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DATA DRIVEN LOCAL COORDINATES: SOME NEW TOPOLOGICAL AND GEOMETRICAL RESULTS

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Abstract: Certain topological and geometrical properties of data driven local coordinates (DDLC) for state-space systems as introduced in (Wolodkin *et al.*, 1997) and (McKelvey and Helmersson, 1999) are derived. First the special case of SISO systems with McMillan degree $n = 1$ is discussed in order to provide some insights into the geometry of the DDLC construction. Then for the MIMO case with general n it is shown that the set of transfer functions corresponding to the parameter space contains a nonvoid open subset of the manifold of transfer functions of order n and that the estimation problem is locally well posed. Moreover, it is stated that the parameter space always contains points corresponding to non minimal systems and a result on the number of disconnected components of the equivalence classes in the space $\mathbb{R}^{n^2+n(m+s)}$ (obtained by an embedding of the system matrices (A,B,C)) concludes this contribution.

Keywords: Parametrization, Linear multivariable systems, State-space models, Identifiability, System Identification.

1. INTRODUCTION

In this paper certain topological and geometrical properties of a novel parametrization for classes of linear systems are analyzed. One step towards an investigation of this type of parametrization called "data driven local coordinates" (DDLC) was given in (Deistler and Ribarits, 2001) and this investigation will be extended here. DDLC has been introduced by (Wolodkin *et al.*, 1997) and (McKelvey and Helmersson, 1999) and is claimed to be advantageous from a numerical point of view.

The models considered are of the form

$$x(t+1) = Ax(t) + Bu(t) \tag{1}$$

$$y(t) = Cx(t) + \varepsilon(t) \tag{2}$$

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Here $x(t)$ is the n-dimensional state vector and $y(t)$ is the s-dimensional observed output. In this paper, $u(t)$ is either a m-dimensional deterministic input or equal to $\varepsilon(t)$. In any case, the process $(\varepsilon(t))$ is white noise. $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{s\times n}$ are parameter matrices. The stability condition is usually imposed, i.e. $|\lambda_{max}(A)| < 1$ where λ_{max} denotes the eigenvalue of maximal modulus. The transfer function (from $u(t)$ to $y(t)$) is of the form

$$k(z) = \sum_{j=1}^{\infty} K_j z^j = \sum_{j=1}^{\infty} CA^{j-1}Bz^j = Cz(I-zA)^{-1}B \tag{3}$$

where z stands for the backward shift operator as well as for a complex variable. In case of $u(t) = \varepsilon(t)$, the identity matrix has to be added to the right hand side of (3).

Let U_A be the set of all transfer functions (3) for arbitrary state dimension n. U_A is infinite dimensional and may be broken into finite dimensional bits. Usually these bits are described by a subset of an Euclidean space (the set of free parameters), and the case where these bits are subsets of the class of causal systems of fixed McMillan degree n, which is called $M(n) \subset U_A = \cup_{n\in\mathbb{N}\cup\{0\}} M(n)$, is considered here. The subset of $M(n)$ corresponding to stable transfer functions is denoted by $M_s(n)$. It is well-known that $M(n)$ is a real analytic manifold of dimension $n(m+s)$; see e.g. (Hannan and Deistler, 1988). $M_s(n)$ is a submanifold of $M(n)$ of the same dimension.

The set U_A is endowed with the so called pointwise topology which corresponds to the relative topology in the product space $(\mathbb{R}^{s\times m})^{\mathbb{N}}$ for the coefficients $(K_j | j \in \mathbb{N})$. Note that $\bar{M}(n)$, which denotes the closure of the set $M(n)$, satisfies: $\bar{M}(n) = \cup_{i\leq n} M(i)$:

For given m, s and n one can embed the matrix triple (A,B,C) in $\mathbb{R}^{n^2+n(m+s)}$. By π the mapping attaching transfer functions to such matrix triples is denoted (or, with slight abuse of notation, also the mapping attaching transfer functions to free parameters):

$$\begin{aligned} \pi:\ \mathbb{R}^{n^2+n(m+s)} &\rightarrow \bar{M}(n) \\ (A,B,C) &\mapsto Cz(I-zA)^{-1}B \end{aligned}$$

The paper is organized as follows: In section 2, DDLC is briefly introduced. A list of results for the special case $s=m=n=1$ is presented in section 3. Then new results for the case of general n and arbitrary input and output dimensions are given in section 4. Finally, in section 5 a result about non minimal systems corresponding to T_D is stated, and the equivalence classes for minimal systems are shown to consist of two disconnected components. Section 6 concludes this contribution.

