Copyright © 2002 IFAC
15th Triennial World Congress, Barcelona, Spain

PRESERVATION OF STABILITY AND SPECTRUM FOR A CLASS OF INFINITE-DIMENSIONAL SYSTEMS

Yutaka Yamamoto *

* *Department of Applied Analysis and Complex Dynamical Systems, Graduate School of Informatics, Kyoto University, Kyoto 606-8501, JAPAN; yy@i.kyoto-u.ac.jp*

Abstract: For distributed or infinite-dimensional systems, stability raises various difficulties. One anomaly is that there can be two realizations, both approximately reachable and observable, but one is exponentially stable and the other is not (unstable). Another is that even the spectrum is not preserved among such realizations. The paper gives conditions under which stability and spectrum are preserved for approximately reachable and observable realizations. The result have much bearing on robust controller designs based on external system description, e.g., H-infinity designs.

Keywords: Infinite-dimensional systems; Spectrum; Stability; Pseudorational impulse responses

1. INTRODUCTION

There are many anomalies for distributed parameter systems arising from the freedom of endowing different topologies on the state space. It is known that even the spectrum is not preserved under various realizations, albeit approximately reachable and observable (Fuhrmann, 1981). Another is the question of stability: it is generally not determined by the location of spectrum, nor is it preserved for various different realizations, again all of them being approximately reachable and observable (Zabczyk, 1975; Zwart et. al, 1995). This is certainly a very undesirable situation for many design methods based on external data such as transfer functions, for example, H^∞ control theory. For example, suppose that one has designed a controller that guarantees L^2 input/output stability. This is typically the case with many H^∞ design methods. While one may expect that it yields internal stability of the realization he is dealing with, it need not be guaranteed for distributed parameter systems. This is the case even for such a familiar class of delay systems (Logemann, 1987).

In (Yamamoto, 1988; Yamamoto, 1989) we have shown that for a certain class of systems called *pseudorational*, by requiring a strong notion of canonicity, one can prove that the spectrum determines stability. It is however left open whether other approximately reachable and observable still remain stable. The best scenario in this situation is that once the system is guaranteed to be externally (input/output) stable, then all of its approximately reachable and observable realizations are stable. One can then safely discuss internal stability based on its external behavior (impulse response, transfer function, etc.).

This paper gives conditions under which such notions as spectrum and stability are preserved.

2. PSEUDORATIONAL IMPULSE RESPONSES AND THEIR REALIZATIONS

We start by defining the notions of time-invariant linear systems, pseudorational impulse responses, and their realizations.

In what follows, we confine ourselves, without loss of generality, to the single-input single-output case. Generalization to the multivariable case can be easily obtained by considering each component of the impulse response.

Let $\mathcal{E}'(\mathbb{R}_-)$ denote the space of distributions having compact support contained in the negative half line $(-\infty, 0]$. Distributions such as Dirac's delta δ_a placed at $a \leq 0$, its derivative δ'_a are examples of elements in $\mathcal{E}'(\mathbb{R}_-)$. An impulse response function W ($\operatorname{supp} W \subset [0, \infty)$) is said to be *pseudorational* if it satisfies the following two conditions:

(1) $W = q^{-1} * p$ for some $q, p \in \mathcal{E}'(\mathbb{R}_-)$, where the inverse is taken with respect to convolution;
(2) $\operatorname{ord} q^{-1} = -\operatorname{ord} q$, where $\operatorname{ord} q$ denotes the order of a distribution q (Schwartz, 1966).

Let $\Omega := \varinjlim L^2[-n, 0]$ denote the *inductive limit* of the spaces $\{L^2[-n, 0]\}_{n>0}$; it is the union of all these spaces endowed with the finest topology that makes all injections $j_n : L^2[-n, 0] \to \Omega$ continuous; see, e.g., (Treves, 1967). Dually, $\Gamma := L^2_{loc}[0, \infty)$ is the space of all *locally* Lebesgue square integrable functions with obvious family of seminorms:

$$\|\phi\|_n := \left\{ \int_0^n |\phi(t)|^2 dt \right\}^{1/2}.$$

This is the *projective limit* of spaces $\{L^2[0, n]\}_{n>0}$. Ω is the space of past inputs, and Γ is the space of future outputs, with the understanding that the present time is 0. These spaces are equipped with the following natural *left shift* semigroups:

$$(\sigma_t \omega)(s) := \begin{cases} \omega(s+t), & s \leq -t, \\ 0, & -t < s \leq 0, \end{cases} \qquad (1)$$
$$\omega \in \Omega, t \geq 0, s \leq 0.$$

$$(\sigma_t \gamma)(s) := \gamma(s+t), \quad \gamma \in \Gamma, t \geq 0, s \geq 0. \qquad (2)$$

