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AN $\mathcal{H}_\infty$ DYNAMIC ANTI-WINDUP SCHEME FOR INPUT-CONSTRAINED LINEAR SYSTEMS

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Abstract: This paper proposes a dynamic compensation scheme for input-constrained linear systems to cope with the windup phenomenon. Given a linear controller for such a linear system designed without considering its input constraints, an additional dynamic compensator is proposed to account for the constraints. This dynamic anti-windup scheme is based on $\mathcal{H}_\infty$ optimization, and some stability properties of the resulting closed-loop system are given.

Keywords: Saturation, Constraints, Windup, $\mathcal{H}_\infty$ optimization, Compensation

1. INTRODUCTION

Actuator saturation is a nonlinear problem that needs to be dealt with in almost all practical control systems. Feedback loops are broken when the actuators saturate. Performance deterioration and even instability may result especially when the plants or controllers are unstable. A general term for these phenomena is referred to as windup, and compensation for preventing this windup is called anti-windup (Åström and Wittenmark, 1995). Recently, a rigorous definition of the anti-windup is presented on the basis of an $\mathcal{L}_2$ criterion, and it is shown that all static observer-based compensation schemes satisfy the definition at least locally (Kapoor *et al.*, 1998).

Generally, the following strategy is adopted for anti-windup: design a controller ignoring the saturation, and then add an appropriate compensator to account for the saturation. Some important anti-windup schemes include conventional anti-windup (Doyle *et al.*, 1987), observer-type techniques (Åström and Wittenmark, 1984; Walgama and Sternby, 1990), conditioning techniques (Hanus *et al.*, 1987; Peng *et al.*, 1998), unified framework by coprime factorization (Kothare *et al.*, 1994), finite gain techniques (Teel and Kapoor, 1997; Kapoor *et al.*, 1998), and optimization based methods (Park and Choi, 1995; Park and Choi, 1997; Miyamoto and Vinnicombe, 1996; Miyamoto, 1997; Edwards and Postlethwaite, 1998; Crawshaw and Vinnicombe, 2000).

This paper proposes an $\mathcal{H}_\infty$ dynamic anti-windup compensation scheme for input-constrained linear control systems. A new definition of $\mathcal{H}_\infty$ anti-windup is given, which takes account of the gain from $\mathrm{sat}(u)-u$ to $y_p-\bar{y}_p$ where $\mathrm{sat}(u)$ and u are the actual saturated input and the unsaturated control input, and y_p and $\bar{y}_p$ are the real output and the fictitious output computed assuming the absence of the saturation. To design a compensator satisfying this definition, an error dynam-

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ics between the state variables of systems with and without saturating actuators is derived and rewritten in a standard form for the so-called $\mathcal{H}_\infty$ disturbance feedforward problem. The proposed compensator is then obtained by applying the standard $\mathcal{H}_\infty$ optimization procedure to the resulting linear error model, which guarantees that the induced $\mathcal{L}_2$ gain from $\text{sat}(u)-u$ to $y_p-\bar{y}_p$ is less than a given number γ, with the hope of making y_p close to its desired value, i.e., $\bar{y}_p$. It is shown that this number γ can actually be chosen arbitrarily. Some stability properties of the resulting closed-loop system are given, and the superiority of the proposed design method is illustrated by simulation.

The following notation is used in this paper:

- For a vector $u \in \Re^m$, $\|u\|$ is the Euclidean norm of u.
- For a vetor $u \in \Re^m$ and a set $\mathcal{U} \subset \Re^m$, the distance from u to $\mathcal{U}$ is defined as

$$dist(u,\mathcal{U}) := \inf_{w\in\mathcal{U}} \|u-w\|. \tag{1}$$

- For a vector-valued signal $f(t)$, its 2-norm is defined as

$$\|f\|_2 := \left(\int_0^\infty \|f(t)\|^2 dt\right)^{\frac{1}{2}}, \tag{2}$$

and $f(t)$ is said to belong to $\mathcal{L}_2$ if $\|f\|_2 < \infty$.