2. PARAMETRIZATION BY DDLC

The parametrization by DDLC will be described in this section; for further details see e.g. (McKelvey and Helmersson, 1999).

Taking all entries in (A,B,C) one could use $\mathbb{R}^{n^2+n(m+s)}$ as a parameter space. For $k(z) \in M(n)$ the classes of observational equivalence within $\mathbb{R}^{n^2+n(m+s)}$ are real analytic manifolds of dimension n^2. The idea now is to avoid the drawback of n^2 essentially "unnecessary" coordinates by only parametrizing the $n(m+s)$ dimensional ortho-complement to the tangent space to a certain equivalence class in $\mathbb{R}^{n^2+n(m+s)}$ at a given (A,B,C). The system (A,B,C) is obtained by an initial estimate, therefore the term "data driven local coordinates". Then the corresponding parameter space is of dimension $n(m+s)$ rather than $n^2+n(m+s)$ and thus has no "unnecessary" coordinates.

The construction of the tangent space to the n^2 dimensional equivalence class at a given minimal realization (A,B,C) is obtained by considering a state transformation $T=(I_n+\Delta)$ with $||\Delta|| < 1$, where $||.||$ denotes some matrix norm. Note that $T^{-1} = (I_n - \Delta + \Delta^2 - \dots)$:

$$\begin{aligned} \tilde{A} &= TAT^{-1} = A + I_n\Delta A - A\Delta I_n + \mathrm{O}(||\Delta||^2) \\ \tilde{B} &= TB = B + I_n\Delta B \\ \tilde{C} &= CT^{-1} = C - C\Delta I_n + \mathrm{O}(||\Delta||^2) \end{aligned}$$

Using the relation $vec(XYZ) = Z^T \otimes X vec(Y)$, where

$$X \otimes Y = \begin{pmatrix} X_{11}Y & \dots & X_{1q}Y \\ \vdots & & \vdots \\ X_{p1}Y & \dots & X_{pq}Y \end{pmatrix}$$

with $X \in \mathbb{R}^{p\times q}$ and $Y \in \mathbb{R}^{r\times s}$ one obtains

$$\begin{pmatrix} vec(\tilde{A}) \\ vec(\tilde{B}) \\ vec(\tilde{C}) \end{pmatrix} = \begin{pmatrix} vec(A) \\ vec(B) \\ vec(C) \end{pmatrix} + Qvec(\Delta) + \mathrm{O}(||\Delta^2||)$$
$$Q = \begin{pmatrix} A^T \otimes I_n - I_n \otimes A \\ B^T \otimes I_n \\ -I_n \otimes C \end{pmatrix} \in \mathbb{R}^{n^2+n(m+s)\times n^2} \qquad (4)$$

As the columns of Q span the tangent space to the equivalence class at (A,B,C), the orthogonal complement is spanned by the columns of some $Q^{\perp}$ (which can be obtained e.g. via a singular value decomposition of Q) and the DDLC are then obtained as follows:

$$\begin{pmatrix} vec(A(\xi)) \\ vec(B(\xi)) \\ vec(C(\xi)) \end{pmatrix} = \begin{pmatrix} vec(A) \\ vec(B) \\ vec(C) \end{pmatrix} + Q^{\perp}\xi$$
$$\xi \in T_D = \mathbb{R}^{n(m+s)} \qquad (5)$$

Let V_D denote the set of all transfer functions of the form $\pi(A(\xi),B(\xi),C(\xi))$ corresponding to T_D via (5). Clearly, for given (A,B,C), T_D can be identified via (5) with an affine subspace in $\mathbb{R}^{n^2+n(m+s)}$. With slight abuse of notation we will use T_D for both, the parameter space $\mathbb{R}^{n(m+s)}$ and the affine subspace in $\mathbb{R}^{n^2+n(m+s)}$.

3. THE SISO CASE WITH $N = 1$

In this section some results for the special case $n = s = m = 1$ obtained in (Deistler and Ribarits, 2001) are listed as they provide useful insights into geometrical and topological properties of the DDLC parametrization.

