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H^∞ FILTERING OF SIGNALS SUBJECTED TO MULTIPLICATIVE WHITE NOISE

Adrian Stoica

University "Politehnica" of Bucharest, Faculty of Aerospace Engineering,
Str. Splaiul Independentei, no. 313, Ro-77206, Bucharest, Romania
e-mail: amstoica@fx.ro

Abstract: In the present paper an H^∞-type filtering approach for continuous-time stochastic systems with multiplicative white noise is presented. The design procedure uses the Bounded Real Lemma type result. The paper considers the case when the filter oder is fixed. Necessary and sufficient solvability conditions are derived in terms of two linear matrix inequalities coupled by a rank constrain condition. A numerical example illustrates the theoretical developments.

Keywords: Stochastic systems, Multiplicative white noise, H^∞-filtering, γ-attenuation, Linear matrix inequalities.

1. INTRODUCTION

One of the direct applications of the H^∞ control theory is the filtering of signals generated by systems corrupted with stochastic noise. These filtering techniques are mainly based on the γ-attenuation type results derived for different classes of stochastic systems. Much attention has received in the last years the study of stochastic systems with multiplicative noise (*e.g.* (El Ghaoui, 1991), (Boyd *et al.*, 1994), (Dragan *et al.*, 1997)) due to their wide of applications ((Costa and Kubrusly, 1996), (Ugrinovskii, 1998)). The γ -attenuation problem for continuous-time case has been studied in (Hinrichsen and Pritchard, 1998) where necessary and sufficient conditions for solving the problem are expressed in terms of two coupled nonlinear matrix inequalities. The discrete-time conterpart has been investigated in (Dragan and Stoica, 1998). Robust stabilization problems for stochastic systems with multiplicative white noise can be found in (Dragan *et al.*, 1997), (Ugrinovskii, 1998), (Gershon *et al.*, 1999), (El Ghaoui, 1991). In the present paper an H^∞ type filtering approach for continuous-time stochastic systems with multiplicative white noise is presented. The design procedure uses the Bounded Real Lemma type result derived in (Morozan, 1995) and differs from the developments derived in (Gershon *et al.*, 2001), where a similar problem is considered. Unlike (Gershon *et al.*, 2001) where the order of the obtained filter equals the order of the plant, the present paper deals with the more general situation when the filter has an apriori imposed order. This formulation is particularly useful in applications where the plant generating the filtered signal has high order. The paper is organized as follows. In Section 2 some introductive preliminaries and notations are introduced related to the class of stochastic systems under investigation and the problem formulation. The main result is derived in Section 3 where necessary and sufficient conditions for solving the filtering problem are derived. A numerical example illustrating the theoretical developments is presented in Section 4.

2. PRELIMINARIES AND PROBLEM FORMULATION

2.1 *Notations and definitions*

Throughout the paper $\{\Omega, \mathcal{F}, \mathcal{P}\}$ denotes a given probability space. Consider the stochastic system subjected to multiplicative white noise:

$$
\begin{aligned}
dx(t) &= [A_0 x(t) + Bu(t)]dt + \sum_{j=1}^{r} A_j x(t) dw_j(t) \\
y(t) &= Cx(t) + Du(t)
\end{aligned} \tag{1}
$$

where $x \in \mathbf{R}^n$ denotes the state vector, $u \in \mathbf{R}^m$ is the control variable, $y \in \mathbf{R}^p$ is the output and $w_1(t), \ldots, w_r(t)$, $t \in \mathbf{R}$ are independent standard Wiener processes. With $\mathfrak{F}_t \subset \mathfrak{F}$ will be denoted the smallest σ-algebra with respect to which all random functions $w_j(t_2) - w_j(t_1)$, $t_1 \le t_2 \le t$, $1 \le j \le r$ are measurable and they also contain all sets $S \in \mathfrak{F}$ with $\mathcal{P}(S) = 0$. By $L^2(\mathbf{R} \times \Omega, \mathbf{R}^m)$ it is denoted the space of measurable functions $u : \mathbf{R} \times \Omega \to \mathbf{R}^m$ with

$$
\|u\|^2 = \int_{\mathbf{R}} E|u(t)|^2 dt < \infty ,
$$

where E is the expectation and $|u|^2 = \sum_{i=1}^{m} u_i^2$. The space of all $u \in L^2(\mathbf{R} \times \Omega, \mathbf{R}^m)$ with the property that $u(t)$ is measurable with respect to $\mathfrak{F}_t$ for every $t \in \mathbf{R}$ is denoted by $\tilde{L}^2(\mathbf{R} \times \Omega, \mathbf{R}^m)$. The space $\tilde{L}^2(\mathbf{R} \times \Omega, \mathbf{R}^m)$ is closed in L^2 and hence it is a Hilbert space.