An *input/output* or a *Hankel* operator associated with an impulse response function W is defined to be the continuous linear mapping $\mathcal{H}_W : \Omega \to \Gamma$ defined by

$$\mathcal{H}_W(\omega)(t) := \int_{-\infty}^0 W(t - \tau)\omega(\tau) d\tau.$$

Let us now introduce the notion of a (linear, time-invariant) system.

Definition 2.1. A *(linear, time-invariant) system* Σ is a quadruple (X, Φ, g, h) such that

- X is a Banach space, and $\Phi(t)$ is a strongly continuous semigroup defined on it;
- $g : \Omega \to X$ is a continuous linear mapping such that $g\sigma_t = \Phi(t)g$ for all $t \geq 0$;
- $h : X \to \Gamma$ is also a continuous linear map satisfying $h\Phi(t) = \sigma_t h$ for all $t \geq 0$.

$$\begin{tikzcd} \Omega \arrow[rr, "\mathcal{H}_W"] \arrow[dr, "g"'] & & \Gamma \\ & X \arrow[ur, "h"'] & \end{tikzcd}$$

The mappings g and h are called *reachability map* and *observability map*, respectively. Σ is said to be *approximately reachable* if g has dense image, and *observable* if h is one to one. It is *topologically observable* if h gives a topological homomorphism (i.e., continuously invertible when its codomain is restricted to $\operatorname{im} h$). Σ is *weakly canonical* if it is approximately reachable and observable; it is *canonical* if it is further topologically observable. Σ is said to be a *realization* of an impulse response W if $\mathcal{H}_W = hg$.

The definition above looks a little abstract and appears to have little information needed to analyze linear systems. However, when there are certain "smoothness hypotheses" satisfied, then it is immediate to write down a differential equation description in the following form (Yamamoto, 1988):

$$\frac{dx}{dt} = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

where A is the infinitesimal generator of $\Phi(t)$, and

$$g(\omega)(t) = \int_{-\infty}^0 \exp(-At)B\omega(t)dt$$
$$h(x)(t) = C\exp(At)x.$$

These properties justify the terms reachability and observability maps.

A system $\Sigma = (X, \Phi, g, h)$ is said to be *exponentially stable* if there exist positive constant C, β such that

$$\|\Phi(t)x\| \leq Ce^{-\beta t}\|x\| \qquad (3)$$

For a pseudorational impulse response $W = q^{-1} * p$, one can always associate with it a topologically observable realization $\Sigma^{q,p}$ as follows (Yamamoto, 1988):

Define X^q as follows:

$$X^q := \{x \in \Gamma \mid \pi(q * x) = 0\}$$

where π is the truncation to $(0, \infty)$. It is easy to check X^q is a σ_t-invariant closed subspace of Γ. To define $\Sigma^{q,p}$, take this X^q as the state space with σ_t (restricted to X^q) as its semigroup. Then define $g : \Omega \to X^q$ and $h : X^q \to \Gamma$ as follows.

$$g(\omega) := \pi(q^{-1} * p * \omega)$$
$$h(x) = x \text{ (injection).}$$

Since h is clearly a topological homomorphism, $\Sigma^{q,p}$ is topologically observable. It is approximately reachable if the pair (q, p) is further approximately coprime (Yamamoto, 1988).

Facts 2.2. (1) Let $\Sigma^{q,p}$ be as above. The spectrum of the infinitesimal generator A^q of system $\Sigma^{q,p}$ is given by

$$\sigma(A^q) = \{\lambda \,|\, \hat{q}(\lambda) = 0\}. \tag{4}$$

Furthermore, every point in $\sigma(A^q)$ is an eigenvalue with finite multiplicity. The resolvent set $\rho(A^q)$ is its complement.