Note that for a given minimal realization (a,b,c) corresponding to $k(z) \in M(1)$, the scalar a is unique and the corresponding equivalence class in $\mathbb{R}^3$ is a hyperbola (with 2 branches) which is determined by a fixed a and $bc = const$; see the thick curve in figure 1. Non minimal systems (corresponding to the trivial transfer function $k(z) = 0$) are represented by the union of the planes given by $b = 0$ and $c = 0$, respectively. Figure 1 also includes two other types of

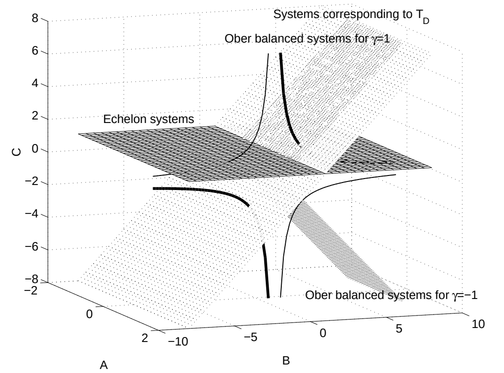

Fig. 1. Equivalence classes and parametrized subsets in $\mathbb{R}^3$.

parametrizations which are commonly used for linear systems, namely the echelon and the Ober balanced form. Both of them are canonical forms and are known to satisfy certain desirable topological properties. The system matrices for these two parametrizations are given as follows:

▶ In figure 1 the set $T_{(1)}$ of all echelon parameters corresponds to the plane given by $c = 1$ except for the line given by $b = 0$, $c = 1$. This is because the echelon systems are parametrized by

$$\begin{pmatrix} a(\theta_1,\theta_2) \\ b(\theta_1,\theta_2) \\ c(\theta_1,\theta_2) \end{pmatrix} = \begin{pmatrix} \theta_1 \\ \theta_2 \\ 1 \end{pmatrix}, (\theta_1,\theta_2) \in T_{(1)} = \mathbb{R} \times \mathbb{R} \setminus \{0\}$$

▶ Ober's balanced canonical form only parametrizes stable systems, i.e. $k(z) \in M_s(n)$. The sets $T^o_{(1)}$ and $T^o_{(-1)}$ of free Ober parameters correspond to the two open half plane segments $b = c, b > 0$ for $\gamma = (1)$ and $b = -c, b > 0$ for $\gamma = (-1)$, respectively, with a restricted in both cases by $-1 < a < 1$. Note that γ is a structural index which determines the piece of $M_s(n)$ to be parametrized and can only take the values (1) and (-1) in the case considered here. If $\gamma = (1)$, for instance, the discrete time system matrices are explicitely given by

$$\begin{pmatrix} a(\sigma,b_c) \\ b(\sigma,b_c) \\ c(\sigma,b_c) \end{pmatrix} = \begin{pmatrix} \frac{2\sigma - b_c^2}{2\sigma + b_c^2} \\ 2\frac{\sqrt{2\sigma}b_c}{2\sigma + b_c^2} \\ 2\frac{\sqrt{2\sigma}b_c}{2\sigma + b_c^2} \end{pmatrix}, (\sigma,b_c) \in T^o_{(1)} = \mathbb{R}^+ \times \mathbb{R}^+$$

The fact that $|a| < 1$ is easily seen and $b = c$ can take any positive value for a suitable choice of $\sigma, b_c > 0$.

▶ In the case of DDLC, commencing from a minimal system (a,b,c), one obtains $Q = (0,b,-c)^T$ and thus the columns of $Q^\perp$ can be chosen as $(1,0,0)^T$ and $(0,c,b)^T$. Hence, T_D can be identified with the whole plane given by $(a,b,c)^T + Q^\perp\xi$. Starting with an Ober balanced initial system $(a,b,c) = (a,b,\gamma b)$, e.g. for $\gamma = 1$, T_D becomes the whole plane given by $b = c$.

From figure 1 the following *geometrical and topological properties of DDLC* can be seen:

▶ Note that $V_D \not\subseteq M(1)$. This is an immediate consequence of the fact that T_D intersects the planes given by $b = 0$ and $c = 0$ yielding two straight lines. In case of an Ober balanced initial system, this intersection becomes the a-axis only.

▶ On the other hand, $V_D \not\supseteq M(1)$ because the plane corresponding to T_D does not intersect or touch certain hyperbolae: see the thin curve in figure 1.

▶ Also note that T_D^{min}, which is the subset of T_D corresponding to all minimal systems, is open and dense in T_D.

