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DYNAMIC RECONSTRUCTION OF LINEAR IMAGES USING VISUAL DATA

Satoru Takahashi * **Clyde F. Martin** **

* *Department of Information Sciences, Tokyo Denki University, Hatoyama-Machi, Saitama 350-0394, Japan*
** *Department of Mathematics and Statistics, Texas Tech University, Lubbock, Texas 79409, U.S.A.*

Abstract: The theory of dynamic splines to reconstruct linear images from noisy directional data is shown in this paper. The dynamic splines that are used are periodic and are obtained by optimizing over both control and initial data.

Keywords: Optimal control, Splines, Interpolation

1. INTRODUCTION

Splines have been investigated in 1970s as a basic tool of applied mathematics (de Boor, 1978), (Schumaker, 1981). The dynamic splines based on linear control theory were first used in trajectory planning by Crouch and collaborators (Crouch and Jackson, 1991). For statistical applications, an important result was made by Wahba (Wahba, 1990). Further, in a series of papers (Agwu and Martin, 1998), (Egerstedt and Martin, 2001), (Martin and Egerstedt, 2001), (Sun and Martin, 2000), Martin and his collaborators have studied some of the basic properties of smoothing splines from the view point of optimal control. Based on these papers, it is possible to reconstruct noisy distance data using visual information. We will use this data to reconstruct unknown boundaries using smoothing splines. For purpose of this paper our data was collected using a laser range finder. This results in less noisy data, then would have been obtained using strictly visual information.

2. NOTATION AND PROBLEM DESCRIPTION

In this section, we will establish some notation that will be used throughout this paper. We will assume as given a controllable and observable linear system of the form

$$\begin{aligned} \dot{x} &= Ax + bu, \quad x(0) = x_0 \\ y &= cx \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, y, $u \in \mathbb{R}$, and A, b and c are constant matrices of compatible dimensions. Further, the data we consider in this paper is given in terms of time and output as

$$D = \{(t_i, \alpha_i) : \ i = 1, \cdots, N\} \tag{2}$$

where we assume that $0 < t_1 < t_2 < \cdots < t_N$.

The general problem of dynamic splines that we want to solve is the following.

Problem 1. The problem is to produce a control $u(t)$ which drives the output function $y(t)$ either through or close to the data α_i at time t_i while minimizing some cost function $J(u, x_0)$.

Solving the differential equation (1), we have

$$y(t) = ce^{At}x_0 + \int_0^t ce^{A(t-s)}bu(s)ds.$$

Then let

$$y(t_i) = ce^{At_i}x_0 + \int_0^{t_i} ce^{A(t_i-s)}bu(s)ds \tag{3}$$

where the t_i's are the interpolation times. We now define a set of linearly independent basis functions (see, (Sun and Martin, 2000))

$$g_t(s) = \begin{cases} ce^{A(t-s)}b & t > s \\ 0 & t \le s \end{cases}$$

and define the linear functional in terms of $g_t(s)$ as

$$L_t(u) = \int_0^T g_t(s)u(s)ds$$

where the value of T is at least as large as t_N. Then we can rewrite (3) as

$$y(t_i) = ce^{At_i}x_0 + L_{t_i}(u). \tag{4}$$

This notation is useful in that $L_{t_i}(u)$ is expressed as an inner product in an appropriate space.

3. INTERPOLATING SPLINES

In this section, we will establish the form of interpolating splines that are optimized over initial data as well as control. We consider the following cost functional

$$J(u, x_0) = \int_0^T u^2(t)dt$$

and solve the following optimal control problem in the Hilbert space $\mathcal{H} = L_2[0,T] \times \mathbb{R}^n$. That is, we now ask the following modification to Problem 1 as follows:

Problem 2. For $u \in L_2[0,T]$ and $x_0 \in \mathbb{R}^n$,

$$\min_{u,x_0} J(u, x_0)$$

subject to the constraints

$$y(t_i) = \alpha_i.$$

Using (4) we now construct the Lagrangian

$$\begin{aligned} &L(u, x_0, \lambda) \\ &= J(u, x_0) + \sum_{i=1}^N \lambda_i (y(t_i) - \alpha_i) \\ &= \int_0^T u^2(t)dt + \sum_{i=1}^N \lambda_i (ce^{At_i}x_0 + L_{t_i}(u) - \alpha_i) \end{aligned}$$

where λ_i shows the Lagrangian parameter. In order to obtain the necessary conditions for a local minimum (Luenberger, 1969) we take partial derivatives in the sense of the Gateaux. Namely, now our goal is to minimize this functional $L(u, x_0, \lambda)$ over the Hilbert space $L_2[0,T] \times \mathbb{R}^n$. We calculate the Gateaux derivative as

$$\begin{aligned} &\frac{\partial L(u, x_0, \lambda)}{\partial u}(w) \\ &\qquad = 2\int_0^T u(t)w(t)dt + \sum_{i=1}^N \lambda_i L_{t_i}(w), \\ &\frac{\partial L(u, x_0, \lambda)}{\partial x_0}(z_0) = \sum_{i=1}^N \lambda_i ce^{At_i} z_0, \\ &\frac{\partial L(u, x_0, \lambda)}{\partial \lambda}(\gamma) \\ &\qquad = \sum_{i=1}^N \gamma_i (ce^{At_i}x_0 + L_{t_i}(u) - \alpha_i). \end{aligned}$$

Setting these three equations equal to zero we have the necessary conditions such that

$$u(t) = -\frac{1}{2}\sum_{i=1}^N \lambda_i g_{t_i}(t), \tag{5}$$

$$\sum_{i=1}^N \lambda_i ce^{At_i} = 0, \tag{6}$$

$$ce^{At_i}x_0 + L_{t_i}(u) = \alpha_i, \quad i = 1, \cdots, N. \tag{7}$$

Note that the problem is finite dimensional because it follows from (5) that the space of controls is finite dimensional. Substituting u from (5) into (7) we have N equations in $N + n$ unknowns. However from (6) we have n additional constraints and hence we have $N+n$ linear equations in $N+n$ unknowns. Making the substitution we have

$$-\frac{1}{2}\sum_{i=1}^N \lambda_i \langle g_{t_j}, g_{t_i}\rangle + ce^{At_j}x_0 = \alpha_j, \quad j = 1, \cdots, N.$$

Let

$$G = (\langle g_{t_j}, g_{t_i}\rangle)$$

and note that G is the Gramian matrix. Further, let

$$P = \begin{pmatrix} ce^{At_1} \\ \vdots \\ ce^{At_N} \end{pmatrix}. \tag{8}$$

We can rewrite the equations in the form

$$-\frac{1}{2}G_i\lambda + P_i x_0 = \alpha_i, \quad i = 1, \cdots, N$$

and

$$P^T\rho = 0$$

where $\lambda = (\lambda_1, \cdots, \lambda_N)^T$, G_i denotes the *ith* row of the matrix G, and P_i denotes the *ith* row of the matrix P. In terms of a system of equations we can write

$$\begin{pmatrix} G & -2P \\ P^T & 0 \end{pmatrix} \begin{pmatrix} \lambda \\ x_0 \end{pmatrix} = \begin{pmatrix} -2\alpha \\ 0 \end{pmatrix}$$

where $\alpha = (\alpha_1, \cdots, \alpha_N)^T$. Using elementary row operations we can reduce the system to the following form

$$\begin{pmatrix} I & -2G^{-1}P \\ 0 & -P^TG^{-1}P \end{pmatrix} \begin{pmatrix} \lambda \\ x_0 \end{pmatrix} = \begin{pmatrix} -2G^{-1}\alpha \\ -P^TG^{-1}\alpha \end{pmatrix}.$$

It follows from this form that the system has a unique solution if and only if $P^TG^{-1}P$ is positive definite. Since G is positive definite, the matrix $P^TG^{-1}P$ fails to be positive definite if and only if there exists an x_0 such that $Px_0 = 0$. This can happen if and only if for each i, $ce^{At_i}x_0 = 0$ and this can happen if and only if

$$\text{span}\{ce^{At_i} : \ i = 1, \cdots, N\} \neq \mathbb{R}^n.$$

The question of uniqueness is very difficult to answer explicitly (Martin and Smith, 1987).

