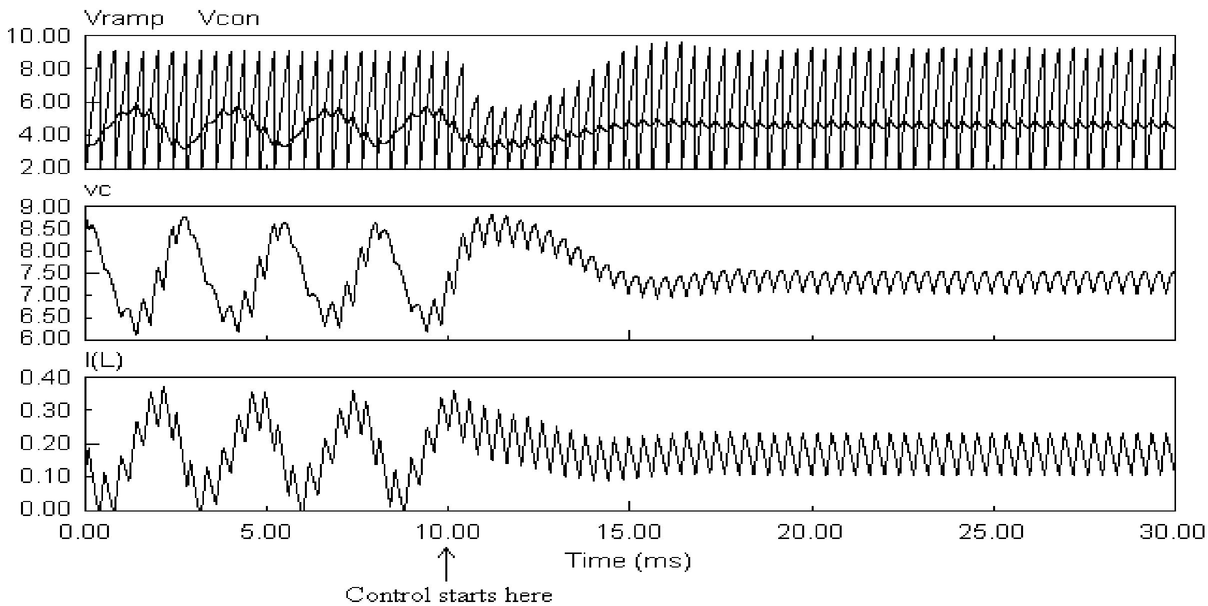

Fig. 6. Stabilization of the 1-periodic orbit when the converter is working in the quasiperiodic regime

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