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GENERALIZED MINIMUM VARIANCE CONTROL WITH TIME SHIFT MULTIPLICATIONS

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Abstract: This paper describes a Generalized Minimum Variance Control (GMVC) strategy for time varying systems (TVS). If GMVC applies to TVS, be conscious of the time varying multiplication, such as multiplying that comprises of more than two polynomials. In TVS, the time varying multiplication includes the time shift operator. The calculation of time shift multiplication was proposed by Zhen. Li. However, GMVC must be designed for servo systems. Then, the time varying multiplication is composed of double formed time varying multiplication. This paper improves the calculation of the double formed time varying multiplication. Furthermore, the proposed GMVC is verified with the servo characteristic and the noise characteristic. The simulation compares the proposed GMVC with the conventional GMVC for TVS.

Keywords: Coloured Noise, Discrete Time Systems, Disturbance Rejection, Minimum Variance Control, Polynomials, Predictive Control, Servo Systems, Time Delay, Time Varying Systems

1. INTRODUCTION

Predictive control (Åström *et al.*, 1977) is an effective control method for plants including a time delay. Minimum Variance Control (MVC) (Wellstead and Zarrop, 1991), Generalized Minimum Variance Control (GMVC) (Clarke and Gawthrop, 1979) and Generalized Predictive Control (GPC) (Clarke *et al.*, 1987; Clarke and Mohdadi, 1989) are predictive control strategies calculating the output prediction by using polynomials of plant model.

In Time Varying Systems (TVS), these strategies can not achieve the desirable control object, because these strategies have mostly developed in time invariant systems (TIS). Therefore, a control law, which is matched to the time varying polynomials, is necessary. However, by only matching with time varying polynomials that are on-line, the output can not track the reference signal yet. The reason is that the control strategy has time varying multiplications. These multiplications can not accurately be calculated in TVS. O.P.Palsson et. al., proposed a GPC (Palsson *et al.*, 1993; Palsson *et al.*, 1994) that avoids the multiplication by without using the Diophantine equation for TVS. Furthermore, M. Doi and Y. Mori designed a servo GMVC (Doi and Mori, 2000) that based on Palsson's method. On the other hand, Z. Li et. al., proposed MVC (Li *et al.*, 1997; Li and Evans, 1997) whose multiplication can be calculated by providing a time shift operator for TVS.

This paper designs a servo GMVC based on Z. Li's time shift calculation for TVS. In the case that applied Z. Li's method to GMVC, two further problems must be solved: The first problem is that GMVC, based on Z. Li's method for TVS, must calculate a double formed multiplication. The cause of this double multiplication is that a pre-compensator (Takahashi *et al.*, 1998), based on the internal model principle, is installed in the GMVC to eliminate these offsets. However, the double formed multiplication can not be yet calculated by using the Li's method that deals with the single multiplication. The second problem is that the unique solution of the Diophantine equation can not be derived for all arrangements of the polynomials in the Diophantine equation, because the polynomials become noncommutative in TVS (Koivo, 1980).

This paper proposes an algorithm for the double formed multiplication and argues the existence of the Diophantine solution in TVS.

Furthermore, in the case of TVS with coloured noise, the Palsson type GMVC (Doi and Mori, 2000) does not entirely minimize the cost function. The Palsson type GMVC is the predictive method (Palsson *et al.*, 1993; Palsson *et al.*, 1994) without the Diophantine equation. This paper compares the characteristic of coloured noise with the Palsson type GMVC and the proposed GMVC that improved the multiplication. The proposed GMVC can sufficiently minimize the cost function.

2. PROBLEM FORMULATION

This paper adapts a pre-compensator (Takahashi *et al.*, 1998), based on the internal model principle, for the servo systems. In TVS, a Diophantine equation of the conventional form is expressed as follows:

$$C_k(q^{-1})P(q^{-1}) = E_k(q^{-1})A_k(q^{-1}) + q^{-j}F_k(q^{-1}), \tag{1}$$

$$\begin{aligned} E_k(q^{-1}) &= 1 + e_1(k)q^{-1} + \cdots + e_{j-1}(k)q^{-(j-1)}, \\ F_k(q^{-1}) &= f_0(k) + f_1(k)q^{-1} + \cdots + f_h(k)q^{-h}, \\ h &= \max\{n, n_p + l - j\}, \end{aligned}$$

where $E_k(q^{-1})A_k(q^{-1})$ of equation (1) is the time varying multiplication. Z. Li introduced the algorithm (Li *et al.*, 1997; Li and Evans, 1997) for the TVS multiplications.