2. ANTI-WINDUP CONFIGURATION AND PROBLEM FORMULATION

Consider an unconstrained linear time invariant plant of the form

$$\dot{\bar{x}}_p = A_p\bar{x}_p + B_p\bar{u} \tag{3}$$

$$\bar{y}_p = C_p\bar{x}_p \tag{4}$$

where $\bar{x}_p \in \Re^n$ is the state, $\bar{y}_p \in \Re^m$ is the output, $\bar{u} \in \Re^m$ is the control, and A_p, B_p, C_p are constant matrices of appropriate dimensions. In this paper, $\bar{(\cdot)}$ represents the unconstrained signal (i.e. in the absence of input saturation). For this plant, a nominal controller of the form

$$\dot{\bar{x}}_c = A_c\bar{x}_c + B_c(r-\bar{y}_p) \tag{5}$$

$$\bar{u} = C_c\bar{x}_c + D_c(r-\bar{y}_p) \tag{6}$$

is assumed to be given such that the closed-loop system without input saturation is stable and well-behaved, where $r \in \Re^m$ is the set-point, $\bar{x}_c \in \Re^{n_c}$ is the controller state in the absence of input saturation, and A_c, B_c, C_c are constant matrices of appropriate dimensions. Because of the stability of the unconstrained closed-loop (5) and (6), we have the following:

$$A := \begin{bmatrix} A_p - B_pD_cC_p & B_pC_c \\ -B_cC_p & A_c \end{bmatrix} \tag{7}$$

is Hurwitz.

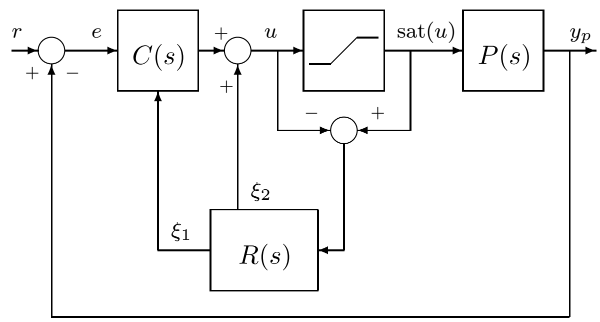

Fig. 1. The proposed closed-loop with dynamic anti-windup

Now consider the same plant but with input saturation

$$\dot{x}_p = A_px_p + B_p\text{sat}(u) \tag{8}$$

$$y_p = C_px_p. \tag{9}$$

In this description,

$$\text{sat}(u) := [\text{sat}_1(u_1)\ \text{sat}_2(u_2)\ \cdots\ \text{sat}_m(u_m)]^T \tag{10}$$

where u_i (i=1,2,...,m) is the i-th element of u and

$$\text{sat}_i(u_i) = \begin{cases} u_{i,min} & \text{if } u_i < u_{i,min} \\ u_i & \text{if } u_{i,min} \le u_i \le u_{i,max} \\ u_{i,max} & \text{if } u_i > u_{i,max} \end{cases} \tag{11}$$

with $u_{i,min}$ and $u_{i,max}$ being the lower and upper limits for u_i. For this input-constrained system, we propose, on the basis of the nominal controller (5)-(6), the following control scheme with anti-windup compensation

$$\dot{x}_c = A_cx_c + B_c(r-y_p) + \xi_1 \tag{12}$$

$$u = C_cx_c + D_c(r-y_p) + \xi_2 \tag{13}$$

where ξ_1, ξ_2 are compensation signals for anti-windup, which are obtained from the dynamic compensator

$$\begin{aligned} \dot{x}_{aw} &= A_{aw}x_{aw} + B_{aw}(\text{sat}(u)-u) \\ \xi_1 &= C_{1aw}x_{aw} + D_{1aw}(\text{sat}(u)-u) \\ \xi_2 &= C_{2aw}x_{aw} + D_{2aw}(\text{sat}(u)-u) \end{aligned} \tag{14}$$

where A_{aw}, B_{aw}, C_{1aw}, D_{1aw}, C_{2aw}, D_{2aw} are parameters to be designed and x_{aw} is the state of the dynamic anti-windup compensator. The resulting closed-loop system is depicted in Fig. 1, where $C(s)$ and $R(s)$ represent the controller (12)-(13) and the anti-windup compensator (14), respectively.

For the design purposes of this paper, we define the $\mathcal{H}_\infty$ anti-windup problem as follows:

Definition 1: The compensator $R(s)$ is said to solve the $\mathcal{H}_\infty$ anti-windup problem if the following three conditions are satisfied:

- *Condition* 1: For a performance level γ, the following norm bound is achieved:

$$\|y_p - \bar{y}_p\|_2 \le \gamma\|\text{sat}(u)-u\|_2$$

for $\text{sat}(u) - u \in \mathcal{L}_2$.

- *Condition* 2: Let $\mathcal{U}$ be any strict compact subset of the set $[u_{1,min}, u_{1,max}] \times \cdots \times [u_{m,min}, u_{m,max}]$, and X_a be

$$X_a := \begin{bmatrix} x_p - \bar{x}_p \\ x_c - \bar{x}_c \\ x_{aw} \end{bmatrix}.$$

Then $X_a \in \mathcal{L}_2$ if $\|x_{aw}(0)\|$ and $\|dist(\bar{u}, \mathcal{U})\|_2$ are sufficiently small.
- *Condition* 3: If $\mathrm{sat}(u(t)) - u(t) = 0$, then $X_a = 0$ is asymptotically stable.

