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IMPROVING TRANSIENT WITH COMPENSATION LAW

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Abstract: In the synthesis of tracking control systems, the compensation signal, which is applied in the finite-horizon time, is effective for improving the performance of controlled system. In this paper, a design method of finite-horizon compensation signal and optimal internal state of controller are discussed for stabilized systems. By characterizing the singular-value problem for correspondingly defined Hankel operator, it is shown that the internal state and the compensation signal, which attains favorable transient, is constructively given based on the combination of singular vectors. The strength and the limitation of applying the compensation signal are illustrated with numerical examples.

Keywords: compensation law, initial value setting, servo-mechanism

1. INTRODUCTION

In the synthesis of tracking control systems, the compensation signal and initial value setting, which are applied in the finite-horizon time, are effective for improving the performance of controlled system. In this view point, various compensation method are discussed: a design method of preview compensation signal for attenuating the error to a target signal(Hayase, *et al.*, 1969; Tomizuka, 1975), a feed-forward compensation and an initial value of controller (Ikeda, *et al.*, 1988; Egami, *et al.*, 1990).

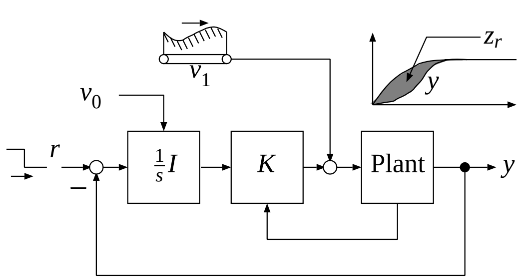


Fig. 1. Servo-mechanism with compensation

In this paper, a design method of finite-horizon compensation signal and optimal internal state of controller is discussed for stabilized systems. Clarifing the relation of the input/output signal and internal state of the system, we provide a design method of the compensation signal for the tracking problem. Let us first highlight out problem with a simplified problem (Fig. 1). In the servo system Fig. 1, r denotes the reference signal and y denotes the output response driven by r. In order to obtain required output transient y_r, it is effective to set the initial value v_0 of integrator and introduce a compensation input v_1 together with the reference signal r. The generalized design problem of compensation signals is depicted in Fig. 2. The objective here is to design (v_0, v_1) in order to generate the response z_r. If the relation of the compensation input (v_0, v_1) and z_r is clarified, the calculation of (v_0, v_1) is possible to generate the required output response z_r.

In the following, we clarify the relations between compensation input (v_0, v_1) and the output z based on the input-output mapping, then we derive the design method of compensation law. In Fig. 1, v_0 denotes the internal state of controller and v_1 denotes preview compensation input in the finite-horizon. Finally, we investigate the effectiveness of the compensation law with numerical examples.

2. FORMULATION

Define the controlled system Σ (Fig. 2):

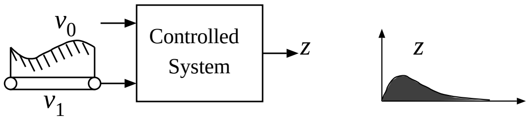


Fig. 2. System with compensation

$$\Sigma: \begin{cases} \dot{x}(t) = Ax(t) + Dv(t) \\ z(t) = Ex(t) \end{cases}, \qquad (1)$$

$$v(\tau) = \begin{cases} v_1(\tau) & 0 \le \tau \le h \\ 0 & h < \tau \end{cases},$$

where $v_1(t) \in L_2(0,\ h; R^m)$ is a compensation input in the finite-horizon $[0,h]$, $x(t) \in R^n$, $z(t) \in R^p$ are the state, the output respectively. The initial value of system Σ is described by

$$x(0) = Vv_0,\ V \in R^{n\times r}, \qquad (2)$$

where $v_0 \in R^r$ is an operatable part of the state. We make following assumptions for matrices A, D, E.

(A1) A is stable.

(A2) (E, A, D) is observable and controllable.

We define the compensation law $(v_0,\ v_1)$ in Hilbert space $R^r \times L_2(0,h;R^m)$. $v_0 \in R^r$ is an operatable initial state and $v_1 \in L_2(0,h;R^m)$ is compensation input.