On the other hand, the Diophantine equation for servo systems is

$$\begin{aligned} &C_k(q^{-1})P(q^{-1}) \\ &= E_k(q^{-1})\Delta A_k(q^{-1}) + q^{-j}F_k(q^{-1}), \end{aligned} \tag{2}$$

which includes a filter ($\Delta = 1 - q^{-1}$). However, the multiplication of equation (2) is double form, such as $E_k(q^{-1})\Delta A_k(q^{-1})$. This paper develops the calculation for the double formed time varying multiplication.

Incidentally, the conventional Diophantine equation for TIS is expressed as follows:

$$\begin{aligned} &C(q^{-1})P(q^{-1}) \\ &= \Delta A(q^{-1})E(q^{-1}) + q^{-j}F(q^{-1}), \end{aligned} \tag{3}$$

which sets the order (Takahashi *et al.*, 1998) as $\Delta, A(q^{-1}), E(q^{-1})$. However, in TVS, this order, shown as equation (3), complicates deriving the unique solutions $E_k(q^{-1})$, $F_k(q^{-1})$ for TVS from this Diophantine equation. Therefore, the Diophantine equation in TVS should be considered the order of polynomials.

3. SERVO GMVC

Consider a Single Input Single Output (SISO) system described by Controlled Auto Regressive and Moving Average (CARMA) model

$$A_k(q^{-1})y(k) = q^{-j}B_k(q^{-1})u(k) + C_k(q^{-1})\xi(k), \tag{4}$$

$$\begin{aligned} A_k(q^{-1}) &= 1 + a_1(k)q^{-1} + \cdots + a_n(k)q^{-n}, \\ B_k(q^{-1}) &= b_0(k) + b_1(k)q^{-1} \\ &\quad + \cdots + b_m(k)q^{-m}, \\ C_k(q^{-1}) &= 1 + c_1(k)q^{-1} + \cdots + c_l(k)q^{-l}, \end{aligned}$$

where $u(k)$ and $y(k)$ are the control signal and the output signal, q^{-j} stands for a delay time of plant and $\xi(k)$ is a white noise with the zero mean and the variance σ^2. In TVS, the coefficients of polynomial of equation (4) are expressed as a function of time k.

The generalized output in the cost function $J = E\{h(k+j)^2\}$ of the servo GMVC, based on the internal model principle, is expressed as follows:

$$\begin{aligned} h(k+j) &= P(q^{-1})y(k+j) - R(q^{-1})w(k+j) \\ &\quad + S(q^{-1})\Delta u(k), \end{aligned} \tag{5}$$

$$\begin{aligned} P(q^{-1}) &= 1 + p_1 q^{-1} + \cdots + p_{n_p} q^{-n_p}, \\ R(q^{-1}) &= r_0 + r_1 q^{-1} + \cdots + r_{n_r} q^{-n_r}, \\ S(q^{-1}) &= s_0 + s_1 q^{-1} + \cdots + s_{n_s} q^{-n_s}, \end{aligned}$$

where $w(k+j)$ is the reference signal. If a pre-compensator is not set, GMVC generates offsets due to the control weight $S(q^{-1})$. The GMVC law is derived by minimizing the variance of the cost function $J = E\{h(k+j)^2\}$. At this point, $y(k+j)$ in the generalized output (5) must be predicted, because it can not be observed at time k. Then, the Diophantine equation (2) is introduced for the prediction $y(k+j)$. However, the Diophantine equation (2) has the double formed time varying multiplication $E_k(q^{-1})\Delta A_k(q^{-1})$. Z. Li proposed the algorithm (Li *et al.*, 1997; Li and Evans, 1997) for time varying multiplication. However it can not deal with the double formed multiplication. Therefore, this paper describes an algorithm for the double formed time varying multiplication.

