15th Triennial World Congress, Barcelona, Spain

A HISTORY OF ANALYTIC FEEDBACK SYSTEM DESIGN

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Abstract: In this paper the history of *analytic*, or *synthesis*, methods for linear feedback system design will be reviewed. The term *analytic design* is used to define design methods where an existence theorem is available, and when a solution is known to exist, a computable algorithm for computing a solution is available. Examples of analytic design discussed in this paper include, pole-placement, linear-quadratic control, mean-square control, Youla parameterization, gain and phase-margin optimization, and robust stabilization.

Keywords: Feedback system; linear systems; analytic design, synthesis, history.

1. INTRODUCTION

In this paper we will review the history of analytic feedback design. This review is limited to linear time-invariant systems, characterized by transfer functions or state-space equations. The term *feedback design* implies the determination of a compensator which meets specifications, given a mathematical model of the system to be controlled (plant), with compensator and plant in a feedback configuration. The term *analytic design* is defined to be any design technique which includes the following two elements:

1. An *existence theorem* which can be used to test for the existence of a solution to the design problem.
2. A *computable algorithm* which finds the solution, when one is known to exist.

This is a fairly standard definition, and the term *synthesis* is often substituted for *analytic design* in the the control literature. The two terms will be used interchangeably in this paper. In contrast to *analytic design* methods, we have *ad hoc design* methods, where typically no existence theorem is available, and solution are sought for using trial-and-error methods, with no guarantee of convergence.

Unfortunately, many "practical" design problems do not have "analytic" solutions, and one has no choice but to resort to some ad hoc design method. In additions most analytic methods result in high order compensators. However it is often valuable to know about analytic solutions, and when they are available. A simple example of analytic design is the use of the Nyquist stability criterion, to design a stabilizing constant-gain compensator for a single-input-single-output (SISO) plant. In this case, a Nyquist plot may be used to determine whether a stabilizing gain K exists, and by selecting the gain to result in the proper number of encirclements of the $-1/K$ point, a stabilizing solution can be found. However, if a dynamic compensator is to be explored, the Nyquist criterion can be used only in a trial-and-error way.

A third design class, not reviewed here, is what is sometimes called, *numerical design*. In numerical design there generally is no explicit existence theorem, but a numerical algorithm is available which is know to converge to a solution when one exists. An example of a numerical design is the use of Linear Matrix Inequalities (LMIs) to determine a state-feedback control law which stabilizes a given plant. See, for example, Boyd et al. (1994).

2. EARLY RESULTS

Perhaps one of first analytic-feedback-design results appeared in a study by James Clarke Maxwell of the fly-ball steam-engine governor, published in 1868 Maxwell (1868). In this paper Maxwell derived conditions on the a linearized model of the closed-loop system for stability, which are sometimes referred to Dorato (2000a) as *Maxwell's stability criterion* for cubic polynomials. In particular he showed that the closed-loop system is stable if all the coefficients of the cubic polynomial

$$p(s) = a_1 s^3 + a_2 s^2 + a_3 s + a_4$$

are positive and,

$$a_2 a_3 - a_1 a_4 > 0.$$

The coefficients a_i are functions of the steam-engine and governor parameters. The above inequalities can then be used to determine if a stabilizing fly-ball governor exits, and by selecting parameter values that satisfy the inequalities, a stabilizing governor can be designed. Although this was an analytic result for a very special problem, stable control of steam engines played a critical role in the industrial revolution of the twentieth century.

In the period from 1870 to 1930, feedback control was used in a number of systems, perhaps most notable being the vacuum tube amplifier, but most designs were done by ad hoc methods. In 1932 Harry Nyquist Nyquist (1932) published a frequency-domain based stability criterion which, as noted in the Introduction, may be used as the basis for analytic design of a stable feedback system, if the the compensator is limited to a pure gain. If a dynamic compensator is required, the Nyquist stability criterion provides only ad hoc guidance to a final design. The text of Isaac Horowitz Horowitz (1963) has the term *synthesis* in its title, but it is based on ad hoc use of the Nyquist criterion, and does fit the definition of *synthesis* given here.