(2) For each $\lambda \in \sigma(A^q)$, the generalized eigenfunctions are of the form $\{e^{\lambda t}, te^{\lambda t}, \ldots, t^{n-1}e^{\lambda t}\}$, where n is the geometric multiplicity.

(3) The state space X^q is decomposed as

$$X^q \cong L^2[0,T] \oplus \overline{X_0}$$

where X_0 is the linear subspace spanned by the generalized eigenfunctions given as above.

3. PRESERVATION OF SPECTRUM

Let $W = q^{-1} * p$ be pseudorational, and let $\Sigma = (X, \Phi, g, h)$ be a weakly canonical realization of W. We then have the following commutative diagram:

$$\begin{tikzcd} \Omega \arrow[r, "\mathcal{H}_W"] \arrow[d, "g"'] & \Gamma \\ X \arrow[r, "h"'] & X^q \arrow[u, "j"'] \end{tikzcd}$$

Since Σ is observable, h is injective, and further j is a topological embedding. We may thus consider X as a subspace of X^q (with finer topology) which in turn is a subspace of Γ.

Our question here is that under what conditions the spectrum of A^q or the stability of $\Sigma^{q,p}$ is preserved. In what follows, for simplicity of discussions, we always assume that (q, p) is approximately coprime so that $\Sigma^{q,p}$ is canonical.

Let us start with the invariance of spectrum.

Theorem 3.1. Let Σ be a weakly canonical realization as above, and A the infinitesimal generator of $\Phi(t)$. Suppose that for any $\lambda \in \rho(A^q)$, X (considered as a vector subspace of X^q as above) is invariant under $(\lambda I - A^q)^{-1}$. Then

$$\sigma(A) = \sigma(A^q) = \{\lambda \,|\, \hat{q}(\lambda) = 0\}, \tag{5}$$

i.e., the spectrum is invariant.

Proof Let us first assume $\hat{q}(\lambda) = 0$. By extracting the factor $(s - \lambda)^n$ (n is the algebraic multiplicity of λ in $\hat{q}(s)$) from $\hat{q}(s)$, we see that the generalized eigenspace corresponding to λ constitutes a finite-dimensional subsystem in $\Sigma^{q,p}$. Since $\Sigma^{q,p}$ is approximately reachable, this subspace is approximately reachable, but because of the finite-dimensionality, it is also exactly reachable. Due to approximate reachability of Σ, this subspace must be contained in X. Hence the characteristic equation $(\lambda I - A)x = 0$ also admits a solution in X. Thus if $\hat{q}(\lambda) = 0$, then $\lambda \in \sigma(A)$.

It suffices to prove that if $\hat{q}(\lambda) \neq 0$ then it belongs to the resolvent set $\rho(A)$. Note that X is invariant $(\lambda I - A^q)^{-1}$, and the induced mapping of $(\lambda I - A^q)^{-1}$ on X is precisely $(\lambda I - A)^{-1}$ (see the next block diagram).

$$\begin{tikzcd} X^q \arrow[r, "(\lambda I - A^q)^{-1}"] & X^q \\ X \arrow[u, "h"] \arrow[r, "(\lambda I - A)^{-1}"'] & X \arrow[u, "h"'] \end{tikzcd}$$

It suffices to prove the continuity of $(\lambda I - A)^{-1}$. Suppose that $x_n \to x$ in X and $(\lambda I - A)^{-1}x_n \to y$ also in X. Then $h(x_n) \to h(x)$ and $h((\lambda I - A)^{-1}x_n) \to h(y)$ in X^q. Since h is a continuous embedding that commutes with shifts, $h((\lambda I - A)^{-1}x_n) = (\lambda I - A^q)^{-1}h(x_n)$. Hence by the continuity of $(\lambda I - A^q)^{-1}$ (because $\lambda \in \rho(A^q)$), $h((\lambda I - A)^{-1}x_n) \to (\lambda I - A^q)^{-1}h(x) = h((\lambda I - A)^{-1}x)$. By the uniqueness of a limit, $h((\lambda I - A)^{-1}x) = h(y)$. Since h is injective, $(\lambda I - A)^{-1}x = y$. This means that $(\lambda I - A)^{-1}$ has closed graph. By the closed graph theorem (Yosida, 1964), $(\lambda I - A)^{-1}$ is also continuous. □

The preservation of spectrum depends on the invariance of X under $(\lambda I - A^q)^{-1}$. A condition that guarantees this is given as follows. In the following lemma, we consider X as a subset of X^q.