Theorem 1 The algorithm of the double formed time varying multiplication $E_k(q^{-1})\Delta A_k(q^{-1})$ in the Diophantine equation (2) is defined as follows:

$$E_k(q^{-1})\Delta A_k(q^{-1}) = \sum_{r=0}^{j-1}\sum_{s=0}^{1}\sum_{t=0}^{n}\{e_r(k)(-1)^s a_t(k-s-r)q^{-(r+s+t)}\}. \tag{6}$$

Proof: Z. Li's algorithm (Li *et al.*, 1997; Li and Evans, 1997) is applied two times for the double formed time varying multiplication. First, $\Delta A_k(q^{-1})$, which is the single formed multiplication into $E_k(q^{-1})\Delta A_k(q^{-1})$, is calculated by applying Z. Li's time shift calculation

$$A'_k(q^{-1}) = \Delta A_k(q^{-1}) = \sum_{s=0}^{1}\sum_{t=0}^{n}\{(-1)^s a_t(k-s)q^{-(s+t)}\}, \tag{7}$$

where a new polynomial $A'_k(q^{-1})$ is defined, such as $A'_k(q^{-1}) = \Delta A_k(q^{-1})$. The signature of Z. Li's algorithm is that the time varying polynomials are shifted regarding time, such as $a_t(k-s)$ of equation (7). Figure 1 shows the structure of multiplying the time varying multiplication with $\Delta A_k(q^{-1})$ and the output $y(k)$. The coefficients of polynomials, which are used in the conventional multiplication without the time shift, are expressed as the domain surrounded by the dotted line. Calculating each y_2 in figure 1 can not derived from the polynomials that are surrounded by the dotted line. Figure 1 confirms that the time shift operator is necessary for the time varying multiplication. Second, $E_k(q^{-1})A'_k(q^{-1})$ is calculated again by time shift multiplication

$$E_k(q^{-1})A'_k(q^{-1}) = \sum_{s=0}^{j-1}\sum_{t=0}^{n+1}\{e_r(k)a'_t(k-s)q^{-(s+t)}\}. \tag{8}$$

Figure 2 shows the multiplication $E_k(q^{-1})A'_k(q^{-1})y(k)$. Figure 2 also confirms that the time shift operator should be installed into the calculation of the time varying multiplication (8).

Finally, equation (6), that is expressed as a compact form, is derived by is substituting equation (7) for $A'_k(q^{-1})$ in equation (8).

□

The j steps ahead output $P(q^{-1})y(k+j)$ of equation (5) is

$$\begin{aligned}&P(q^{-1})y(k+j)\\ &= C_{k+j}^{-1}(q^{-1})\{E_{k+j}(q^{-1})\Delta B_{k+j}(q^{-1})u(k)\\ &\quad + F_{k+j}(q^{-1})y(k)\} + E_{k+j}(q^{-1})\Delta\xi(k+j),\end{aligned} \tag{9}$$

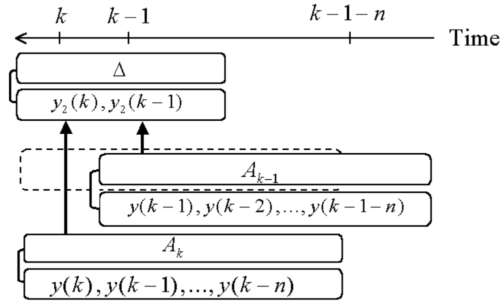


Fig. 1. The structure of multiplying ΔA in TVS

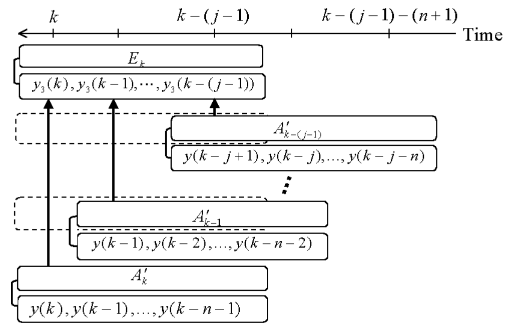


Fig. 2. The structure of multiplying EA' in TVS

which is calculated by using the Diophantine equation (2). the control strategy of servo GMVC is

$$\begin{aligned}&\{E_{k+j}(q^{-1})\Delta B_{k+j}(q^{-1})\\ &\quad + C_{k+j}(q^{-1})S(q^{-1})\Delta\}u(k)\\ &= C_{k+j}(q^{-1})R(q^{-1})w(k+j) - F_{k+j}(q^{-1})y(k),\end{aligned} \tag{10}$$

which is derived by substituting the prediction $P(q^{-1})\widehat{y}(k+j)$ without the noise term $E_{k+j}(q^{-1})\Delta\xi(k+j)$ of equation (9) for the cost function. Furthermore, the servo GMVC (10) has a double formed time varying multiplication $E_{k+j}(q^{-1})\Delta B_{k+j}(q^{-1})$. This multiplication is calculated with

$$\begin{aligned}&E_{k+j}(q^{-1})\Delta B_{k+j}(q^{-1})\\ &= \sum_{r=0}^{j-1}\sum_{s=0}^{1}\sum_{t=0}^{m}\{e_r(k+j)(-1)^s\\ &\quad \times b_t(k+j-s-r)q^{-(r+s+t)}\},\end{aligned} \tag{11}$$

which is also solved by applying Theorem 1.