By introducing the compensation signal $(v_0,\ v_1)$, the system response of (1) can be improved in the following points:

(1) transfer the state $x(h)$ to obtain the required response

(2) reform the response $z_{[0,h]}$ in $[0,h]$

So we define the required response by

$$\hat{z} := \begin{bmatrix} Fx(h) \\ z_{[0,h]} \end{bmatrix} \in \mathscr{Z}, \qquad (3)$$

$$\mathscr{Z} := R^n \times L_2(0,h;R^p).$$

$\mathscr{Z} := R^n \times L_2(-h,0;R^l)$ is a Hilbert space endowed with the inner product:

$$\langle \psi, \phi \rangle_{\mathscr{Z}} := \psi^{0\mathrm{T}}\phi^0 + \int_0^h \psi^{1\mathrm{T}}(\beta)\phi^1(\beta)\mathrm{d}\beta, \quad (4)$$

$$\psi = \begin{bmatrix} \psi^0 \\ \psi^1 \end{bmatrix},\ \phi = \begin{bmatrix} \phi^0 \\ \phi^1 \end{bmatrix} \in \mathscr{Z},$$

where $F \in R^{n\times n}$ is defined by the solution $M =: F^{\mathrm{T}}F$ to the equation

$$MA + A^{\mathrm{T}}M + E^{\mathrm{T}}E = 0. \qquad (5)$$

Since M is an observability gramian of the system (1), the equality

$$\|\hat{z}\|^2_{\mathscr{Z}} = \int_0^h z^{\mathrm{T}}(t)z(t)\mathrm{d}t + x^{\mathrm{T}}(h)Mx(h)$$

$$= \int_0^\infty z^{\mathrm{T}}(t)z(t)\mathrm{d}t = \|z\|^2_{L_2(0,\infty;R^p)},$$

holds for the evaluation of system response.

In the following, we denote the required response by $\hat{z}_r \in \mathscr{Z}$ and clarify the design method of compensation law which satisfies the condition:

$$\forall \varepsilon > 0:\ \|\hat{z}_r - \hat{z}\|_{\mathscr{Z}} \le \varepsilon. \qquad (6)$$

3. COMPENSATION LAW AND OUTPUT

In this section, we elaborate the relation between the compensation law and the response based on input-output mapping. The compensation law and the output are expressed by the singular values/vectors of Hankel operator. In the same manner to (3),(4), we denote the compensation law of system Σ by

$$\hat{v} := \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} \in \mathscr{V},\ \mathscr{V} := R^r \times L_2(0,h;R^m), \quad (7)$$

and define the Hilbert space $\mathscr{V}$ endowed with the inner product:

$$\langle \hat{\psi}, \hat{\phi} \rangle_{\mathscr{V}} := \hat{\psi}^{0\mathrm{T}}\hat{\phi}^0 + \int_0^h \hat{\psi}^{1\mathrm{T}}(\beta)\hat{\phi}^1(\beta)\mathrm{d}\beta \quad (8)$$

$$\hat{\psi} = \begin{bmatrix} \hat{\psi}^0 \\ \hat{\psi}^1 \end{bmatrix},\ \hat{\phi} = \begin{bmatrix} \hat{\phi}^0 \\ \hat{\phi}^1 \end{bmatrix} \in \mathscr{V}.$$

The relation between $\hat{v}$ and $\hat{z}$ are described by

$$\hat{z} = \Gamma\hat{v},\ \Gamma \in \mathscr{L}(\mathscr{V}, \mathscr{Z}), \qquad (9)$$

$$\begin{bmatrix} (\Gamma\hat{v})^0 \\ (\Gamma\hat{v})^1(\xi) \end{bmatrix}$$

$$:= \begin{bmatrix} F\left(e^{Ah}Vv^0 + \int_0^h e^{A(h-\beta)}Dv^1(\beta)\mathrm{d}\beta\right) \\ E\left(e^{A\xi}Vv^0 + \int_0^\xi e^{A(\xi-\beta)}Dv^1(\beta)\mathrm{d}\beta\right) \end{bmatrix}, \quad (10)$$

$$\hat{v} = (v^0, v^1) \in \mathscr{V} \quad (0 \le \xi \le h).$$

We first provide a solution to the singular value problem of Γ and clarify the relation between the achievable output and compensation law. For the operator $\Gamma \in \mathscr{L}(\mathscr{V}, \mathscr{Z})$, the singular value $\sigma > 0$ and vectors (f, g)

$$\sigma g = \Gamma f,\ \sigma f = \Gamma^* g,\ f \ne 0,\ g \ne 0. \qquad (11)$$

are obtained by the following theorem.