In 1942, Norbert Wiener developed his theory on optimal mean-square filtering. The result were finally published in 1949 Wiener (1949). Based on these result, analytic techniques were developed for the design of feedback systems which minimized a mean-squared performance measure using frequency domain concepts (e.g. spectral factorization). In the text of Newton, Gould, and Kaiser, G.C. Newton et al. (1957) is devoted entirely to analytic mean-square design. The text of John G. Truxal Truxal (1955) includes the term *synthesis* in the title, however the only chapter that deals with synthesis, as defined here, is a chapter on optimal mean-square design.

An excellent review of the history of control engineering for the period 1800-1955 may be found in the two volumes of Stuart Bennett, Bennett (1979), Bennett (1993).

3. THE STATE-SPACE REVOLUTION

At the First IFAC Congress of 1960, Rudy Kalman Kalman (1960) introduced the control community to state-space concepts, where the system to be controlled is characterized by matrix equations of the form,

$$\dot{x} = Ax + Bu, \quad y = Cx + Du \tag{1}$$

where the vector x represents the state of the system, u the control input, and y the output.

Kalman also introduced the concepts of *controllability* and *observability* for state-space equation of the form in (1). In particular the system is controllable if

$$rank[B \; AB \; A^2B ... A^{n-1}B] = n$$

and observable if

$$rank[C' \; A'C' \; (A')^2 C' ... (A')^{n-1} C']' = n$$

where n is the size of the square system matrix A.

One of the first analytic design techniques that resulted from the new state-space approach was that, if a system is controllable, the closed-loop eigenvalues can be arbitrarily located in the complex plane (pole assignment). See, for example Wonham (1967), for the general multi-input result. This result may be summarized as follows:

Design Problem: Find a state-feedback control law, $u = -Kx$d, such that the eigenvalues of $A - BK$ take on arbitrary given values.
Existence Theorem: A solution exists if the system in Equation (1) is controllable.
Solution: The MATLAB function *place* directly computes the matrix K, given a vector of desired eigenvalues p. The algorithm for *place* is based on the computation of generalized eigenvectors.

With the introduction of Pontryagin's maximum principle and Bellman's dynamic programming, see for example Athans and Falb (1966), in the late fifties, analytic design techniques for optimization of integral-quadratic performance measures became available. This type of design is referred to as *Linear Quadratic (LQ)* design. A typical analytic LQ design result is the following:

Design Problem: Find a state-feedback control law, $u = -Kx$, which yields a stable close-loop system and minimizes the performance measure,

$$V = \int_0^\infty (y'(t)y(t) + u'(t)Ru(t))dt \qquad (2)$$

Existence Theorem: A solution exists if the system is controllable and observable, and the matrix R is positive definite.
Solution: The matrix K may be computed the solution of an algebraic Riccati equation (ARE), which in turn may be solved by the computation of eigenvalues and eigenvectors. The MATLAB function *lqr* directly computes K.

In 1961, Kalman and Bucy Kalman and Bucy (1961), solved the Wiener filtering problem using state-space methods. They were able to design a filter which generated a mean-square optimal estimate, $\hat{x}$, of the state x when the output was corrupted with white noise. The optimal filter required a solution of a Riccati equation, and the filter was of the same order as the plant.
In 1963, Gunckle and Franklin Gunckel and Franklin (1963) solved the problem of optimizing the expected value of an integral-quadratic performance measure, using state-estimate feedback for discrete-time systems. It was shown that the state-estimate feedback control law could be computed from $u = -K\hat{x}$, where K was computed from the deterministic LQ problem (separation principle). The net result was that an analytic design procedure became available for mean-square optimization of systems with noisy output-feedback. The theory for linear-quadratic design was presented in detail in two texts which appeared in the early seventies, i.e. Anderson and Moore (1970) and Kwakernaak and Sivan (1972), and some updated material appears in the recently published text Dorato et al. (2000).

4. RETURN TO THE FREQUENCY DOMAIN

The mid-seventies witnessed a return to frequency domain methods for analytic design. In 1974, Youla et al., Youla et al. (1974), investigated the the problem of stabilizing a feedback system with a *stable* compensator. their analytic design results can be summarized as follows:

Design Problem: Given a SISO Plant, with rational transfer function $G(s)$, find a *stable* compensator which stabilizes the feedback system (unity feedback).