Notation: Noting that the Diophantine equation is formed, the order of polynomials are $E_k(q^{-1})\Delta A_k(q^{-1})$. If $E_k(q^{-1})\Delta A_k(q^{-1})$ in Diophantine equation (2) is transformed into $\Delta E_k(q^{-1})A_k(q^{-1})$ as

$$\begin{aligned}&\Delta E_k(q^{-1})A_k(q^{-1})\\ &= \sum_{r=0}^{1}\sum_{s=0}^{j-1}\sum_{t=0}^{n}\{(-1)^r e_s(k-s)\end{aligned}$$

$$\times\, a_t(k-s-r)q^{-(r+s+t)}\}, \qquad (12)$$

the Diophantine equation (12) has time shift operator, such as $e_s(k-s)$. In other words, when the multiplication $\Delta E_k(q^{-1})$ is calculated, the coefficients of $E_k(q^{-1})$ are expressed as

$$\begin{aligned}
&\Delta E_k(q^{-1})\\
&= 1 + \{e_1(k) - 1\}q^{-1} + \{e_2(k) - e_1(k-1)\}q^{-2}\\
&\quad + \cdots + \{e_{j-1}(k) - e_{j-2}(k-1)\}q^{-(j-1)}\\
&\quad - e_{j-1}(k-1)q^{-(j-2)}. \qquad (13)
\end{aligned}$$

Then, the unique solution $E_k(q^{-1})$ of the Diophantine equation can not be derived, because equation (13) has factors $e(k)$, $e(k-1)$ at different times. On the other hand, $E_k(q^{-1})\Delta$ that is included in equation (2) is

$$\begin{aligned}
&E_k(q^{-1})\Delta\\
&= 1 + \{e_1(k) - 1\}q^{-1} + \{e_2(k) - e_1(k)\}q^{-2}\\
&\quad + \cdots + \{e_{j-1}(k) - e_{j-2}(k)\}q^{-(j-1)}\\
&\quad - e_{j-1}(k)q^{-(j-2)}. \qquad (14)
\end{aligned}$$

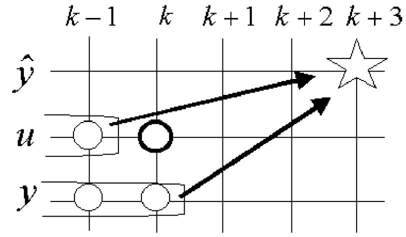
Equation (14) has only factor $e(k)$ at the same time. Therefore, the unique solution of the Diophantine equation as $E_k(q^{-1})\Delta A_k(q^{-1})$ is easily derived than the orders $\Delta E_k(q^{-1})A_k(q^{-1})$.

4. GMVC FOR TVS WITH COLOURED NOISE

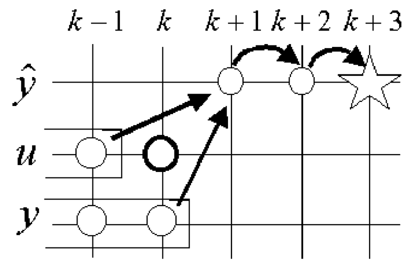
Consider the characteristic of coloured noise in GMVC for TVS. In the coloured noise systems, the degree of polynomial $C_k(q^{-1})$ is expressed as more than 1 degree.

4.1 *Palsson method*

O. P. Palsson proposed another predictive method (Palsson *et al.*, 1993; Palsson *et al.*, 1994; Doi and Mori, 2000) for TVS. This signature is that the Diophantine equation is not used to avoid the time varying multiplication. Figure 3 compares the structure of predicting $\hat{y}(k+3)$ with the Diophantine type method and Palsson type method. Figure 3(a) confirms that the prediction $\hat{y}(k+3)$ is directly derived from the Diophantine equation and the past data. On the other hand, in figure 3(b), the predictions, $\hat{y}(k+1)$ and $\hat{y}(k+2)$ before time $k+3$, must be calculated by refering to the CARMA model (4) to derive the objective prediction $\hat{y}(k+3)$.