Theorem 1. The singular value σ_i of Hankel operator Γ is given by the roots of transcendental equation

$$\det\left\{\begin{bmatrix} -\sigma^{-1}M & I \end{bmatrix} e^{J(\sigma)h} \begin{bmatrix} \sigma^{-1}W \\ I \end{bmatrix}\right\} = 0, \quad (12)$$

$$J(\sigma) := \begin{bmatrix} A & \sigma^{-1}DD^{\mathrm{T}} \\ -\sigma^{-1}E^{\mathrm{T}}E & -A^{\mathrm{T}} \end{bmatrix},$$

$$W := VV^{\mathrm{T}},$$

where matrix $M > 0$ is the solution to (5). And further, corresponding singular vectors (f_i, g_i) are given by

$$f_i^0 = V^{\mathrm{T}}u_i, \quad (13)$$

$$f_i^1(\beta) = \begin{bmatrix} 0 & D^{\mathrm{T}} \end{bmatrix} e^{J(\sigma_i)\beta} \begin{bmatrix} \sigma_i^{-1}W \\ I \end{bmatrix} u_i, \quad (14)$$

$$0 \le \beta \le h,$$

$$g_i^0 = \begin{bmatrix} F & 0 \end{bmatrix} e^{J(\sigma_i)h} \begin{bmatrix} \sigma_i^{-1}W \\ I \end{bmatrix} u_i, \quad (15)$$

$$g_i^1(\xi) = \begin{bmatrix} E & 0 \end{bmatrix} e^{J(\sigma_i)\xi} \begin{bmatrix} \sigma_i^{-1}W \\ I \end{bmatrix} u_i, \quad (16)$$

$$0 \le \xi \le h,$$

where nonzero vector $u_i \neq 0$ is defined by

$$\begin{bmatrix} -\sigma_i^{-1}M & I \end{bmatrix} e^{J(\sigma_i)h} \begin{bmatrix} \sigma_i^{-1}W \\ I \end{bmatrix} u_i = 0. \quad (17)$$

■

The proof of Theorem 1 is found in (Izumi, *et al.*, 1997).

The singular values and vectors of operator Γ clarify the relations between input and output signals. In order to generate the output $\hat{z} = g_i$, the input $\hat{v} = 1/\sigma_i \cdot f_i$ is required by (11). In Theorem 2, we show the characteristics of Γ. The proof is omitted.

Theorem 2. The operator $\Gamma \in \mathscr{L}(\mathscr{V}, \mathscr{Z})$ has following properties.

(a) The singular values $\sigma_1\ \sigma_2 \cdots \sigma_i$ converge to zero, and $\sum_{i=1}^{\infty} \sigma_i^2 < \infty$ where $\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_i \ge \cdots > 0$
(b) $\{f_i\}(f_i \in \mathscr{V})$, $\{g_i\}(g_i \in \mathscr{Z})$ are an orthogonal basis. $\{g_i\}$ is complete in $\mathscr{Z}$.

■

In the sequel, we normalize the singular vectors by $\|f_i\|_{\mathscr{V}} = \|g_i\|_{\mathscr{Z}} = 1$ $(i = 1, 2, \cdots)$

4. THE DESIGN OF COMPENSATION LAW

The input and output signals are decomposed by the singular vectors (f_i, g_i). To obtain the output g_i, $\hat{v} = 1/\sigma_i \cdot f_i$ is required since $\sigma g = \Gamma f$ holds. By evaluating the costs of the input with norm $\|\hat{v}\|_{\mathscr{V}}$, Hence, in order to generate the output $\hat{z}$ which satisfies (6), we can design the signal $\hat{v}$ by the combination of singular vectors. The design method is obtained as follows.