Definition 1 *The system (1) is exponentially stable in mean square (ESMS) if there exist* $\alpha > 0$ *and* $\beta \ge 1$ *such that* $E|\Phi(t, t_0)|^2 \le \beta e^{-\alpha(t - t_0)}$ *for all* $t \ge t_0$, *where* $\Phi(t, t_0)$ *is the fundamental (random) matrix solution associated with the system*:

$$
dx(t) = A_0 x(t) dt + \sum_{j=1}^{r} A_j x(t) dw_j(t).
$$

2.2 *Input-output operators*

In (Morozan, 1995) it is proved that if (1) is ESMS then one can define the following input-output operator:

$$
T : \tilde{L}^2(\mathbf{R} \times \Omega, \mathbf{R}^m) \to \tilde{L}^2(\mathbf{R} \times \Omega, \mathbf{R}^p)
$$

by

$$
(Tu)(t) = y_u(t) = Cx_u(t) + Du(t)
$$

where

$$
x_u(t) = \int_{-\infty}^{t} \Phi(t, \tau) Bu(\tau) d\tau
$$

is a solution of the first equation of system (1). One can prove that T is a linear bounded operator.

The following result is an LMI version of the Bounded Real Lemma type result derived in (Morozan, 1995).

Proposition 1 *Assume that the stochastic system (1) is ESMS; then* $\|T\| < \gamma$ *if and only if it exists a symmetric matrix* $X > 0$ *such that:*

$$
\begin{bmatrix}
A_0^T X + XA_0 + \sum_{j=1}^{r} A_j^T XA_j + C^T C & XB + C^T D \\
B^T X + D^T C & D^T D - \gamma^2 I
\end{bmatrix} < 0. \tag{2}
$$

2.3 *Problem formulation*

Consider the stable system with the output y_1 multiplicatively perturbed with white noise:

$$
\begin{aligned}
dx(t) &= (Ax(t) + Bu(t))dt \\
dy_1(t) &= C_1 x(t)(dt + \sigma dw(t)) \\
y_2(t) &= C_2 x(t)
\end{aligned} \tag{3}
$$

where

$A \in \mathbf{R}^{n \times n}$, $B \in \mathbf{R}^{n \times m}$, $C_1 \in \mathbf{R}^{p_1 \times n}$, $C_2 \in \mathbf{R}^{p_2 \times n}$,

$\sigma \in \mathbf{R}$, $w(t)$ is a scalar standard Wiener process and y_2 is a quality output. Given $\gamma > 0$ and $n_f \in \mathbf{N}$, the problem consists in determining the n_f-order deterministic filter:

$$
\begin{aligned}
\dot{x}_f &= A_f x_f + B_f y_1 \\
y_f &= C_f x_f
\end{aligned} \tag{4}
$$

such that the input-output operator

$$
T : \tilde{L}^2(\mathbf{R} \times \Omega, \mathbf{R}^m) \to \tilde{L}^2(\mathbf{R} \times \Omega, \mathbf{R}^{p_2})
$$

from $u \to z$, where $z(t) = y_2(t) - y_f(t)$ has the norm less than γ.

3. MAIN RESULT

The following result provides necessary and sufficient conditions under which the filtering problem formulated in Section 2.3 has a solution.