▶ The lack of global identifiability of T_D^{min} is easily seen from figure 1: For balanced initial realizations all equivalence classes in T_D^{min} consist of exactly two elements, in any other case the equivalence classes in T_D^{min} consist of two elements except for the points where T_D touches a hyperbola which gives a singleton.

▶ Nevertheless, T_D^{min} is locally identifiable (i.e. $\pi|_{T_D^{min}}$ is locally injective) at any point except for the points where T_D touches a hyperbola.

▶ Note that in case of an Ober balanced initial system T_D contains *all* Ober balanced systems with the same γ. As can be shown easily, this feature is no longer valid for general n, since in this case the Ober balanced systems correspond to a "curved" manifold of the same dimension.

▶ Starting from an Ober balanced initial system, $V_D^{min} = \pi(T_D^{min})$ is open in $M(1)$. On the other hand, for any other initial system, V_D^{min} is *not* open in $M(1)$ because the transfer functions corresponding to the points where hyperbolae touch T_D^{min} are boundary points of V_D^{min}.

4. TOPOLOGICAL RESULTS

It is clear from figure 1 that any hyperbola in the neighborhood of the initial system has a unique intersection with T_D and thus can be represented uniquely by DDLC. More generally, one obtains the following local result:

Theorem 1. Assume that the initial system (A,B,C) is *minimal*. Then for a sufficiently small open neighborhood T_{loc} of $0 \in T_D$ the corresponding $\pi(T_{loc}) = V_{loc}$ is a nonvoid open subset of $M(n)$ and the mapping $\pi|_{T_{loc}}$ is a homeomorphism.

Remark: From Theorem 1 it is seen that V_D contains a thick subset of $\bar{M}(n)$ which contains $\pi(A,B,C)$. Additionally, due to the homeomorphism described above, the estimation problem is *locally* well posed in the sense that consistency of the transfer function estimates in V_{loc} implies consistency of the parameter estimates in T_{loc}.

Remark: As will be shown in the sequel, the result of Theorem 1 is valid for other parametrizations around a minimal (A,B,C) fulfilling some general conditions which are easy to verify.

Before Theorem 1 is proved, a result from (Glover and Willems, 1974) on local identifiability at the point $\xi = \xi_0$ is stated. Let $T^g \subset \mathbb{R}^d$ denote a parameter space which is an open subset of $\mathbb{R}^d$. Note that local identifiability means injectivity of the function $\pi|_{T^g_{loc}}$ with T^g_{loc} being some open neighborhood of ξ_0 in T^g.

Theorem 2. Let $\psi : T^g \to \mathbb{R}^{n^2+n(m+s)}$ be a continuously differentiable mapping attaching the vectorization of the system matrices $(A(\xi),B(\xi),C(\xi))$ to $\xi \in T^g$ and suppose that $(A(\xi_0),B(\xi_0),C(\xi_0))$ is minimal. Then

(1) T^g is locally identifiable at $\xi = \xi_0$ *if and only if*

$$F : GL(n) \times T^g \to \mathbb{R}^{n^2+n(m+s)}$$
$$(T,\xi) \mapsto \begin{pmatrix} vec(TA(\xi)T^{-1}) \\ vec(TB(\xi)) \\ vec(C(\xi)T^{-1}) \end{pmatrix} \quad (6)$$

is locally injective at $(T,\xi) = (I,\xi_0)$.

(2) If the rank of $X(\xi)$ equals r for all $\xi \in U(\xi_0)$, where $X(\xi) = \left[\frac{\partial F}{\partial T}(I,\xi); \frac{\partial F}{\partial \xi}(I,\xi)\right]$ and $U(\xi_0)$ is some open neighborhood of ξ_0, then T^g is locally identifiable at the point $\xi = \xi_0$ *if and only if* $r = n^2 + d$, or, equivalently, *if and only if* $det(X^T(\xi_0)X(\xi_0)) \neq 0$.

PROOF. (1): "$\Rightarrow$" The fact that local identifiability translates to local injectivity of F is clear: In a suitably chosen open neighborhood O of ξ_0 all parameters will correspond to minimal systems (ψ is continuous) and therefore all equivalent systems will be related by the given similarity transformation. Local identifiability therefore implies the existence of an open set $T^g_{loc} \subseteq O$ such that $F|_{GL(n)\times T^g_{loc}}$ is injective. In particular, this also means that $F|_{O_T \times T^g_{loc}}$ is injective where O_T from now on always denotes some open neighborhood of the identity matrix $I \in \mathbb{R}^{n\times n}$.