The objective of this paper is to design the compensator $R(s)$ such that all these conditions are satisfied.

3. ANTI-WINDUP SYNTHESIS AND ANALYSIS

3.1 Derivation of the $\mathcal{H}_\infty$ anti-windup

We first derive an error model between the constrained closed-loop with anti-windup and the unconstrained closed-loop without input saturation. To this end, define the following variables:

$$Z = \begin{bmatrix} y_p - \bar{y}_p \\ D_u U \end{bmatrix}, \ W = \mathrm{sat}(u) - u$$

$$Y = W, \ U = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}, \ X = \begin{bmatrix} x_p - \bar{x}_p \\ x_c - \bar{x}_c \end{bmatrix}$$

where D_u is a weighting matrix for the compensation signal U. The vector Z contains $y_p - \bar{y}_p$ to be minimized; the reason for including $D_u U$ in Z is to ensure the existence of a proper compensator. W and Y are the driving force and measurement for anti-windup compensation, respectively, and are equivalent here. Using these variables, the error model is depicted in Fig. 2, where $G(s)$ is

$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ 0 & D_{21} & 0 \end{array}\right] \tag{15}$$

with A as in (7) and

$$B_1 = \begin{bmatrix} B_p \\ 0 \end{bmatrix}, \ B_2 = \begin{bmatrix} 0 & B_p \\ I & 0 \end{bmatrix}, \ C_1 = \begin{bmatrix} C_p & 0 \\ 0 & 0 \end{bmatrix},$$

$$D_{12} = \begin{bmatrix} 0 \\ D_u \end{bmatrix}, \ D_{21} = I.$$

Note that the model in Fig. 2 is derived in such a way that allows us to apply the results of the $\mathcal{H}_\infty$ disturbance feedforward problem considered in (Doyle *et al.*, 1989), thereby leading to the satisfaction of Condition 1. Hence we derive the following $\mathcal{H}_\infty$ anti-windup compensator

$$\begin{aligned} \dot{x}_{aw} &= (A + \widetilde{B}_2 F_\infty) x_{aw} + B_1(\mathrm{sat}(u) - u) \\ U &= D_u^{-1} F_\infty x_{aw} \end{aligned} \tag{16}$$

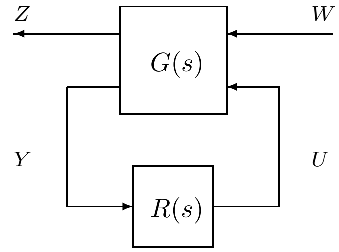

Fig. 2. Block diagram for dynamic anti-windup compensator design

where x_{aw} is the compensator state, $F_\infty = -\widetilde{B}_2^T X_\infty$, $\widetilde{B}_2 = B_2 D_u^{-1}$, and X_∞ satisfies the following ARE (algebraic Riccati equation)

$$X_\infty A + A^T X_\infty + X_\infty(\gamma^{-2} B_1 B_1^T - \widetilde{B}_2 \widetilde{B}_2^T) X_\infty + C_1^T C_1 = 0. \tag{17}$$

It is shown in section 3.2 that the resulting compensator in (16) can be made to satisfy all the conditions given in Definition 1 for any γ.

3.2 Analysis of anti-windup and stability

We first give the following lemma, which concerns the solvability of the ARE in (17).

Lemma 1. Assume that $D_u = d_u I$ and that the matrix A is Hurwitz[2]. If $\gamma \geq d_u > 0$, then the ARE in (17) is guaranteed to have a positive semidefinite solution X_∞, which in turn ensures the existence of the anti-windup compensator (16).

Proof: If $D_u = d_u I$, $\gamma^{-2} B_1 B_1^T - B_2 D_u^{-1}(B_2 D_u^{-1})^T$ is equal to

$$\begin{bmatrix} -(d_u^{-2} - \gamma^{-2}) B_p B_p^T & 0 \\ 0 & -d_u^{-2} I \end{bmatrix},$$

which can then be written in the form of $B_o B_o^T$ for some B_o if $\gamma \geq d_u > 0$. This, together with the stability of the matrix A, ensures the existence of a positive semidefinite solution to the ARE, completing the proof. ■

It follows from lemma 1 that the compensator (16) always exists for any $\gamma > 0$ as d_u can be set to any positive number $\leq \gamma$. The next lemma shows that the proposed compensator satisfies Condition 2 as well under the same condition.