Theorem 3. Let $\alpha_i := \langle \hat{z}_r, g_i \rangle_{\mathscr{Z}}$ $(i = 1, 2, \cdots)$ for given z_r and N is an integer number such that the inequalities (18), (19) hold.

$$\|\hat{z}_r\|_{\mathscr{Z}}^2 - \sum_{i=1}^{N-1} \alpha_i^2 > \varepsilon^2 \quad (18)$$

$$\|\hat{z}_r\|_{\mathscr{Z}}^2 - \sum_{i=1}^{N} \alpha_i^2 \le \varepsilon^2 \quad (19)$$

Then the compensation signal

$$\hat{v} = \sum_{i=1}^{N} \frac{\alpha_i}{\sigma_i} \cdot f_i \quad (20)$$

satisfies the condition (6). ■

Proof *Using the fact the output is given by*

$$\hat{z} = \Gamma\hat{v} = \sum_{i=1}^{N} \alpha_i g_i, \quad (21)$$

we verify that (6) holds. From (21) and Theorem 2,

$$\begin{aligned} \|\hat{z}_r - \hat{z}\|_{\mathscr{Z}}^2 &= \|\hat{z}_r - \sum_{i=1}^{N} \alpha_i g_i\|_{\mathscr{Z}}^2 \\ &= \|\hat{z}_r\|_{\mathscr{Z}}^2 + \sum_{i=1}^{N} \alpha_i^2 - 2\sum_{i=1}^{N} \alpha_i \langle \hat{z}_r, g_i \rangle_{\mathscr{Z}} \\ &= \|\hat{z}_r\|_{\mathscr{Z}}^2 - \sum_{i=1}^{N} \alpha_i^2 \le \varepsilon^2 \end{aligned} \quad (22)$$

is obtained. Therefore

$$\|\hat{z}_r - \hat{z}\|_{\mathscr{Z}} \le \varepsilon \quad (23)$$

holds for given z_r. ■

By Theorem 3, for the case that the controlled system is relaxed, the relations of compensation law and the required output is clarified. In the design of compensation law, we shall pay the attention, because the magnitude of input signals comes larger as many f_i is adopted into the compensation law.

In the next section, we extend the previous results for the case that the controlled system has the initial values.

5. THE CASE OF NONZERO INITIAL VALUE

If we design the compensation law for the case that the controlled system has an initial values, the compensation law can improve the performance of controlled system for the more general case.

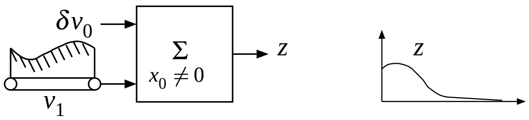


Fig. 3. System with compensation

In this section, we define the controlled system which has the initial value and construct the design method of the compensation law. Consider the controlled system depicted in Fig. 3:

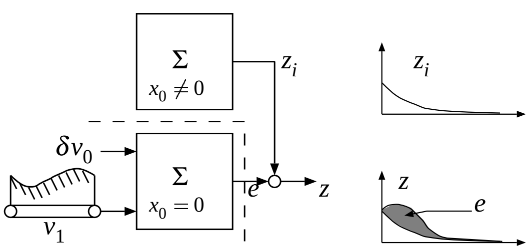
$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Dv(t) \\ z(t) = Ex(t) \end{cases}, \tag{24}$$

$$v(\tau) = \begin{cases} v_1(\tau) & 0 \leq \tau \leq h \\ 0 & h < \tau \end{cases},$$

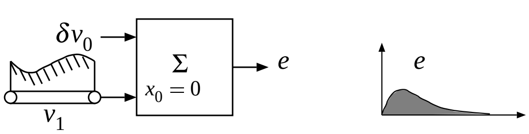
$$x(0) = x_0 + V\delta v_0, \tag{25}$$

$$x(t) \in R^n,\ z(t) \in R^p,\ v(t) \in R^m.$$

where $x(t) \in R^n$ is the state, $z(t) \in R^p$ is the output, $v(t) \in R^m$ is the input for the compensation input $v_1 \in L_2(0,h;R^m)$. In the initial value $x(0)$, x_0 denotes the initial value of the plant and δv_0 denotes the additional change, which is designed to improve the response. The controlled system Fig. 3 is equivalently represented by Fig. 4. In Fig. 4(a), the system in upper block generates the output z_i by the initial value x_0, the system in lower block generates the output e by controller's initial value and compensation input $(\delta v_0, v_1(\cdot))$. Splitting the controlled system Σ by the input and initial value, and re-defining the required output as $e_r = z_r - z_i$, the problem that z pursue z_r is represented by the problem that e pursue e_r.



(a) Step 1



(b) Step 2

Fig. 4. Signal redefinition

Lemma 4. In the case that the system Σ (24) has initial value x_0, the design method of compensation law $\hat{v}$= $(\delta v_0, v_1(\cdot))$ is obtained as follows.