4. PERIODIC SMOOTHING SPLINES

Based on the discussion in the previous section, we now consider the case of periodic smoothing splines. The optimal problem is the following which is just a restatement of the previous section.

Problem 3. For $u \in L_2[0, T]$ and $x_0 \in \mathbb{R}^n$,

$$\min_{u,x_0} J(u, x_0)$$

subject to the constraint

$$y(t_1) = y(t_N) \tag{9}$$

where

$$J(u, x_0) = \tau \int_0^T u^2(t)dt + \sum_{i=1}^N \lambda_i(ce^{At_i}x_0 + L_{t_i}(u) - \alpha_i)^2$$

and the constant τ is assumed to be strictly positive.

The condition (9) is called "periodic". Let the Lagrangian

$$\begin{aligned} L(u, x_0, \mu) &= \tau \int_0^T u^2(t)dt + \sum_{i=1}^N \lambda_i(ce^{At_i}x_0 + L_{t_i}(u) - \alpha_i)^2 \\ &\quad + \mu(y(t_1) - y(t_N)) \end{aligned} \tag{10}$$

be given where $\mu \in \mathbb{R}\backslash\{0\}$. Note that the rate of convergence of the optimal control depends on the choice of the parameter τ (see (Sun and Martin, 2000), (Wahba, 1990)). As before we want to minimize $L(u, x_0, \mu)$ and obtain the optimal control u, and the optimal initial data x_0 as well as μ.

First, calculating the Gateaux derivative of $L(u, x_0, \mu)$ with respect to u we have that

$$\begin{aligned} &\frac{\partial L(u, x_0, \mu)}{\partial u}(w) \\ &= 2\int_0^T \left[\sum_{i=1}^N \lambda_i(ce^{At_i}x_0 + L_{t_i}(u) - \alpha_i)g_{t_i}(t) \right. \\ &\quad \left. + \tau u(t) + \frac{\mu}{2}(g_{t_1}(t) - g_{t_N}(t)) \right] w(t)dt. \end{aligned}$$

Setting this equal to zero we find the condition that

$$\begin{aligned} &\sum_{i=1}^N \lambda_i(ce^{At_i}x_0 + L_{t_i}(u) - \alpha_i)g_{t_i}(t) \\ &\qquad + \tau u(t) + \frac{\mu}{2}(g_{t_1}(t) - g_{t_N}(t)) = 0. \end{aligned}$$

Therefore we see that we must have the optimal u as a linear combination of g_{t_i}'s,

$$u(t) = \sum_{i=1}^N \rho_i g_{t_i}(t). \tag{11}$$

Substituting the u of (11) into (10) we have

$$\begin{aligned} &L(\rho, x_0, \mu) \\ &= \tau\rho^TG\rho + \sum_{i=1}^N \lambda_i(\beta_i + \rho^TGe_i)^2 \\ &\quad + \mu\{c(e^{At_1} - e^{At_N})x_0 + \rho^TG(e_1 - e_N)\} \\ &= \tau\rho^TG\rho + 2\rho^TGD\beta + \rho^TGDG\rho + \beta^TD\beta \\ &\quad + \mu\{c(e^{At_1} - e^{At_N})x_0 + \rho^TG(e_1 - e_N)\} \end{aligned}$$

where $\rho = (\rho_1, \cdots, \rho_N)^T$, D is the diagonal matrix of the weights λ_i, e_i is the ith unit vector, and β denotes the vector $(\beta_1, \cdots, \beta_N)^T$ defined by

$$\beta_i = ce^{At_i}x_0 - \alpha_i, \quad i = 1, \cdots, N.$$

Calculating the Gateaux derivative of $L(\rho, x_0, \mu)$ with respect to ρ, x_0 and μ, respectively, and setting three expressions equal to zero we have the necessary conditions such that

$$\tau G\rho + GD\beta + GDG\rho + \frac{\mu}{2}G(e_1 - e_N) = 0,$$

$$\sum_{i=1}^N \lambda_i\{p_k^i ce^{At_i}x_0 + (-\alpha_i + \theta_i)p_k^i\} + \frac{\mu}{2}\phi_k = 0,$$

$$c(e^{At_1} - e^{At_N})x_0 + \rho^TG(e_1 - e_N) = 0$$

where p_k^i denotes the kth column element of ce^{At_i}, $\phi_k = p_k^1 - p_k^N$, and $\theta_i = \rho^TGe_i$, $i = 1, \cdots N$,