Theorem 1 *The filtering problem has a solution if and only if there exist the symmetric matrices* $P > 0, R > 0, \tilde{R} > 0$ *of dimension* $(n \times n)$, $(n \times n)$, $(n_f \times n_f)$, *respectively and a* $(n \times n_f)$ *matrix* $\tilde{M}$ *such that:*

$$
\begin{bmatrix}
A^T P + PA + \sigma^2 C_1^T U^T \tilde{R} U C_1 & PB \\
B^T P & -\gamma^2 I
\end{bmatrix} < 0 \tag{5}
$$

$$\begin{bmatrix} A^T R + RA + MUC_1 + C_1^T U^T M^T \\ +\sigma^2 C_1^T U^T \tilde{R} U C_1 + C_2^T C_2 & RB \\ B^T R & -\gamma^2 I \end{bmatrix} < 0 \quad (6)$$

$$\begin{bmatrix} R & M \\ M^T & \tilde{R} \end{bmatrix} > 0 \quad (7)$$

$$rank \begin{bmatrix} P - R & M \\ M^T & -\tilde{R} \end{bmatrix} = n_f \quad (8)$$

where $U = \begin{bmatrix} I \\ 0 \end{bmatrix}$ if $n_f \geq p_1$ or $U = [I \quad 0]$ if $n_f < p_1$.

Proof. When coupling the filter (4) to the system (3) one obtains the resulting system:

$$\begin{aligned} dx(t) &= (Ax(t) + Bu(t))dt \\ dx_f(t) &= (A_f x_f(t) + C_1 x(t))dt + \sigma B_f C_1 x(t) dw(t) \\ z(t) &= C_2 x(t) - C_f x_f(t) \end{aligned}$$

or equivalently

$$\begin{aligned} d\begin{bmatrix} x \\ x_f \end{bmatrix} &= \begin{bmatrix} A & 0 \\ B_f C_1 & A_f \end{bmatrix} \begin{bmatrix} x \\ x_f \end{bmatrix} dt + \begin{bmatrix} B \\ 0 \end{bmatrix} u dt \\ &+ \begin{bmatrix} 0 & 0 \\ \sigma B_f C_1 & 0 \end{bmatrix} \begin{bmatrix} x \\ x_f \end{bmatrix} dw \quad (9) \\ z &= [C_2 \quad -C_f] \begin{bmatrix} x \\ x_f \end{bmatrix} \end{aligned}$$

Introduce the following notations:

$$\begin{aligned} A_0 &= \begin{bmatrix} A & 0 \\ B_f C_1 & A_f \end{bmatrix}; \; A_1 = \begin{bmatrix} 0 & 0 \\ \sigma B_f C_1 & 0 \end{bmatrix}; \\ B_0 &= \begin{bmatrix} B \\ 0 \end{bmatrix}; \; C = [C_2 \quad -C_f] \end{aligned} \quad (10)$$

According with Proposition 1 the γ-attenuation condition for the input-output operator associated with the resulting system (9) is satisfied if and only there exists $X > 0$ such that:

$$\begin{bmatrix} A_0^T X + XA_0 + A_1^T X A_1 + C^T C & XB_0 \\ B_0^T X & -\gamma^2 I \end{bmatrix} < 0. \quad (11)$$

Consider the following partition of X :

$$X = \begin{bmatrix} R & M \\ M^T & \tilde{R} \end{bmatrix},$$

where $R \in \mathbf{R}^{n \times n}, \tilde{R} \in \mathbf{R}^{n_f \times n_f}$ and $M \in \mathbf{R}^{n \times n_f}$. Then using expressions (10), the inequality (11) becomes:

$$\mathfrak{L}(R, M, R, A_f, B_f, C_f) = \begin{bmatrix} \mathfrak{L}_{11} & \mathfrak{L}_{12} & \mathfrak{L}_{13} & \mathfrak{L}_{14} \\ \mathfrak{L}_{12}^T & \mathfrak{L}_{22} & \mathfrak{L}_{23} & \mathfrak{L}_{24} \\ \mathfrak{L}_{13}^T & \mathfrak{L}_{23}^T & \mathfrak{L}_{33} & \mathfrak{L}_{34} \\ \mathfrak{L}_{14}^T & \mathfrak{L}_{24}^T & \mathfrak{L}_{34}^T & \mathfrak{L}_{44} \end{bmatrix} < 0 \quad (12)$$

where

$$\begin{aligned} \mathfrak{L}_{11} &= A^T R + RA + MB_f C_1 + C_1^T B_f^T M^T \\ &\quad + \sigma^2 C_1^T B_f^T \tilde{R} B_f C_1 + C_2^T C_2, \\ \mathfrak{L}_{12} &= A^T M + C_1^T B_f^T \tilde{R} + MA_f - C_2^T C_f, \\ \mathfrak{L}_{13} &= RB, \\ \mathfrak{L}_{14} &= 0, \\ \mathfrak{L}_{22} &= A_f^T \tilde{R} + \tilde{R} A_f, \\ \mathfrak{L}_{23} &= M^T B, \\ \mathfrak{L}_{24} &= -C_f^T, \\ \mathfrak{L}_{33} &= -\gamma^2 I, \\ \mathfrak{L}_{34} &= 0, \\ \mathfrak{L}_{44} &= -I. \end{aligned} \quad (13)$$