Lemma 3.2. Let $\Sigma = (X, \Phi, g, h)$ be as above. Suppose $0 \in \rho(A^q)$, $X \subset A^q(X \cap D(A^q))$ and $D((A^q)^2) \subset X$. Then X is invariant under $(\lambda I - A^q)^{-1}$ for any $\lambda \in \rho(A^q)$.

Proof By hypothesis, $y = A^q w$ for some $w \in X \cap D(A^q)$. Take any $\lambda \in \rho(A^q)$ and y in X. We want to solve $(\lambda I - A^q)y = x$ for some x in X. This means that $x := (\lambda I - A^q)^{-1}y \in X$. Observe

$$\begin{aligned} x - w &= (\lambda I - A^q)^{-1}y - {A^q}^{-1}y \\ &= -\lambda(\lambda I - A^q)^{-1}{A^q}^{-1}y \end{aligned}$$

by the well-known resolvent identity (Yosida, 1964). Now $(A^q)^{-1}y \in D(A^q)$ by hypothesis. Then $(\lambda I - A^q)^{-1}{A^q}^{-1}y \in D((A^q)^2)$, which in turn is a subset of X. This implies $x \in X$. □

Remark 3.3. As is clear from the above proof, $0 \in \rho(A^q)$ may be replaced by any $\lambda \in \rho(A^q)$ by suitably shifting the formulas. Since X (or $h(X)$ to be precise) is a shift invariant subspace of X^q, it is usually confined by some regularity assumptions. Since A^q is a differential operator (because σ_t is the left shift (Yamamoto, 1988)), assuming $X \subset A^q(X \cap D(A^q))$ is quite possible in most cases. Also, $D((A^q)^2) \subset X$ may be satisfied in many cases.

Example 3.4. As an example where the condition above is easily satisfied, consider

$$X := \{x(t) \mid x \in X^q \text{ and continuous}\} \subset X^q$$

It is known that the topology of X^q is determined by finite-time data on $[0, T]$ for some $T > 0$ (Yamamoto, 1988). This is a consequence of pseudorationality and easily seen to carry over to X also. Then the hypotheses of Lemma 3.2 are clearly satisfied.

4. PRESERVATION OF STABILITY

Preservation of stability presents a more subtle and difficult problem.

Let us first start with the stability of $\Sigma^{q,p}$:

Theorem 4.1. ((Yamamoto, 1991)) $\Sigma^{q,p}$ is exponentially stable if and only if

$$\sup\{\operatorname{Re}\lambda \mid \lambda \in \sigma(A)\} < 0. \tag{6}$$

In other words, there exists $c > 0$ such that

$$\operatorname{Re}\lambda \le -c \text{ for all } \lambda \text{ such that } \hat{q}(\lambda) = 0.$$

Remark 4.2. Note that for infinite-dimensional systems, $W \in H^\infty(\mathbb{C}_+)$ is not enough to guarantee (6) nor exponential stability; see (Logemann, 1987).

The question here is that under what condition this stability property is preserved. If W were not pseudorational, this is not true. For counterexamples, see (Fuhrmann, 1981; Zabczyk, 1975; Zwart et. al, 1995). Part of the difficulty here is due to the fact that the spectral mapping theorem is incomplete for continuous spectrum for semigroups. This will become clear in what follows.

Let us start with the following lemma:

Lemma 4.3. Let $\Sigma = (X, \Phi(t), g, h)$ be a linear system. Then it is exponentially stable if and only if there exists $T > 0$ such that the spectral radius $r_\sigma(\Phi(T))$ satisfies

$$r_\sigma(\Phi(T)) < 1. \tag{7}$$

Proof Let $\Phi(t)$ be exponentially stable. Then by (3) there exists $T > 0$ such that $\|\Phi(T)\| < 1$. Since $\|\Phi(T)\| \le r_\sigma(\Phi(T))$, (7) follows.