Lemma 2. Assume that $D_u = d_u I, \gamma \geq d_u > 0$ and that the matrix A is Hurwitz. Then the compensator in (16) satisfies Condition 2.

Proof: The proof of this lemma can be obtained as in that of theorem 1 in (Kapoor *et al.*, 1998) and thus is omitted here. ■

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[2] As mentioned in section 2, A is always designed to be stable.

Now we present the main theorem of this paper, which shows that the problem described in Definition 1 can be solved.

Theorem 1. Assume that $D_u = d_u I, \gamma \geq d_u > 0$ and that the matrix A is Hurwitz. Then the compensator given in (16) satisfies all the conditions in Definition 1.

Proof: Having established the existence of the proposed compensator and the satisfaction of Condition 2 from Lemmas 1 and 2, we now prove that conditions 1 and 3 are satisfied. To this end, write the state equation as

$$\dot{X} = AX + B_1 W + B_2 D_u^{-1} \widetilde{U} \tag{18}$$

$$Z = C_1 X + D_{12} D_u^{-1} \widetilde{U}, \; Y = W \tag{19}$$

where $\widetilde{U} = D_u U$ and $D_{12} D_u^{-1} = [0 \; I]^T$. The transfer matrix from $[W^T \; \widetilde{U}^T]^T$ to $[Z^T \; Y^T]^T$, denoted by $\widetilde{G}(s)$, is then given as follows:

$$\widetilde{G}(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 D_u^{-1} \\ \hline C_1 & 0 & \begin{bmatrix} 0 \\ I \end{bmatrix} \\ 0 & I & 0 \end{array}\right]. \tag{20}$$

This realization of $\widetilde{G}(s)$ satisfies the conditions for the disturbance feedforward problem (Zhou *et al.*, 1996). Hence we have

$$\|Z\|_2 \leq \gamma \|\text{sat}(u) - u\|_2.$$

The satisfaction of Condition 1 then follows from the inequality

$$\|y_p - \bar{y}_p\|_2 \leq \|Z\|_2.$$

We now set $W = 0$ to prove the satisfaction of Condition 3, resulting in

$$\dot{x}_{aw} = (A + \widetilde{B}_2 F_\infty) x_{aw}$$
$$U = D_u^{-1} F_\infty x_{aw}$$
$$\dot{X} = AX + B_2 U.$$

Since A is stable and so is $A + \widetilde{B}_2 F_\infty$ by theorem 13.5 in (Zhou *et al.*, 1996), x_{aw}, U and X are bounded and tend to 0. Hence Condition 2 is satisfied as well. ■

In the following theorem, we discuss the local exponential stability of the closed-loop system. This theorem extends theorem 1 in (Kapoor *et al.*, 1998) for the case of dynamic anti-windup compensation.

Theorem 2. Assume that $D_u = d_u I, \gamma \geq d_u > 0$ and that the matrix A is Hurwitz. Suppose also that there exists an initial condition $(\bar{x}_p^{ss}, \; \bar{x}_c^{ss})$ for equation

$$\begin{bmatrix} \dot{\bar{x}}_p \\ \dot{\bar{x}}_c \end{bmatrix} = A \begin{bmatrix} \bar{x}_p \\ \bar{x}_c \end{bmatrix} + \begin{bmatrix} B_p D_c \\ B_c \end{bmatrix} r \tag{21}$$

so that $\bar{u}(t) \equiv u^* \in \mathcal{U}$ and $(\bar{x}_p(t), \; \bar{x}_c(t)) = (\bar{x}_p^{ss}, \; \bar{x}_c^{ss})$. Then the unconstrained closed-loop and the constrained closed-loop with the proposed anti-windup converge to $(\bar{x}_p^{ss}, \; \bar{x}_c^{ss})$ and $(\bar{x}_p^{ss}, \; \bar{x}_c^{ss}, \; 0)$, respectively, if and only if the origin of the system

$$\dot{X}_a = \begin{bmatrix} A & \widetilde{B}_2 F_\infty \\ 0 & A + \widetilde{B}_2 F_\infty \end{bmatrix} X_a + \begin{bmatrix} B_1 \\ B_1 \end{bmatrix} [\text{sat}(v) - v] \tag{22}$$

$$v = u^* + K X_a, \; K = [-D_c C_p \;\; C_c \;\; [0 \; I] D_u^{-1} F_\infty]$$

is locally exponentially stable with domain of attraction containing the point $(x_p(0) - \bar{x}_p^{ss}, \; x_c(0) - \bar{x}_c^{ss}, \; x_{aw}(0))$.