(a)Diophantine method



(b)Palsson method

Fig. 3. The structure of the prediction $\hat{y}(k+j)$

4.2 *Noise characteristic*

This section verifies the effect due to the degree of $C_k(q^{-1})$ for the characteristic of coloured noise. Note that this argument is not based on the servo systems because this section want to confirm only the effect of coloured noise.

First, in the case of applying the Diophantine equation, the future output is

$$\begin{aligned}
&P(q^{-1})y(k+j)\\
&= C_{k+j}^{-1}(q^{-1})\{E_{k+j}(q^{-1})B_{k+j}(q^{-1})u(k)\\
&\quad + F_{k+j}(q^{-1})y(k)\} + E_{k+j}(q^{-1})\xi(k+j),
\end{aligned} \qquad (15)$$

where the noise term includes only the future output because $E_k(q^{-1})$ is $j-1$ degrees.

Second, the Palsson method predicts the future output $y(k+j)$

$$\begin{aligned}
y(k+j) = &-\sum_{r=1}^{min(n,j-1)}\{a_r(k+j)y(k+j-r)\}\\
&+ b_0(k+j)u(k) + h_p(k+j \mid k)\\
&+ \sum_{s=1}^{min(l,j-1)}\{c_s(k+j)\xi(k+j-s)\}\\
&+ \sum_{t=j}^{l}\{c_t(k+j)\xi(k+j-t)\}, \qquad (16)
\end{aligned}$$

which is directly derived from the CARMA model. Equation (16) expresses that $h_p(k+j)$ is the data of the observed output and control signals, the fourth term $c_s(k+j)\xi(k+j-s)$ is the future noises and the fifth term is the past and present noises. In other words, if the degree of $C_{k+j}(q^{-1})$ is more than j degree, the past and present noises remain. Furthermore, the first term $a_r(k+j)y(k+j-r)$ in equation (16) is calculated by again refering to the CARMA model (4). The future output clause vanishs by iterating the above procedure. However the past and present noise clause appear in the case of including the coloured noise. In TVS with coloured noise, GMVC with the Diophantine equation is superior to the Palsson type GMVC because the noise clause of the future output comprises only future values due to the Diophantine equation.

5. SIMULATION RESULTS

This chapter presents some simulation results to highlight the scheme of the proposed servo GMVC. The TVS model is of the form (Li and Evans, 1997):

$$\begin{aligned} &y(k+2) + a_1(k)y(k+1) \\ &= u(k) + b_1(k)u(k-1) + \xi(k+2) + c_1(k) \end{aligned} \tag{17}$$

$$a_1(k) = \begin{cases} 1 + 0.135e^{-k}, \\ \qquad 20i - 2 < k \le 10(2i+1) - 2, \\ -1 - 0.135e^{-k}, \\ \qquad 10(2i+1) - 2 < k \le 20(i+1) - 2, \end{cases}$$

$$b_1(k) = 0.3[2 + \cos(0.2\pi k + 0.4\pi)],$$

$$c_1(k) = \begin{cases} 0.9\dfrac{k+2}{k+3}, \\ \qquad 20(2i-1) + 3 < k \le 40i + 3, \\ -0.9\dfrac{k+2}{k+3}, \\ \qquad 40i + 3 < k \le 20(2i+1) + 3. \end{cases}$$

This model, such that the parameters sharp change as step shape, is chosen to obviously verify the efficient of the time varying polynomials. The reference signal is changing at $k = 20$ from 0 to 1 and a load disturbance of the magnitude 0.2 occurs at $k = 70$. The delay time of the plant is 2 steps and the variance of noise $\xi(k)$ is 0.01. The weights of cost function are $P(q^{-1}) = R(q^{-1}) = 1$ and $S(q^{-1}) = 0.05$. Note that these simulations are calculated at various sample periods by using the varying polynomials. Figure 4 shows the time response based on the conventional GMVC and figure 5 shows the result of the proposed GMVC. While the output in figure 4 can not track the reference signal, Figure 5 tracks it not only the reference changing but also the load disturbance. These results confirms that the multiplications due to the time shift operator and the servo design provide the remarkable effects for TVS. In the following, figure 6 and figure 7 compare the coloured noise characteristic in TVS with the Palsson type GMVC and the proposed GMVC. The results of figure 6 and figure 7 show that the Palsson type GMVC has larger variances. These variances are the output signal and the control signal, such as the input energy. Therefore the results confirms the verification in sec.4.2 this results mean that the proposed GMVC is effective in TVS.

