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A NEW MODEL FOR FLEXIBLE-JOINT ROBOT MANIPULATORS

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Abstract. We propose a new model for robot manipulators with rigid links and flexible joints. Our model is expressed in Cartesian *dependent* coordinates which represent the positions, orientations and velocities of the links and the rotors, as opposed to *independent* generalized coordinates in which classical "Lagrangian" models are defined. The model consists of a set of dynamics equations and holonomic constraints. We believe this model may facilitate the task of observer-design based tracking control.
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1. INTRODUCTION

In this paper we propose a new dynamic-kinematic model for robot manipulators with rigid links and flexible joints. Our motivation to develop this model alternative to the classical 'Lagrangian' models proposed in (Spong 1987, Burkov and Zaremba 1987) and presented in (Marino and Nicosia 1985, Nicosia and Tomei 1990) stems from control applications. In particular time-varying trajectory tracking control problems. The main feature of our model is that it is linear in the *velocities* hence it lends itself to an easier definition of admissible reference trajectories and for output feedback control, it makes the observer design simpler. This is because one can overcome the technical difficulties of the quadratic (in the generalized velocities) terms due to the Coriolis and centrifugal forces.

The price paid for this apparent "simplicity" is that the obtained model is a non minimal realization of the dynamics and kinematics. As a matter of fact the model is represented using Cartesian *constrained* coordinates instead of generalized ones. Hence its state is neither controllable nor observable. Yet, there exists an observable and controllable output which is a function of the state hence, output controllers can be designed and moreover, the controllable observable output is that of physical interest (the generalized coordinates). The physical interpretation of this uncontrollability and unobservability becomes evident if we remark that uncontrollable and unobservable state is the set of Cartesian velocities, positions and orientations of each link *independently*, expressed in the base frame. It is obvious that their movement is mechanically constrained since the links are hinged to one another by the articulations. Opposed to this, the controllable and observable output corresponds to the set of generalized positions and velocities, i.e., the articulations' coordinates.

The derivation of the model follows an energy-based approach as it is usually done for the Lagrangian models except that we consider the kinetic energy as a function of the Cartesian instead of the generalized velocities. This development follows closely that of robot dynamics textbooks (see e.g. (Spong and Vidyasagar 1989, Sciavicco and Siciliano 1996)). In this paper we will use the notations of the second reference. A detailed development regarding the geometry of the robot is omitted here due to lack of space and is presented in (Melhem and Loría 2001). The model we propose resembles to some extent, that of (Jain and Rodriguez 1995) in that the resulting inertia matrix is block diagonal and, in some cases, diagonal. However, as pointed out before, our model is expressed in Cartesian *dependent* coordinates and therefore it is of augmented dimension with respect to any model defined upon generalized coordinates.

Our work has been motivated mainly by solving the problem of position feedback control hence, we seeked at obtaining a model linear in the unmeasurable velocities. See (Loría and Panteley 1999, Besançon *et al.* 1998, Spong 1992). The common point to all these models is that the change of coordinates seeked for is globally invertible, hence, one basically looks for a map which transforms coordinates from $\mathbb{R}^n$ into $\mathbb{R}^n$. In contrast to this, the dimension of the Cartesian coordinates of our model is $12n$ if one considers that the angular velocities of the rotors are due only to their own rotation while it is $15n$ if one considers as well the contributions of the angular velocities of the links to those of the rotors'. The mapping, which is unique in one sense and has multiple solutions in the other, constitutes a *kinemtics* equation. Thus, while a globally invertible mapping may almost never be obtained, our kinematic model can be easily computed following the Denavit-Hartenberg convention.

In the next section we derive our dynamic and kinematic models based on an energetical approach. In section 2.2 we present some important properties of the kinematics model. We provide a discussion related to the Lagrangian model in Section 3 We conclude with some remarks in Section 4.

2. A "NEW" DYNAMIC MODEL

We will consider only manipulators driven by DC motors, thru flexible transmissions. We consider the manipulator as an open kinematic chain of $n+1$ rigid bodies, i.e., the base (0-th link) plus n other links, interconnected by n flexible articulations. The i-th motor is supposed to be placed at the $(i-1)$-th link.