Existence Theorem: A stable stabilizing compensator exists if and only if between each pair of zeros on the non-negative real s-plane axis there is an *even* number of poles (parity- interlacing property, *p.i.p.*).
Solution: The problem of finding a stable compensator is reduced to interpolating at the zeros of the plant in the right-half s-plane with a BIBO unit, i.e. a BIBO stable function whose inverse is also BIBO stable. Details of an algorithm to compute the required unit may be found in Vidyasagar (1985). It should be noted that the stable compensator may be of higher order than the plant.

While a stable compensator may not often be a direct design technique, it is important to know when one cannot stabilize with a stable compensator. Most introductory control text books limit their discussions of ad hoc design procedures to stable compensators, e.g. lead-lag, but for some plants no *stable stabilizing compensator* exists. It helps to know this before spending too much time with trial-and-error procedures. If design specification go beyond simple stabilization, an unstable compensator may be required even if *p.i.p* is satisfied. For stable plants a stable stabilizing compensator always exists, and it is advantages to implement a stable compensator in this case, to preserve a stable open-loop system in case of sensor failures.

In 1976, Youla et al., Youla et al. (1976), presented a frequency domain approach to the mean-square feedback design problem for multi-input-multi-output (MIMO) systems. A very important analytic design result included in this study relates to the parameterization of all stabilizing compensator, now commonly referred to as *Youla* parameterization. This analytic design result may be summarized as follows:

Problem: Given a rational transfer-function matrix $G(s)$, find a parameterization of all compensators $D(s)$ which will stabilize $G(s)$ in a feedback configuration.
Existence Theorem: A parameterization exists for any plant.
Solution: If the plant is expressed as matrix fractions, $G(s) = A^{-1}(s)B(s) = B_1(s)A_1^{-1}(s)$ where A, B and A_1, B_1 are any left-right coprime polynomial matrices, then all stabilizing may be parameterized as follows,

$$D = (Y + A_1Q)(X - B_1Q)^{-1}$$

were X, Y are polynomial matrices which satisfy the matrix Bezout identity,

$$A(s)X(s) + B(s)Y(s) = I$$

and $Q(s)$ is *any* stable rational matrix. The MATLAB function *youla* may be used to do all the required computations. For the simple choice $Q(s) = 0$, the order of the compensator is the same as the order of the plant.

In 1983 Zames and Francis, Zames and Francis (1983), introduced H^∞ norms for frequency domain design. The H^∞-norm of BIBO stable function $G(s)$ is denoted $\|G(s)\|_\infty$ and defined as

$$\|G(s)\|_\infty = \sup_\omega |G(j\omega)|$$

An example of an analytic design result reported in their paper may be stated as follows:

Problem: Find a compensator $D(s)$ such that the closed-loop system is stable and the H^∞-norm of the weighted sensitivity function

$$W(s)(1 + D(s)G(s))^{-1}$$

is minimized.
Existence Theorem: A solution always exists.
Solution: The solution is reduced to finding a BIBO stable function which interpolates at unstable zeros of the plant $G(s)$. The interpolation problem can by solved by the Nevanlinna-Pick interpolation algorithm.

In 1984, Hidenori Kimura, Kimura (1984), applied H^∞ concepts to the analytic design of compensators which guarantee robust closed-loop stability for unstructured plant variations. The existence of a robustly stabilizing compensator depends on the existence of a bounded-real function (BIBO stable function with H^∞ bounded by one) which interpolates at unstable poles of the plant. Details may be found in the cited reference.
Using H^∞ theory, Allen Tannenbaum, Tannenbaum (1980), presented an analytic design approach to gain-margin optimization, and subsequently extended the result to phase-margin optimization, Khargonekar and Tannenbaum (1985). Details may be found in the cited references and the text Doyle et al. (1992). Gain and phase-margin designs are generally presented in introductory control texts, as ad hoc procedures. Reference Dorato (2000a) presents in an introduction to analytic design using interpolation concepts outlined above.
In 1989, Doyle et al. Doyle et al. (1989) presented a state-space solution to the H^∞ control problem, reducing the problem to the solution of two Riccati equations.

5. CONCLUSIONS

Analytic design provides useful answers to a number of specific control problems. However, some problems have no analytic solution, and one most resort to ad hoc, trial-and-error, procedures or to numerical procedures. This is almost always the case when the compensator is required to be of fixed structure and low order. Approaches to design when analytic results are not available are presented in Dorato (2000b).

Acknowledgments: The author is grateful to his colleague Chaouki Abdallah for his inputs and help in preparing this paper

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