Conversely, suppose (7) holds. Since $r_\sigma(\Phi(T)) = \lim_{n\to\infty} \|\Phi(T)^n\|$ (Yosida, 1964), there exists sufficiently large n such that $r := \|\Phi(T)^n\| < 1$. Then $\|\Phi(nT)\| = \|\Phi(T)^n\| < 1$. Let

$$C := \max_{0\le t\le nT} \|\Phi(t)\|.$$

It then follows that, if $t = mnT + \theta$ for $0 \le \theta < nT$,

$$\|\Phi(t)\| = \|\Phi(mnT + \theta)\| = \|\Phi(nT)^m \Phi(\theta)\| \le Cr^m.$$

Take $\beta = -(\log r)/(n+1)T$. Then

$$Cr^m \le Ce^{-\beta(mn+1)T} \le Ce^{-\beta t}.$$

This clearly implies (3). □

Let $\Sigma = (X, \Phi, g, h)$ be a weakly canonical realization of a pseudorational impulse response $W = q^{-1} * p$. Suppose that $\Sigma^{q,p}$ is exponentially stable. Then σ_t satisfies $r_\sigma(\sigma_T) < 1$ for some T. If this number remains the same for $\Phi(T)$, then the exponential stability would follow.

To explore the situation, we start with the following rather simple result.

Proposition 4.4. Let $W = q^{-1} * p$ and Σ be as above. Suppose that Σ satisfies the hypotheses of Theorem 3.1. Suppose further that $\Sigma^{q,p}$ is exponentially stable, and $\Phi(T)$ has no continuous spectrum for some $T > 0$. Then $r_\sigma(\Phi(T)) < 1$, and hence Σ is also exponentially stable.

Proof By hypothesis, the spectrum of A is preserved,

$$\sigma(A) = \sigma(A^q)$$

and every point in $\sigma(A)$ is an eigenvalue. That is, there are no residual or continuous spectrum. By the spectral mapping theorem for semigroups (Pazy, 1983), we must have

$$P_\sigma(\Phi(T)) \subset \exp(P_\sigma(A)T) \cup \{0\},$$

where $P_\sigma(V)$ denotes the point spectrum of an operator V. Since there is no residual spectrum for A, $\Phi(T)$ does not have residual spectrum either (Pazy, 1983). Finally, since $\Phi(T)$ is assumed not to have continuous spectrum,

$$\sigma(\Phi(T)) \subset \exp(P_\sigma(A)T) \cup \{0\}.$$

Since $\Sigma^{q,p}$ is exponentially stable, the right-hand side is contained in $\{z \mid |z| < 1\}$. Hence $r_\sigma(\Phi(T)) < 1$ and the result follows. □

The drawback here is the assumption on the continuous spectrum. We shall elaborate more on this in what follows.

One case is that $\Phi(T)$ is compact. This is the case with retarded delay-differential equations, and an abstract characterization for such systems has been given in (Yamamoto and Hara, 1992). Assuming this, we readily have the following:

Theorem 4.5. Let Σ be as above, and suppose that $\Sigma^{q,p}$ is exponentially stable. Suppose also that Σ satisfies the hypotheses of Theorem 3.1 and that $\Phi(t)$ is compact for some $t > 0$. Then Σ is also exponentially stable.

Proof Since $\Phi(t)$ can have only point spectrum other than 0, the spectral mapping theorem readily implies the conclusion. $\square$

Let us now prove the following main result:

Theorem 4.6. Let Σ be a weakly canonical realization as above. Suppose that $\Sigma^{q,p}$ is exponentially stable, and hence $r_\sigma(\sigma_T) < 1$ for some $T > 0$. Suppose also that $(\lambda I - \Phi(T))^{-1}$ leaves X invariant for any λ with $|\lambda| > r_\sigma(\sigma_T)$. Then Σ is also exponentially stable.

Proof Consider the commutative diagram.

$$\begin{array}{ccc} X^q & \xrightarrow{(\lambda I - \sigma_T)^{-1}} & X^q \\ {\scriptstyle h}\uparrow & & \uparrow{\scriptstyle h} \\ X & \xrightarrow[(\lambda I - \Phi(T))^{-1}]{} & X \end{array}$$

We prove that $(\lambda I - \Phi(T))^{-1}$ is continuous with respect to the topology of X.