Step 1 Devide the output $\hat{z}$ of Σ into the signals: $\hat{z}_i$ is genearted by x_0 and $\hat{e}$ is generated by $\hat{v}$.

$$\hat{z} = \hat{e} + \hat{z}_i$$

$$\hat{e} = \Gamma \hat{v}$$

$$\hat{z}_i = \begin{bmatrix} Fe^{Ah}x_0 \\ Ee^{A\beta}x_0 \end{bmatrix}$$

Step 2 Along with the representation (3), re-define the required output as $\hat{e}_r$ for the output $\hat{e}$

$$\hat{e}_r = \hat{z}_r - \hat{z}_i$$
$$= \begin{bmatrix} F(x_r(h) - e^{Ah}x_0) \\ z_r(\beta) - Ee^{A\beta}x_0 \end{bmatrix} \in \mathscr{Z} \tag{26}$$

$$\hat{z}_r = \begin{bmatrix} Fx_r(h) \\ z_r(\cdot) \end{bmatrix} \in \mathscr{Z}$$

Step 3 By Theorem 3, the compensation law $\hat{v}$ is obtained by

$$\hat{v} = \sum_{i=1}^{N} \frac{\alpha_i}{\sigma_i} \cdot f_i \tag{27}$$

where $\alpha_i = \langle \hat{e}_r, g_i \rangle_{\mathscr{Z}}$.

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6. NUMERICAL EXAMPLE

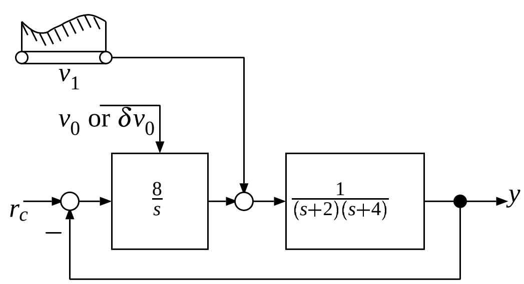

Fig. 5. Servo system

In this section, we investigate the effectiveness of the compensation law for the servo system in Fig. 5(Saito, *et al.*, 1998). The compensation law is designed by Theorem 3, Lemma 4.

Fig. 6 shows the outputs: the required output and step response without the compensation. The required output y_r is described by (28).

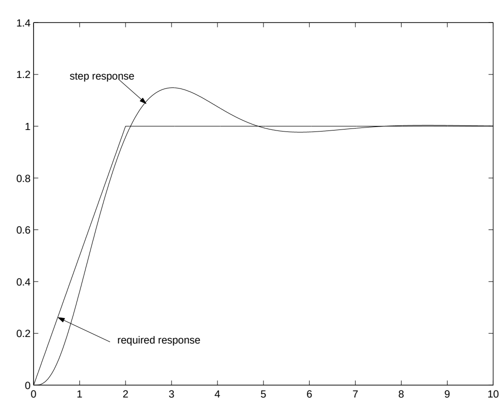

Fig. 6. Step response and required response

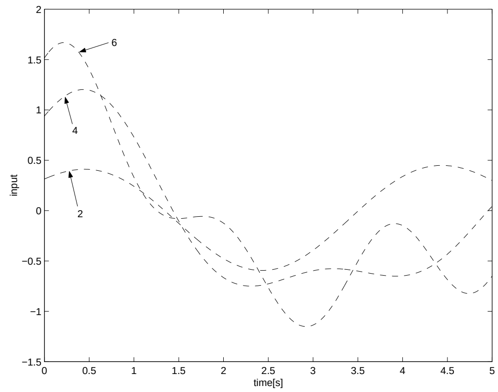

Fig. 7. Compensation inputs $v_1(\cdot)$ (A)

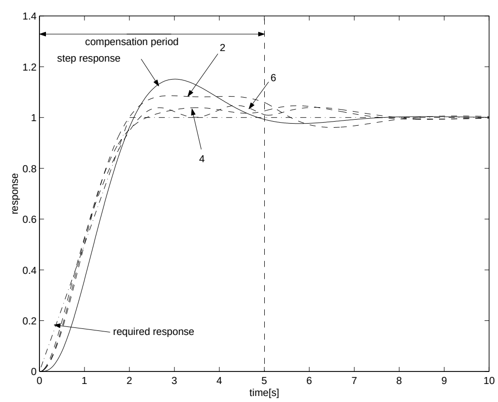

Fig. 8. Responses $y(\cdot)$ (A)

$$y_r(t) = \begin{cases} \dfrac{t}{2} & 0 \le t \le 2 \\ 1 & 2 < t \end{cases} \qquad (28)$$

The servo system in Fig. 5 has the poles -4.65, $-0.68 \pm 1.13j$, the step response rises slowly and has overshoot. For the servo system Fig. 5, we discuss the effectiveness in the folowing two cases.