(1): "$\Leftarrow$" Let us assume that T^g were not locally identifiable at $\xi = \xi_0$. Taking $\xi_i, i = 1,2$ with $\xi_1 \neq \xi_2$ in an *arbitrarily small* neighborhood $T^g_{loc} \subset T^g$ of ξ_0 with $\pi(\psi(\xi_1)) = \pi(\psi(\xi_2))$, one can easily calculate the *unique* state transformation $T(\xi_1,\xi_2) \in GL(n)$ depending continuously on $\xi_i, i = 1,2$ and satisfying $T(\xi_1,\xi_2) \to I$ for $\xi_1 \to \xi_2$ such that $F(T(\xi_1,\xi_2),\xi_1) = F(I,\xi_2)$. Thus, F restricted to $O_T \times T^g_{loc}$ cannot be injective.

(2) Clearly, $n^2 + d \leq n^2 + n(m+s)$ has to be fulfilled. Under the assumption that the rank of $\frac{\partial F}{\partial(T,\xi)}(T,\xi)$ is constant in some neighborhood $O_T \times U(\xi_0)$ it is clear that such an F is locally injective (i.e. $F|_{\tilde{O}_T \times T^g_{loc}}$ is injective with $\tilde{O}_T \times T^g_{loc} \subset O_T \times U(\xi_0)$) *if and only if* the rank of this matrix is equal to $n^2 + d$. This is because F can be locally approximated by the linear function DF the injectivity of which is directly determined by the Jacobian. However, note that the "only if" part is due to the constant rank assumption which excludes cases like $f(x) = x^3$ in a neighborhood of $x = 0$, for instance. What remains to show is that $rk(\frac{\partial F}{\partial(T,\xi)}(T,\xi)) = rk(\frac{\partial F}{\partial(T,\xi)}(I,\xi)) = rk(X(\xi)) \forall T \in GL(n)$. This is clear because $\frac{\partial F}{\partial(T,\xi)}(T,\xi)$ can be written as $G(T)X(\xi)H(T)$ where $G(T)$ and $H(T)$ are regular matrices for all $T \in GL(n)$. $\square$

Remark: Note that the theorem is also applicable for parametrizations resulting from some a priori system knowledge so that d need not be $n(m+s)$; however, for $T^g = T_D$, $X(\xi_0)$ will itself be a square matrix. Moreover, it also deals with parametrizations where the parameter space T^g cannot be identified with an affine subspace in $\mathbb{R}^{n^2+n(m+s)}$. The Ober balanced form for $n > 1$ may serve as an example at this point.

Remark: The "constant and full rank assumption" on $X(\xi)$ (and, equivalently, on $\frac{\partial F}{\partial(T,\xi)}(T,\xi)$) has a nice general interpretation in case of $d = n(m+s)$. As long as it is valid, for any fixed $F(T,\xi)$, there will be no direction in the tangent space to the parametrized manifold (the structure of an affine subspace like T_D is a special case) that locally coincides with any direction in the tangent space to the equivalence classes. Moreover, (T,ξ) can serve as a *local coordinate system* in $\mathbb{R}^{n^2+n(m+s)}$ around $\psi(\xi_0)$, i.e. F is a local homeomorphism: Note that $O = GL(n) \times U(\xi_0)$ is an open subset of $\mathbb{R}^{n^2+n(m+s)}$ and $F : O \subset \mathbb{R}^{n^2+n(m+s)} \to \mathbb{R}^{n^2+n(m+s)}$ is continuously differentiable in O with non singular Jacobian. By the inverse function theorem this implies *local* invertibility of F around each $(T,\xi) \in O$, i.e. bijectivity of F on open (in $\mathbb{R}^{n^2+n(m+s)}$!) neighborhoods $O_1(T,\xi) \subset O$ and $O_2(F(T,\xi)) \subset \mathbb{R}^{n^2+n(m+s)}$. This directly yields the *global* result that F *is an open mapping* (which will not be bijective on the whole open set O, in general). In all, continuity of F, together with local invertibility and global openness turn F into a local homeomorphism.

Remark: Theorem 2 also gives some information about the structure of the equivalence classes. If the "constant rank assumption" in (2) in the theorem above holds true and $r < n^2 + d$, then one cannot have local identifiability and thus certain shapes of the equivalence classes can be excluded a priori.