The configuration of such a chain (or manipulator) can be fully described by a set of $2n$ generalized coordinates $q := \operatorname{col}[q_\ell, q_m]$ where $q_\ell := \operatorname{col}[q_{\ell_1}, \cdots, q_{\ell_n}]$, $q_m := \operatorname{col}[q_{m_1}, \cdots, q_{m_n}]$ and

$$q_{m_i} = 1/k_{r_i}\theta_i \qquad i = 1, \cdots, n \tag{1}$$

where q_{m_i} denotes the i-th articulation variable, θ_i the corresponding angular position of the i-th rotor, k_{r_i} the reduction ratio and q_{ℓ_i} denotes the i-th link position.

It is assumed as usual that transmissions are elastic and the elasticity can be modeled, in the space of the q_ℓ and q_m coordinates, by a linear force according to Hooke's law. The base link variables are indexed by 0 and the elasticity force magnitude at the i-th joint is denoted by $k_i > 0$.

All the results of this paper are developed upon the following basic assumption which is commonly used in the literature.

Assumption 1. The rotors of the actuators are modeled as uniform solid bodies having their centers of mass on their axes of rotation.

As for many physical systems, the approach we follow to derive the model is based on the definitions of its kinetic and potential energy functions. This is done in the following subsections. The development is similar to that for the classical Lagrangian models except that we will use Cartesian coordinates.

2.1 *Derivation of the kinetic energy*

We start by defining a notational system to describe the geometry of the robot.

Following the notations of (Sciavicco and Siciliano 1996), let us introduce the following variables. Let $\dot{p}_{\ell_i} \in \mathbb{R}^3$ and $\omega_i \in \mathbb{R}^3$ denote respectively, the vectors of linear and angular velocities of the i-th link's center of mass, expressed on the base frame. Similarly, $\dot{p}_{m_i} \in \mathbb{R}^3$ and $\omega_{m_i} \in \mathbb{R}^3$ are the vectors of linear and angular velocities of the i-th rotor. The *constant* m_{ℓ_i} denotes the mass of the i-th link (including the mass of the $(i+1)$-th stator [1]), m_{m_i} is the mass of the i-th rotor. $I^i_{\ell_i}$ corresponds to the *constant* inertia tensor of i-th link relative to its center of mass, expressed on the *fixed* frame of the i-th link (in accordance with Denavit-Hartenberg's convention). Correspondingly, $\omega^i_i \in \mathbb{R}^3$ is the angular velocity of the i-th link expressed in the same frame as $I^i_{\ell_i}$. $I^{m_i}_{m_i}$ is the constant inertia tensor of the i-th rotor relative to its center of mass expressed on a frame fixed to the rotor. Under assumption 1 this tensor is diagonal. Correspondingly, $\omega^{m_i}_{m_i} \in \mathbb{R}^3$ is the angular velocity of the i-th rotor expressed in the same frame as $I^{m_i}_{m_i}$. $R_i \in \mathbb{R}^{3\times 3}$ corresponds to the rotation matrix expressing the orientation of the i-th link's frame with respect to the base frame. Similarly, $R_{m_i} \in \mathbb{R}^{3\times 3}$ corresponds to the rotation matrix of the i-th rotor's frame with respect to the base frame. Thus, we have that $\omega^i_i = R_i^\top(q_\ell)\omega_i$ and $\omega^{m_i}_{m_i} = R^\top_{m_i}(q)\omega_{m_i}$.

Having introduced all these variables, we can now write the kinetic energy function

$$\begin{aligned}\mathcal{T}(\dot{p}, \omega) = \frac{1}{2}\sum_{i=1}^{n}(m_{\ell_i}\dot{p}^\top_{\ell_i}\dot{p}_{\ell_i} + \omega^{i\top}_i I^i_{\ell_i}\omega^i_i + \\ m_{m_i}\dot{p}^\top_{m_i}\dot{p}_{m_i} + \omega^{m_i\top}_{m_i} I^{m_i}_{m_i}\omega^{m_i}_{m_i})\,.\end{aligned} \tag{2}$$

The kinetic energy expression (2) is the same as for a robot with rigid or elastic joints and can be found in the cited textbooks. The variables in the expression above are Cartesian velocities of each link and rotor (the stators' variables are assimilated to those of the links). These variables are obviously constrained owing to the fact that the links are "hinged".