(A) the compensation input from $t = 0$ to $t = 5$, set the state of integrator at $t = 0$.

(B) the compensation input from $t = 2$ to $t = 5$, set the state of integrator at $t = 2$.

The case (B), where the controller state is reset, is equivalent to the nonzero initial value case in Section 5

For the case (A), the results are shown in Fig. 7, 8. The compensation input v_1 is depicted in Fig. 7, where v_1 is constructed by the $N = 2, 4, 6$ singular vectors. Corresponding to N, the initial state v_0 $(t = 0)$ of the integrator is set to 0.162, 0.100, 0.0971. As the number of the singular vector is increased for 2, 4, 6, the compensation intput v_1 is larger in Fig. 7.

The output y is obtained in Fig. 8. The solid line shows the step response without the compensation, the dash-dotted line is the required response, the dashed lines are with the compensation $(N = 2, 4, 6)$. In Fig. 8, the responses with the compensation rise quickly than the no-compensation case and have low overshoot. As the number of the singular vector increases, the response comes to meet the required response.

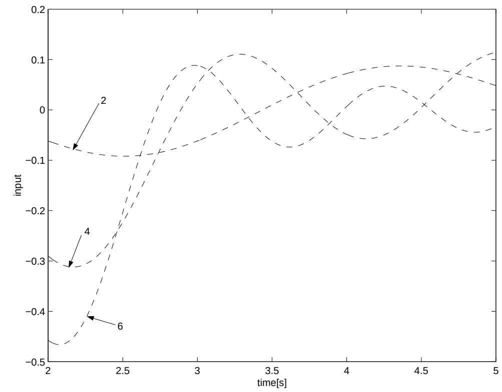

Fig. 9. Compensation inputs $v_1(\cdot)$ (B)

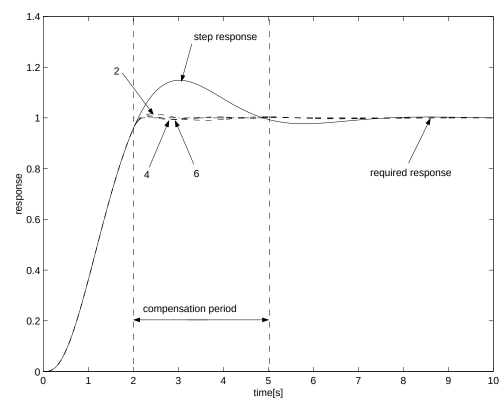

Fig. 10. Responses $y(\cdot)$ (B)

Nextly, we investigate for the case (B). The results are shown in Fig. 9, 10. The compensation input v_1 is in Fig. 9, where v_1 are constructed with 2, 4, 6 singular vectors respectively. Corresponding to N, the initial state δv_0 $(t = 2)$ of the integrator is changed by -0.211, -0.210, -0.208. As the number of the singular vector increases, the compensation intput v_1 become larger in Fig. 9.

The output y is obtained in Fig. 10. The solid line shows the step response without the compensation, the dash-dotted line is the required response, the dashed lines are with the compensation $(N = 2, 4, 6)$. In Fig. 10, the outputs with the compensation are improved from $t = 2$ and have low overshoot.

In the case (B), the response is well improved in spite of short input period compensation. Thus, it is important to discuss the time when the state of the integrator is changed and the time interval of compensation input.

7. CONCLUSION

In this paper, a design method of finite-horizon compensation signal and optimal internal state of controller are discussed for stabilized systems. Firstly, we derive the design mothod of compensation law based on the relation of input/output signal of stabilized system. And the result is extended for the case that the controlled system has a nonzero initial value; to move the state of running plant. For the actual design of compensation law, we should discuss changing the state from the view point of hardware. In the numerical examples, the time of compensation input and moving the state has the large influence. The decision method of the time of compensation input and moving the state should be discussed in the future.

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