Let $\mathbb{R}_n^{n^2+n(m+s)}$ denote the set of points corresponding to minimal systems. Theorem 1 is proved with the help of the next theorem:

Theorem 3. Let O be *any* open subset of $\mathbb{R}_n^{n^2+n(m+s)}$. Then $\pi(O)$ is open in $M(n)$.

PROOF. Let us assume that $\pi(O)$ is not open in $M(n)$. Then one can find a transfer function $k_0(z) \in \pi(O)$ and a sequence $k_t(z) \to k_0(z)$ with $k_t(z) \in M(n) \setminus \pi(O)$. Thus, the inverse image $\pi^{-1}(k_t)$ –

$\pi^{-1}(k_t)$ is an equivalence class in $\mathbb{R}_n^{n^2+n(m+s)}$ – satisfies: $\pi^{-1}(k_t) \cap O = \varnothing$. Let us now consider an arbitrary point $K_0 = (vec(A_0)^T, vec(B_0)^T, vec(C_0)^T)^T \in \pi^{-1}(k_0) \cap O$. Clearly one can find an $\varepsilon > 0$ such that the open ball $K_\varepsilon(K_0)$ with radius ε and center K_0 satisfies $K_\varepsilon(K_0) \subset O$. The condition above then means that $\inf_{K_t \in \pi^{-1}(k_t)} \|K_0 - K_t\| \geq \varepsilon$ where K_t is given analogously. This is shown to yield a contradiction: As $M(n)$ has the structure of a real analytic manifold one can choose a coordinate neighborhood U_α containing the transfer function k_0 together with a homeomorphism ϕ_α from U_α onto an open subset T_α of $\mathbb{R}^{n(m+s)}$; see chapter 2.6 in (Hannan and Deistler, 1988). Convergence of $k_t \to k_0$ therefore implies convergence of the corresponding parameters in T_α and thus convergence of the corresponding system matrices. Next one considers the unique transformation $T_0 \in GL(n)$ mapping the system matrices corresponding to $\phi_\alpha(k_0)$ to the representative $K_0 \in \pi^{-1}(k_0) \cap O$. This mapping is also continuous, which finally implies convergence of the corresponding representatives K_t to K_0, clearly yielding a contradiction to $\inf_{K_t \in \pi^{-1}(k_t)} \|K_0 - K_t\| \geq \varepsilon$. $\square$

PROOF of Theorem 1 First, the rank condition for $X(\xi)$ of Theorem 2 (2) is verified for $\xi_0 = 0$. The first part $\frac{\partial F}{\partial T}(I,\xi)$ is independent of the parametrization and has already been computed in (4). The second part $\frac{\partial F}{\partial \xi}(I,\xi)$ is particularly simple for DDLC because the parameters enter the vectorization of $(A(\xi), B(\xi), C(\xi))$ linearly via $Q^\perp$ (see (5)). Thus

$$X(\xi) = \begin{bmatrix} A(\xi)^T \otimes I_n - I_n \otimes A(\xi) & \vdots & Q^\perp \\ B(\xi)^T \otimes I_n & & \\ -I_n \otimes C(\xi) & & \end{bmatrix} \quad (7)$$

and for $\xi = \xi_0 = 0$ one gets $X(0) = [Q \vdots Q^\perp]$. Moreover, $X(\xi)$ has constant rank $n^2 + n(m+s)$ in a neighborhood T_{loc} of $\xi = 0$ because $X(\xi)$ depends continuously on ξ and the determinant is a continuous function of the entries in $X(\xi)$. Note that $\pi|_{T_{loc}} = \pi \circ F|_{GL(n) \times T_{loc}}$ attaching $k(z) \in M(n)$ to $\xi \in T_{loc} \subset \mathbb{R}^{n(m+s)}$ – independent of the choice of $T \in GL(n)$ – is:

- clearly continuous.
- an open mapping because F is open by a remark above and π is open by Theorem 3. Thus $\pi(T_{loc}) = V_{loc}$ is open in $M(n)$.
- bijective when considered as a function from T_{loc} to V_{loc} because of the injectivity of $F|_{(GL(n) \times T_{loc})}$.

Hence, $\pi|_{T_{loc}}$ is a homeomorphism. $\square$

Considering the topological structure of the sets V_D and V_D^{min} yields a "global" result:

Theorem 4. Assume that the initial system (A,B,C) is minimal. Then T_D^{min} is open and dense in T_D and V_D^{min} is open and dense in V_D.