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[1] Clearly there is no loss of generality in the case that physically the actuators are concentrated in the same place instead of being distributed at each joint.

A mapping between the generalized velocities $\dot{q}_\ell$, $\dot{q}_m$ and the constrained Cartesian velocities $\dot{p}$ and ω can be established. The common use of this mapping is in rewriting the kinetic energy (2) as a function of the generalized positions and velocities to derive the classical Lagrangian model. These mappings can be computed following the convention of Denavit-Hartenberg and are given by the following expressions

$$\dot{p}_{\ell_i} = \mathbf{J}_P^{(\ell_i)}(q_\ell)\dot{q}_\ell \tag{3}$$
$$\omega_i = \mathbf{J}_O^{(\ell_i)}(q_\ell)\dot{q}_\ell \tag{4}$$
$$\dot{p}_{m_i} = \mathbf{J}_P^{(m_i)}(q_\ell)\dot{q}_\ell \tag{5}$$
$$\begin{aligned} \omega_{m_i} &= \omega_{i-1} + \omega_{i-1,m_i} &(6)\\ &= \omega_{i-1} + \mathbf{J}_O^{(m_i)}(q_\ell)\dot{q}_m &(7) \end{aligned}$$

where 1 has been used to compute (5). The Jacobian $(3 \times n)$-matrices $\mathbf{J}_P^{(\ell_i)}(q_\ell), \mathbf{J}_O^{(\ell_i)}(q_\ell)$, $\mathbf{J}_P^{(m_i)}(q_\ell)$ and $\mathbf{J}_O^{(m_i)}(q_\ell)$ are defined closely to those for rigid manipulators as for instance in (Sciavicco and Siciliano 1996). See (Melhem and Loría 2001) for more details.

As mentioned before, instead of using (3)-(7) to derive the kinetic energy as function of generalized coordinates we will use (6) to develop the last term of (2) so that, defining

$$\begin{aligned} p &= \mathrm{col}[p_{\ell_1}, \cdots, p_{\ell_n}, p_{m_1}, \cdots, p_{m_n}] &(8a)\\ \omega &= \mathrm{col}[\omega_1^1, \cdots, \omega_n^n, \omega_0^{m_1}, \cdots, \omega_{n-1}^{m_n}, \\ &\qquad \omega_{0,m_1}^{m_1}, \cdots, \omega_{n-1,m_n}^{m_n}]\,. &(8b) \end{aligned}$$

the kinetic energy function becomes

$$\begin{aligned} \mathcal{T}(\dot{p}, \omega) = \frac{1}{2}\sum_{i=1}^{n} (&m_{\ell_i}\dot{p}_{\ell_i}^\top \dot{p}_{\ell_i} + m_{m_i}\dot{p}_{m_i}^\top \dot{p}_{m_i} + \omega_i^{i\top} I_{\ell_i}^i \omega_i^i \\ &+\omega_{i-1}^{m_i\top} I_{m_i}^{m_i}\omega_{i-1}^{m_i} + \omega_{i-1,m_i}^{m_i\top} I_{m_i}^{m_i}\omega_{i-1,m_i}^{m_i} + \\ &\omega_{i-1}^{m_i\top} I_{m_i}^{m_i}\omega_{i-1,m_i}^{m_i} + \omega_{i-1,m_i}^{m_i\top} I_{m_i}^{m_i}\omega_{i-1}^{m_i})\,. \end{aligned} \tag{9}$$

At this point, to compact the notation we will introduce the state variables

$$\begin{aligned} \nu_1 &= \mathrm{col}[\dot{p}_{\ell_1}, \cdots, \dot{p}_{\ell_n}, \dot{p}_{m_1}, \cdots, \dot{p}_{m_n}, \omega_1^1, \cdots, \omega_n^n] \\ \nu_2 &= \mathrm{col}[0_{3\times 1}, \omega_1^{m_2}, \cdots, \omega_{n-1}^{m_n}] \in \mathbb{R}^{3n} \\ \nu_3 &= \mathrm{col}[\omega_{0,m_1}^{m_1}, \cdots, \omega_{n-1,m_n}^{m_n}] \in \mathbb{R}^{3n} \\ \nu &= \mathrm{col}[\nu_1, \nu_2, \nu_3] \in \mathbb{R}^{15n} \end{aligned}$$