6. CONCLUSIONS

This paper designed the servo GMVC for TVS. The double formed time varying multiplication, which appears in the servo mechanism, is solved with the double time shift summation. The sequence of the time shift multiplication was clarified by the diagram. In the Diophantine equation, this paper pointed out that the structure should be considered to easily derive the unique solution. Furthermore, this paper showed the condition that GMVC without the Diophantine equation can not minimize the cost function, while the proposed GMVC entirely minimizes it. Then, the coloured noise characteristic of the proposed GMVC is superior to GMVC without the Diophantine equation. The simulation confirmed that the proposed GMVC realizes the sufficient response property for TVS.

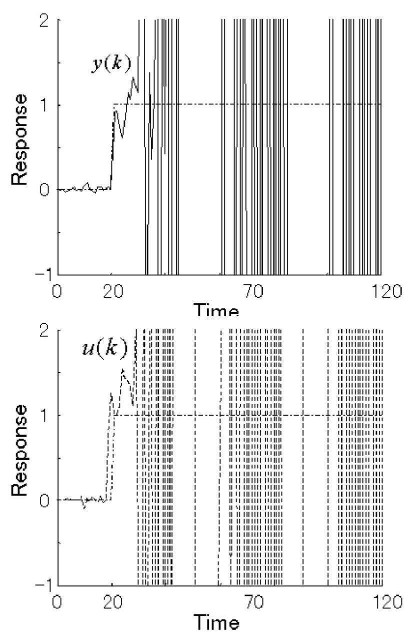


Fig. 4. The conventional GMVC in TVS

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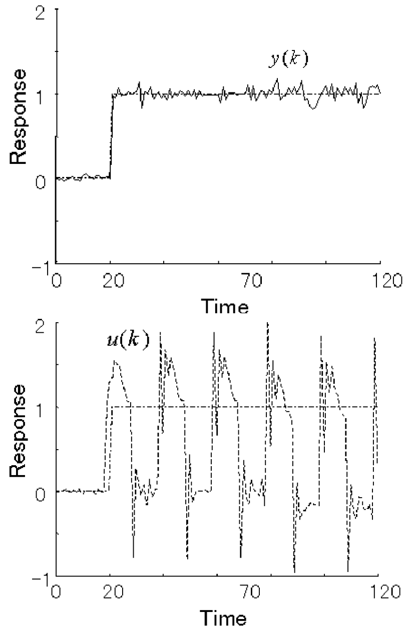

Fig. 5. GMVC with time-shift operator in TVS

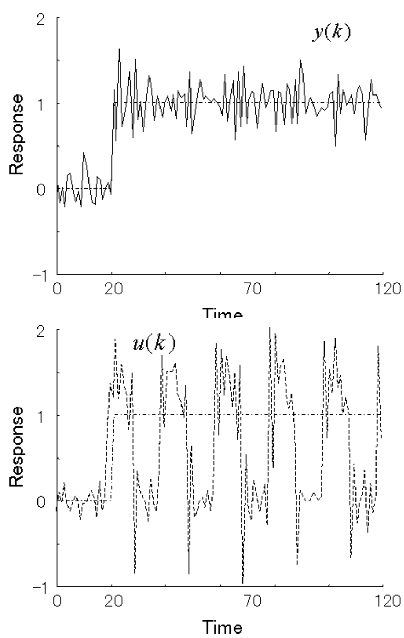

Fig. 6. The noise characteristic of Palsson type GMVC

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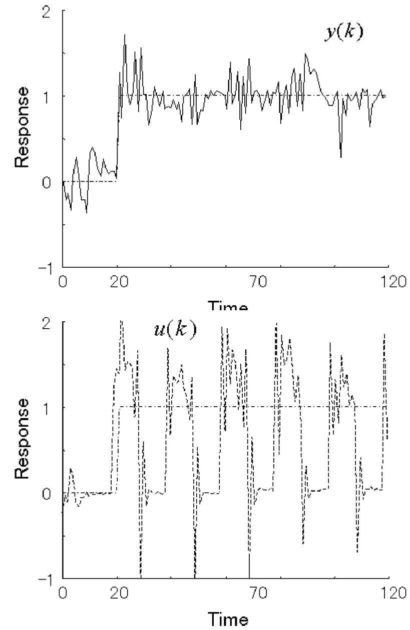

Fig. 7. The noise characteristic of Diophantine type GMVC

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