Assume that B_f is full rank column. This is not a restrictive assumption since in the case when the solution of the filtering problem is with B_f non-full column, from the structure of (12) with (13) it follows that there exists a small enough perturbation of B_f such that the perturbed matrix is full rank and it verifies (12). Morover for B_f full rank column always exists a nonsingular transformation T such that $TB_f = \begin{bmatrix} I \\ 0 \end{bmatrix}$. Therefore without loosing the generality one can choose $B_f = \begin{bmatrix} I \\ 0 \end{bmatrix}$. A similar reasoning can be made is when B_f is full rank row.

The condition (12) can be expressed as:

$$Z + \hat{P}^T \Omega \hat{Q} + \hat{Q}^T \Omega^T \hat{P} < 0 \quad (14)$$

where

$$Z = \begin{bmatrix} \mathfrak{L}_{11} & A^T M + C_1^T B_f^T \tilde{R} & RB & 0 \\ M^T A + \tilde{R} B_f C_1 & 0 & M^T B & 0 \\ B^T R & B^T M & -\gamma^2 I & 0 \\ 0 & 0 & 0 & -I \end{bmatrix} \quad (15)$$

$$\hat{P} = \begin{bmatrix} M^T & \tilde{R} & 0 & 0 \\ -C_2 & 0 & 0 & -I \end{bmatrix}; \; \hat{Q} = [0 \quad I \quad 0 \quad 0]; \; \Omega = \begin{bmatrix} A_f \\ C_f \end{bmatrix}.$$

Based on the Projection Lemma (Boyd *et al.*, 1994), it follows that (14) has a solution Ω if and only if

$$W_{\hat{P}}^{T} Z W_{\hat{P}} < 0 \qquad (16)$$

$$W_{\hat{Q}}^{T} Z W_{\hat{Q}} < 0 \qquad (17)$$

where $W_{\hat{P}}$ and $W_{\hat{Q}}$ denote bases of the null spaces of $\hat{P}$ and $\hat{Q}$ respectively.

Further, perform the following partition of X^{-1} according with the partition of X:

$$X^{-1} = \begin{bmatrix} S & N \\ N^T & \tilde{S} \end{bmatrix}.$$

With these notations one obtains that

$$W_{\hat{P}}^{T} = \begin{bmatrix} S & N & 0 & -SC_2^{T} \\ 0 & 0 & -I & 0 \end{bmatrix}$$

and

$$W_{\hat{Q}} = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}.$$

Direct algebraic computations based on the fact that $S^{-1} - R = MN^{T} S^{-1}$ show that (16) is equivalent with (5), where $P = S^{-1}$ and (17) is equivalent with (6). The rank condition (8) follows from the relationship between X and X^{-1} and shows that $S^{-1} = R - M\tilde{R}^{-1}M^{T}$. €

If the necessary and sufficient conditions of Theorem 1 are accomplished then a solution of the filtering problem can be easily obtained by solving the basic LMI (14) with respect to Ω.

Remark 1 *The rank condition (8) leads to a nonconvex optimization problem that is much harder to be solbed than a convex one. Such situation also appear in control problems with controller rank constraints. Due to the importance of optimization problem with rank constraints, the recent literature offers certain effective procedures to solve such problems (e.g. Grigoriadis and Skelton, 1996, Gubin et. al., 1967).*

4. A NUMERICAL EXAMPLE

In the present section a numerical example illustrates the previous theoretical developments. The Instrumental Landing System (ILS) is a radioelectronic equipment which provides at the bord of the aircrafts on-line informations concerning the aircraft position relative to some glideslope references in the landing phase of the flight. The glideslope signal ($[\mu A]$) can be expressed as:

$$i_{gs} = K i_0 \qquad (18)$$

where the multiplicative factor K depends on the glideslope sensitivity and i_0 denotes the nominal signal. International standards limit maximum deviation from the niminal glideslope sensitivity at $\pm 25\%$, $\pm 20\%$, $\pm 10\%$, respectively, function of the performance category I, II or III of ILS system (Rauw, 1998).