Suppose that $x_n \to x$ in X and $(\lambda I - \Phi(T))^{-1}x_n \to y$ also in X. Then $h(x_n) \to h(x)$ and $h((\lambda I - \sigma_T)^{-1}x_n) \to h(y)$ in X^q. Since h is a continuous embedding that commutes with shifts, $h((\lambda I - \Phi(T))^{-1}x_n) = (\lambda I - \sigma_T)^{-1}h(x_n)$. Hence by the continuity of $(\lambda I - \sigma_T)^{-1}$ (because $\lambda \in \rho((\lambda I - \sigma_T)^{-1})$), $h((\lambda I - \Phi(T))^{-1}x_n) \to (\lambda I - \sigma_T)^{-1}h(x) = h((\lambda I - \Phi(T))^{-1}x)$. By the uniqueness of a limit, $h((\lambda I - \Phi(T))^{-1}x) = h(y)$. Since h is injective, $(\lambda I - \Phi(T))^{-1}x = y$ follows. This means that $(\lambda I - \Phi(T))^{-1}$ has closed graph. Then, again by the closed graph theorem (Yosida, 1964), $(\lambda I - \Phi(T))^{-1}$ is also continuous. $\square$

The following example shows an easy case where the hypothesis of the above theorem is satisfied.

Example 4.7. Suppose there exists a closed operator F such that $F\sigma_T = \sigma_T F$, and $X = D(F)$, i.e.,

$$X = \{x \in X^q : Fx \in X^q\}.$$

Since $F\sigma_T x = \sigma_T F x \in X^q$, X is σ_T-invariant. So let $\Phi(T) = \sigma|_X$, and denote it by the same symbol σ_T since no confusion can arise. It also follows that $(\lambda I - \sigma_T)F = F(\lambda I \sigma_T)$ and hence $F(\lambda I \sigma_T)^{-1} = (\lambda I - \sigma_T)^{-1}F$. Then for $x \in X$,

$$F(\lambda I \sigma_T)^{-1}x = (\lambda I - \sigma_T)^{-1}Fx \in X^q$$

because $Fx \in X^q$. Thus X is $(\lambda I \sigma_T)^{-1}$-invariant, and the condition of Theorem 4.6 is satisfied.

A typical case for F is the differential operator d/dt which happens to be the infinitesimal generator of σ_t.

5. DIFFICULTY IN SPECTRAL RESULTS

We needed an extra condition of the invariance of X under $(\lambda I - \Phi(T))^{-1}$ in Theorem 4.6. In this section we see where the difficulty is in removing this assumption.

Recall from Facts 2.2 that the state space X^q is decomposed as the direct sum of $L^2[0,T]$ and the closure of the space X^0 spanned by the eigenfunctions of form $t^m e^{\lambda t}$. Since the semigroup σ_t is the left shift, the component $L^2[0,T]$ is irrelevant to stability, since any of its element will vanish after the transition in some time. We thus assume $X = \overline{X_0}$, i.e., it is eigenfunction complete. The following lemma then shows that eigenfunction completeness is preserved among weakly canonical realizations.

Lemma 5.1. Let $\Sigma = (X, \Phi, g, h)$ be a weakly canonical realization. Suppose that $\Sigma^{q,p}$ is eigenfunction complete. Then Σ is also eigenfunction complete.

Proof We know that X_0 is contained in X. In particular, it is a subset of $\Omega/\ker \mathcal{H}_W$, since such a quotient gives a weakly canonical realization (Yamamoto, 1988). This realization, with the quotient topology induced from Ω, has the finest topology among all weakly canonical realizations. Furthermore, X_0 is also dense there. This can be seen by following the same procedure in (Yamamoto, 1989). Then, since $\Omega/\ker \mathcal{H}_W$ can be densely embedded in X, X_0 is also dense in X. $\square$

We now assume that $\hat{q}(s)$ has globally bounded multiplicity, say 1. This implies

$$X_0 = \mathrm{span}\{e^{\lambda_1 t}, e^{\lambda_2 t}, \ldots\}.$$

Now let us see what can be said about the stability of Σ under the assumption of the stability of $\Sigma^{q,p}$, but not necessarily invariance of X under $(\lambda I - \Phi(t))^{-1}$.