PROOF. Let us consider the mapping $\Delta : T_D \to \mathbb{R}$ attaching $det(W_o^n(\xi) W_c^n(\xi))$ to ξ where

$$\begin{aligned} \xi &\mapsto \left(B(\xi), A(\xi)B(\xi), \ldots, A(\xi)^{n-1}B(\xi)\right) = \mathcal{C}_n(\xi) \\ &\mapsto \mathcal{C}_n(\xi)\mathcal{C}_n(\xi)^T = W_c^n(\xi) \in \mathbb{R}^{n \times n} \end{aligned}$$

and $W_o^n(\xi)$ is obtained analogously. Note that W_o^n and W_c^n have full rank if and only if $(A(\xi), B(\xi), C(\xi))$ is minimal. Moreover, the determinant of $W_o^n(\xi) W_c^n(\xi)$ is a polynomial in the parameters $\xi_i, i = 1, \ldots n(m+s)$ and thus analytic (and therefore trivially continuous). Openness of T_D^{min} in T_D is straightforward as $T_D^{min} = \Delta^{-1}(\mathbb{R} \backslash \{0\})$ is the inverse image of an open set and Δ is continuous. Denseness of T_D^{min} in T_D follows from a well known result for analytic functions: $\Delta(\xi) = 0$ can only hold true on a thin subset of T_D (Δ cannot vanish anywhere in T_D). Openness of V_D^{min} in V_D follows from the definition of relative openness. Here $V_D^{min} = V_D \cap \underline{M}(n)$ and $\underline{M}(n)$ is known to be open in $\overline{M}(n)$; see (Hannan and Deistler, 1988). Denseness of V_D^{min} in V_D is shown easily: Any $k_0 \in V_D$ can be approximated arbitrarily closely by $k_t \in V_D^{min}$ because every point $\xi_0 \in \pi^{-1}(k_0) \cup T_D$ can be approximated arbitrarily closely by parameter values in T_D^{min} (T_D^{min} is dense in T_D) and π (and thus $\pi|_{T_D}$) is continuous. $\square$

Remark: Note that V_D^{min} is not necessarily open in $M(n)$. This has been discussed in detail for the special case $n = s = m = 1$ in section (3).

5. GEOMETRICAL RESULTS

According to the next theorem, for $n > 0$, T_D^{min} is a proper subset of T_D. The proof is omitted for reasons of brevity, but can be found in (Ribarits and Deistler, 2001).

Theorem 5. Assume that the initial system (A,B,C) is minimal. Then V_D contains transfer functions of lower McMillan degree, i.e. T_D contains non minimal systems.

For the special case $m = s = n = 1$ the set of observationally equivalent minimal systems had the form of hyperbolae (with two branches). The fact that the equivalence class is not connected is valid in general:

Theorem 6. Assume that the system (A,B,C) is minimal and s, m and n are arbitrary. Then the set of observationally equivalent systems constitutes a n^2 dimensional real analytic manifold consisting of two disconnected components.

PROOF. The first statement is well known since for fixed *minimal* (A,B,C) the mapping ϕ attaching the vector $(vec(TAT^{-1})^T, vec(TB)^T, vec(CT^{-1})^T)^T$ to $T \in GL(n)$ is a homeomorphism (injectivity of ϕ on $GL(n)$ is due to the uniqueness of the construction of state transformations). For the second part, it suffices to show that the set $GL(n)$ consists of two disconnected components; see Lemma 7 below. These two components are open in $\mathbb{R}^{n^2}$, their images under ϕ are open and disjoint subsets of the equivalence class (ϕ is a homeomorphism) and thus they are disconnected. However, both components are connected as the image of a connnected set under a continuous mapping is connected. $\square$

In order to make the paper more self-contained, the following two lemmas are included:

Lemma 7. The set $GL(n)$ consists of two disconnected components comprising nonsingular matrices that can be continuously transformed into either the identity I_n or $diag(I_{n-1},-1)$ within $GL(n)$.