Proof: As both A and $A + \widetilde{B}_2 F_\infty$ are stable, the proof of this theorem can be obtained as in that of theorem 1 in (Kapoor *et al.*, 1998), and thus is omitted here. ■

4. ILLUSTRATIVE EXAMPLE

Consider the following plant and controller considered in (Miyamoto, 1997):

$$\dot{x}_p = \begin{bmatrix} -0.1 & 0 \\ 0 & -0.1 \end{bmatrix} x_p + \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} \text{sat}(u)$$

$$y_p = \begin{bmatrix} 4 & -5 \\ -3 & 4 \end{bmatrix} x_p$$

$$\dot{x}_c = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (r - y_p)$$

$$u = \begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix} x_c + \begin{bmatrix} 40 & 50 \\ 30 & 40 \end{bmatrix} (r - y_p).$$

Simulations are performed assuming that $r(t) = [0.6 \; 0.8]^T$. Fig. 3 shows the response of the system without saturation. When the control inputs are constrained to be in $[-15, 15]$, the responses of the saturated system without compensation get much worse as shown in Fig. 4. Now we apply the proposed dynamic compensation method to improve the performance. The design parameter $\gamma (= d_u)$ [3] is tuned to 0.01. Fig. 5 compares the proposed scheme with other major anti-windup schemes in the case where the so-called AN (artificial nonlinearity) (Campo and Morari, 1990; Park and Choi, 1995) is not used. The superiority of the proposed method is clearly illustrated in this figure.

5. CONCLUSION

This paper proposes a new dynamic $\mathcal{H}_\infty$ anti-windup scheme for input-constrained linear systems. An error dynamics between the constrained

[3] The choice of $\gamma = d_u$ can be interpreted as that leading to the least active compensation whilst achieving a given norm bound, which seems quite reasonable. If this is the case, then the user is given just one design parameter, i.e., γ as d_u is just fixed to γ.

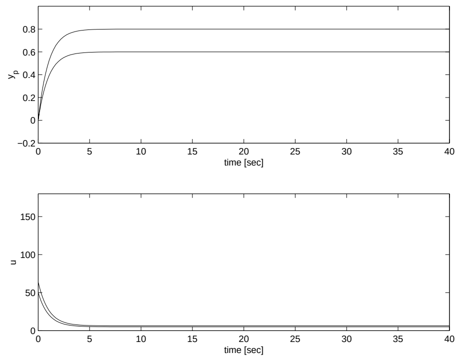

Fig. 3. Responses of the unsaturated system

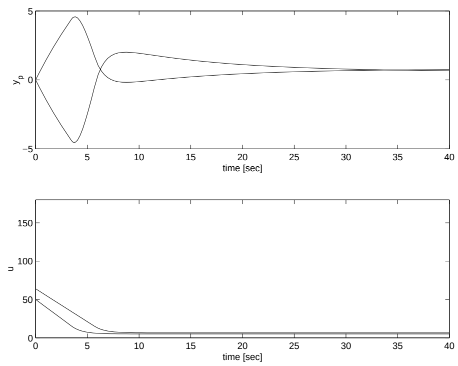

Fig. 4. Responses of the saturated system without compensation

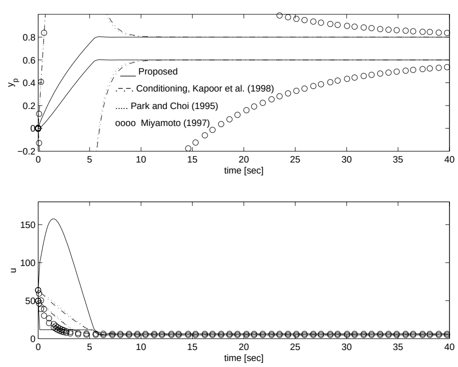

Fig. 5. Responses of the saturated system with various compensation schemes; solid lines (proposed anti-windup), dashdotted lines (conditioning technique), dashdotted lines (Kapoor *et al.*, 1998), dotted lines (Park and Choi, 1995), circled lines (Miyamoto, 1997)

and unconstrained state variables is derived and rewritten in a standard form for the so-called $\mathcal{H}_\infty$ disturbance feedforward problem. The proposed compensator is then obtained by applying the standard $\mathcal{H}_\infty$ optimization procedure to the resulting linear error model. It is shown that the compensator can be obtained for any given norm bound. Some stability properties of the resulting closed-loop systems are also given, and the superiority of the proposed design method is illustrated by simulation.

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