If σ denotes the mean square deviation of K then $P(|K(t) - K_0| < 3\sigma) \geq 0.997$, where K_0 denotes the nominal value of the multiplicative factor. This probability increases when $\sigma \to 0$. Then, taking $\sigma = 0.06$ for which $3\sigma = 0.18$ one can accomplish the standards requirements for Cat. II of ILS. Thus the multiplication factor K in (18) can be replaced by:

$$K = K_0 + \sigma\xi \qquad (19)$$

where ξ is a white noise with unitary covariance. If the altitude dynamics is given by $\dot{x} = Ax + Bu$ with $i_0 = Cx$, then according with (18) and (19) the glide-slope measured signal is $i_{gs} = (K_0 + \sigma\xi)Cx$. Hence one obtains a stochastic system of form (3) for which a deterministic filter is designed. For

$$A = -\frac{1}{30},\; B = \frac{50}{30}, C_1 = 1, C_2 = 1, K_0 = 1$$

applying the result stated in Theorem 1, one obtains for the level of attenuation $\gamma = 5$, $R = 1.9457, M = -0.6692, \tilde{R} = 0.3132; P = 0.5161$ and the filter

$$\dot{x}_f = -0.4073 x_f + y_1$$
$$y_f = 0.4450 x_f.$$

In Figure 1 the unfiltered and the filtered signals are

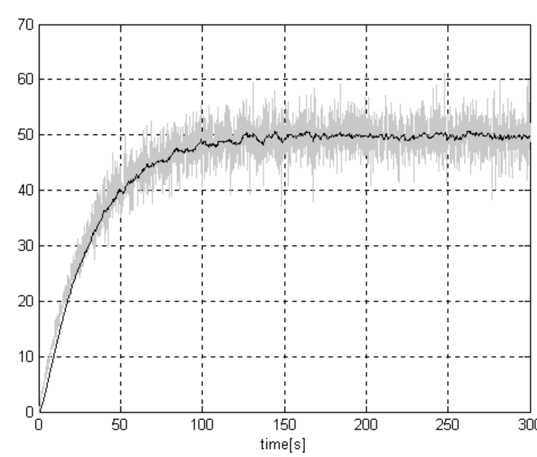

Figure 1. Filtered (black) and unfiltered signal

represented, respectively.

Then a Kalman filter for the attitude dynamics has been designed by tuning the covariance matrices Q_0 and R_0 of the control and output additive white noise perturbations. For $Q_0 = 100$ and $R_0 = 0.1$ the resulting Kalman filter provides the results shown in Figure 2 where the filtered and the unfiltered signals are represented.

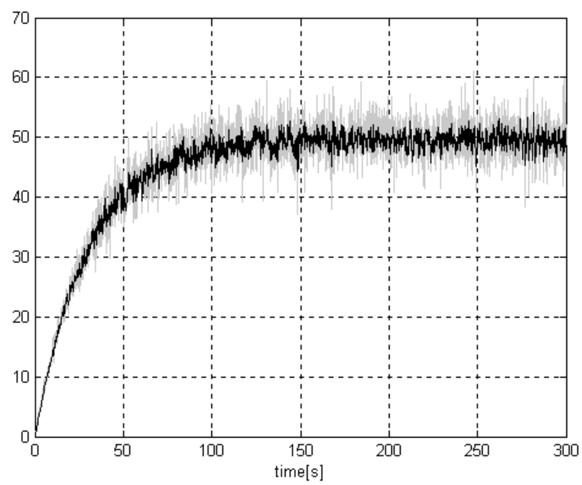

Figure 2. Filtered signal by Kalman filter (black) and unfiltered signal

Analysing the numerical results illustrated in Figures 1 and 2 one concludes that a filter designed using the specific multiplicative character of the stochastic perturbations provides improved results than the ones given by Kalman filters which are appropriate in the case of additive stochastic perturbations.

5. CONCLUSIONS

The present paper describes an H^∞ type filtering approach for stochastic systems subjected to multiplicative white noise. The order of the deterministic filter is fixed. Necessary and sufficient conditions of solvability are derived in terms of two linear inequalities coupled by a rank constraint condition. A numerical example illustrates the theoretical developments.

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