By Lemma 4.3 there exists $T > 0$ such that $r := r_\sigma(\sigma_T) < 1$. Again by Lemma 4.3 it would suffice to show that $r_\sigma(\Phi(T)) < 1$.

Put

$$X_n := \mathrm{span}\{e^{\lambda_1 t}, e^{\lambda_2 t}, \ldots, e^{\lambda_n t}\} \subset X.$$

Take $x := \sum_{i=1}^n x_i e^{\lambda_i t}$. Then

$$\Phi(T) = \sum_{i=1}^n x_i e^{\lambda_i (t+T)}.$$

If we take $\{e^{\lambda_1 t}, e^{\lambda_2 t}, \ldots, e^{\lambda_n t}\}$ as a basis, this operator is expressed by matrix multiplication

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \mapsto \begin{bmatrix} e^{\lambda_1 T} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 T} & & \\ 0 & 0 & \ddots & 0 \\ 0 & \cdots & & e^{\lambda_n T} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad (8)$$

Observe that the norm induced on $[x_1, \ldots, x_n]^T$ from that of X need not be Euclidean nor of any familiar form, in general, but since the *topology* of a finite-dimensional subspace is invariant, the spectral radius of the above matrix representation of $\Phi(T)$ remains invariant under any change of such a topology endowed on X. Thus,

$$r_\sigma(\Phi(T)|_{X_n}) \le \max_{1\le i\le n} |e^{\lambda_i T}| \le r_\sigma(\sigma_T).$$

It looks as though this would imply $r_\sigma(\Phi(T)) < 1$, but actually not. This guarantees that there is no point in the spectrum greater than $r_\sigma(\sigma_T)$ *when the operator is restricted to* X_0, but X_0 is only a dense subspace, and this does not guarantee that there exists no continuous spectrum point for $\Phi(T)$ with $\lambda \ge r_\sigma(\sigma_T)$ when this operator is considered on the whole space X. In other words, the dense subspace X_0 has little control over the whole spectrum, although the operator itself looks rather innocent as (8). This is why we need some extra conditions to prove stability as in Theorem 4.6 or Proposition 4.4.

6. CONCLUDING REMARKS

We have given some conditions under which spectrum and/or stability is preserved. It is desirable to replace the assumptions on the invariance of X under pertinent operators, but it is a theme for future study.

Acknowledgment The author would like to thank Dr. M. Kubo of Kyoto University for valuable discussions.

7. REFERENCES

Fuhrmann, P. A. (1981). *Linear Systems and Operators in Hilbert Space*. McGraw-Hill.

Zwart, H., Y. Yamamoto and Y. Gotoh (1995). Stability is realization-dependent: some examples. *Syst. Control Lett.* **24**, 25–31.

Logemann, H. (1987). On the transfer matrix of a neutral system: Characterizations of exponential stability in input-output terms. *Syst. Control Lett.* **9**, 393–400.

Pazy, A. (1983). *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer-Verlag.

Schwartz, L. (1966). *Théorie des Distributions*. Hermann.

Treves, F. (1967). *Topological Vector Spaces, Distributions and Kernels*. Academic Press.

Yamamoto, Y. (1988). Pseudo-rational input/output maps and their realizations: a fractional representation approach to infinite-dimensional systems. *SIAM J. Control & Optimiz.* **26**, 1415–1430.

Yamamoto, Y. (1989). Reachability of a class of infinite-dimensional linear systems: an external approach with applications to general neutral systems. *SIAM J. Control & Optimiz.* **27**, 217–234.

Yamamoto, Y. (1991). Equivalence of internal and external stability for a class of distributed systems. *MCSS* **4**, 391–409.

Yamamoto, Y. and S. Hara (1992). Internal and external stability and robust stability condition for a class of infinite-dimensional systems. *Automatica* **28**, 81–93.

Yosida, K. (1964). *Functional Analysis*. Springer-Verlag.

Zabczyk, J. (1975). A note on c_0-semigroups. *Bull. l'Acad. Pol. de Sc. Serie Math.* **23**, 895–898.