$k = 1, \cdots, n$. Then, by deformation of the above equations we obtain that

$$\begin{pmatrix} G + \tau D^{-1} & P & \frac{1}{2} D^{-1}(e_1 - e_N) \\ P^T D G & P^T D P & \frac{1}{2}\phi \\ (e_1^T - e_N^T)G & \phi^T & 0 \end{pmatrix} \times \begin{pmatrix} \rho \\ x_0 \\ \mu \end{pmatrix} = \begin{pmatrix} \alpha \\ P^T D \alpha \\ 0 \end{pmatrix} \quad (12)$$

where P is defined in (8) and $\phi = (\phi_1 \cdots \phi_n)^T$. Let us abbreviate (12) as $\mathcal{AB} = \mathcal{C}$. As we mention in the previous section, it may be difficult to see that (12) has a unique solution. That is, the conditions may be inconsistent. But it may have a unique solution in which case the minimization problem is irrelevant. Namely, $\mathcal{A}$ is invertible and then we can obtain a unique optimal control u by using the optimal solution ρ. The resulting output $y(t)$ is a "periodic smoothing spline". In order to solve (12) we can use Gaussian elimination.

Note that when $x_0 = 0$ and non-periodic case, there always exists a unique optimal solution ρ (see, (Egerstedt and Martin, 2001), (Sun and Martin, 2000)). Since G is positive definite it follows that $G + \tau D^{-1}$ is positive definite. Thus, we have the optimal solution u as

$$u(t) = ((G + \tau D^{-1})^{-1}\alpha)^T g(t)$$

where $g(t) = (g_{t_1}(t), \cdots, g_{t_N}(t))^T$.

5. SIMULATION RESULTS

In this section, we present two examples. We will see how our approach can be applied to the reconstruction of the boundaries of a room.

First, assume that a dynamical control system which corresponds to (1) is given by

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad c = (1\ 0)\,.$$

We now turn our attention to the laser range finder, which observes the range along a laser plane (see, Figs. 1, 2). It measures the distance between the robot and the obstacles or the wall of a room at each resolution degree for one scan. That is, as a given data (2), let time t_i be each resolution degree and the output α_i denote a measurement distance, respectively.

What we want to do is to derive the distance either through or close to measurement distance at each resolution degree. We now consider a simple room shape which the corners locate at $(0, 0)$, $(10, 0)$, $(10, 10)$, $(15, 10)$, $(15, 20)$, $(0, 20)$ in the world coordinates. Then assume that the mobile robot is setting at some known position and detects the distance data every 10 degrees for one laser scan and that data is noisy. Further, for one run we put the robot in a place where it can see the entire room but for another run we place it so that it cannot see one of the corners of the offset.

Fig. 1. The mobile robot with a laser range finder.

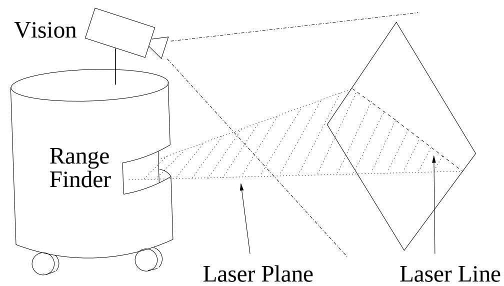

Fig. 2. The laser plane and the laser line.

In Figs. 3 and 5, we have chosen two values of τ to compare the trajectory of $y(t)$. It follows from these two figures that there is the tradeoff between control power and fit. When there is missing information as shown in Fig. 5, that is when the robot cannot see the interior corner, we can see that there is a significant Gibbs' phenomena appearing at the discontinuity points.