PROOF. Note that because the sets $det^{-1}((0,\infty))$ and $det^{-1}((-\infty,0))$ are disjoint and open, $GL(n)$ consists of at least two disconnected components. To show that there are exactly two components, consider the SVD of $T \in GL(n)$: $T = U_1 \Sigma V_1^T$. Here $\Sigma = diag(\sigma_1, \ldots \sigma_n)$ with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$ denoting the singular values and $U_1, V_1 \in O(n)$ where $O(n)$ is the orthogonal group. The singular values are unique, and assume that $U_1 \in O(n)$ is fixed. Hence, the corresponding $V_1 \in O(n)$ will be uniquely determined. Clearly, the matrix Σ can be continuously transformed within $GL(n)$ into the identity I_n. It therefore remains to show that any $U \in O(n)$ can be continuously transformed into either I_n or $diag(I_{n-1},-1)$ because one could then apply such a continuous transformation to the orthogonal matrix $U_1 V_1^T$. Hence, use of Lemma 8 completes the proof. $\square$

Lemma 8. The set $O(n)$ consists of two disconnected components comprising orthogonal matrices that can be continuously transformed into either the identity I_n or $diag(I_{n-1},-1)$ within $O(n)$.

PROOF. For $n = 1$ this is trivial because $O(1) = \{1,-1\}$. For $n = 2$ one can write $U = (u_{i,j}), i,j \in \{1,2\}$. As $U^T U = I$, one can set $u_{11} = cos(\phi)$ and $u_{12} = sin(\phi)$ and gets $u_{21} = \pm u_{12}$ and $u_{22} = \mp u_{11}$ with $\phi \in [0,2\pi)$. Hence, $O(2)$ consists of rotations ($u_{21} = -u_{12}, u_{22} = u_{11}$, i.e. $det = 1$) and products of a rotation and a reflection ($u_{21} = u_{12}, u_{22} = -u_{11}$, i.e. $det = -1$). Now for an *arbitrary* matrix $U = (u_{ij})_{i,j \in \{1,\ldots n\}} \in \mathbb{R}^{n \times n}$ one can always find a continuously parametrized matrix $R_{12} = R_{12}(\phi_1)$, which performs a rotation of the angle ϕ_1 in the plane spanned by the first and second coordinate axis, such that $u_{21} = 0$:

$$\underbrace{\begin{pmatrix} cos(\phi_1) & sin(\phi_1) & & & \\ -sin(\phi_1) & cos(\phi_1) & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}}_{R_{12}(\phi_1)} U = \begin{pmatrix} u & * & \ldots & * \\ 0 & * & \ldots & * \\ \vdots & \vdots & & \vdots \\ * & * & \ldots & * \end{pmatrix}$$

The angle ϕ_1 is then given by $tan^{-1}(\frac{u_{21}}{u_{11}}) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Note that $R_{12}(\phi_1) \in O(n) \subset GL(n)$. Next, one premultiplies by $R_{13}(\phi_2)$ (which is defined accordingly; trigonometric entries appear in the $(1,1),(3,1),(1,3)$ and $(3,3)$ positions) and chooses ϕ_2 such that the $(3,1)$ element cancels out. This will leave the second row unchanged. Writing $R_{12}(\phi_1)R_{13}(\phi_2)\ldots R_{1n}(\phi_{n-1})U = R_1 U = \tilde{U} \in O(n)$ finally yields a matrix with a first column of the form $(\tilde{u}_{11}, 0, \ldots, 0)^T$. Note that $\tilde{u}_{11} = \pm 1$ because $\tilde{U} \in O(n)$. If $\tilde{u}_{11} = -1$, one applies a final rotation $R_{12}(\pi)$ in order to get $\tilde{u}_{11} = 1$. Clearly, the first row must then also be zero everywhere except for the $(1,1)$ position. Next, $R_{23}(\rho_1)R_{24}(\rho_2)\ldots R_{2n}(\rho_{n-2})\tilde{U} = R_2 \tilde{U}$, and after $n-1$ iterations one gets $R_{n-1}\ldots R_2 R_1 U = diag(I_{n-1}, \bar{u}_{nn})$ with $\bar{u}_{nn} = \pm 1$ (because the product is again an element of $O(n)$), dependent on whether $det(U) = 1$ or $det(U) = -1$. $\square$

6. CONCLUSIONS

In this paper some geometrical and topological properties of data driven local coordinates (DDLC) are derived. It is shown that the set of transfer functions described by DDLC has the same topological dimension as the manifold $M(n)$ and that the parameter estimation problem is locally well posed. Moreover, it is stated that the parameter space always contains points corresponding to non minimal systems and it is proved that the equivalence classes in $\mathbb{R}^{n^2+n(m+s)}$ always consist of two disconnected components.

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