In order to reconstruct the wall of a room as shown in Figs. 4 and 6 we calculate polar coordinates given by $(y(t)\cos t,\ y(t)\sin t)$ using the output $y(t)$ in Figs. 3 and 5, respectively. Then we can see good reconstructions as shown in Figs. 4 and 6, respectively. From Fig. 6 we would see that the Gibbs' phenomena indicates to the robot that it should move and take another set of readings. That is, it may suggest that if the robot knows that there is Gibbs' phenomena present it should move from the robot position in Fig. 6 to the next position in Fig. 4. It is a significant open problem of how the robot should autonomously detect the Gibbs' phenomena.

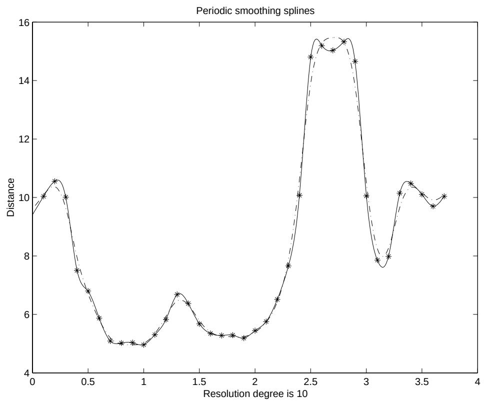

Fig. 3. Periodic smoothing splines with different τ. τ_i's are 1e-06 (dash-dotted) and 1e-08 (solid). Here λ_i's $= 0.5$. The stars correspond to the different α_i at a given resolution t_i.

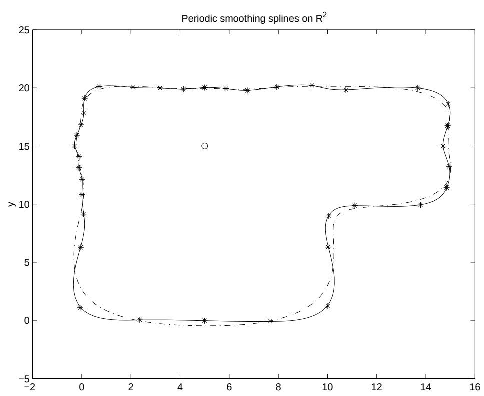

Fig. 4. Reconstruction of periodic smoothing splines by polar coordinates. The circle shows the location of a mobile robot.

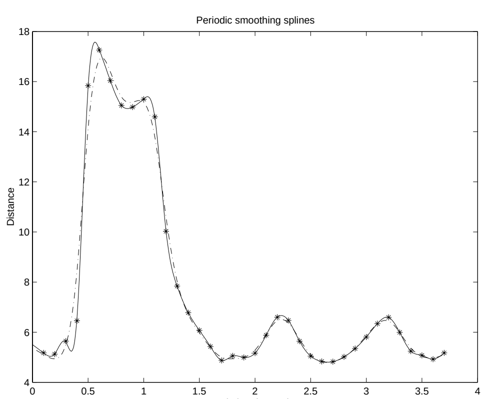

Fig. 5. Periodic smoothing splines with different τ. τ_i's are 1e-06 (dash-dotted) and 1e-08 (solid). Here λ_i's $= 0.5$. The stars correspond to the different α_i at a given resolution t_i.

6. CONCLUSIONS

In this paper, there are two important contributions. First, we introduced a new periodic smoothing splines approach by optimizing over not only the control but also over the initial data. Further, we have developed a method for reconstructing

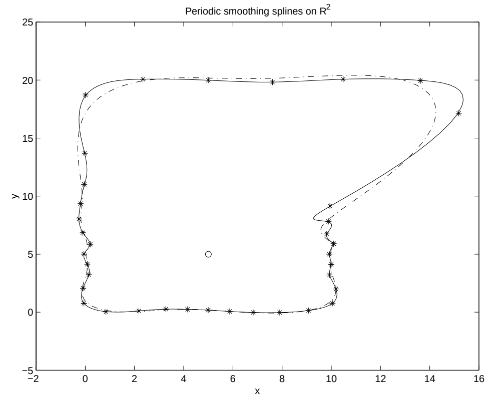

Fig. 6. Reconstruction of periodic smoothing splines by polar coordinates. The circle shows the location of a mobile robot.

the boundaries of a closed room from noisy directional data obtained by the laser range finder by using periodic smoothing splines.

We propose a available method to reconstruct the boundaries of a room or the shape of objects as an application of a range in the area of mobile robotics.

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