that is, the *Cartesian* velocities. We also introduce the *constant* inertia matrix

$$\mathcal{M} = \begin{pmatrix} M_1 & 0 & 0 \\ 0 & M_2 & M_2 \\ 0 & M_2 & M_2 \end{pmatrix} \in \mathbb{R}^{15n\times 15n} \tag{10}$$

where $M_1 \in \mathbb{R}^{9n\times 9n}$ and $M_2 \in \mathbb{R}^{3n\times 3n}$ are symmetric positive definite and are given by

$$\begin{aligned} M_1 &= \text{block-diag}\{m_{\ell_1} I_{3\times 3}, \cdots, m_{\ell_n} I_{3\times 3} \\ &\qquad m_{m_1} I_{3\times 3}, \cdots, m_{m_n} I_{3\times 3}\,,\ I_{\ell_1}^1, \cdots, I_{\ell_n}^n\}\,, \\ M_2 &= \text{block-diag}\{I_{m_1}^{m_1}, \cdots, I_{m_n}^{m_n}\} \end{aligned} \tag{11}$$

hence, $\mathcal{M}$ is positive *semi*definite. Moreover, under 1 the matrix M_2 is diagonal.

Next, we define $\dot{q} \mapsto \nu$, to that end we collect all the Jacobians defined above in

$$\begin{aligned} \mathcal{J}_1(q_\ell) &:= \mathrm{col}[\mathbf{J}_P^{(\ell_1)}, \cdots, \mathbf{J}_P^{(\ell_n)},\ \mathbf{J}_P^{(m_1)}, \cdots, \mathbf{J}_P^{(m_n)}, \\ &\qquad R_1^\top \mathbf{J}_O^{(\ell_1)}, \cdots, R_n^\top \mathbf{J}_O^{(\ell_n)}] \in \mathbb{R}^{9n\times n} &(12a)\\ \mathcal{J}_2(q) &:= \mathrm{col}[R_{m_1}^\top \mathbf{J}_O^{(\ell_0)}, \cdots, R_{m_n}^\top \mathbf{J}_O^{(\ell_{n-1})}] &(12b)\\ \mathcal{J}_3 &= \mathrm{col}[R_{m_1}^\top \mathbf{J}_O^{(m_1)}, \cdots, R_{m_n}^\top \mathbf{J}_O^{(m_n)}] \in \mathbb{R}^{3n\times n} &(12c)\\ \mathcal{J}(q) &:= \begin{pmatrix} \mathcal{J}_1(q_\ell) & 0 \\ \mathcal{J}_2(q) & 0 \\ 0 & \mathcal{J}_3 \end{pmatrix} \in \mathbb{R}^{15n\times 2n}\,. &(12d) \end{aligned}$$

Using (3-7), (10) and the definitions of $\omega_{m_i}^{m_i}$ and ω_i^i, we can write in compact form: $\nu_1 = \mathcal{J}_1(q_\ell)\dot{q}_\ell$, $\nu_2 = \mathcal{J}_2(q)\dot{q}_\ell$, $\nu_3 = \mathcal{J}_3\dot{q}_m$ and

$$\nu = \mathcal{J}(q)\dot{q} \tag{13}$$

hence the kinetic energy (9) becomes

$$\begin{aligned} \mathcal{T}(\nu) := \frac{1}{2}\,\nu_1^\top M_1\nu_1 + \frac{1}{2}\nu_2^\top M_2\nu_2 + \frac{1}{2}\,\nu_3^\top M_2\nu_3 + \\ \frac{1}{2}\,\nu_2^\top M_2\nu_3 + \frac{1}{2}\,\nu_3^\top M_2\nu_2 = \frac{1}{2}\,\nu^\top \mathcal{M}\nu \end{aligned} \tag{14}$$

which, due to (13) is a positive quantity for all [2] $\dot{q} \neq 0$.

In the sequel, we will refer to the equation (13) as the *kinemtic model* or the *kinematics equation*. This equation establishes the *holonomic* constraints on the coordinates of the kinematic chain. Even though, for convenience we have expressed these constraints in a form involving the velocities, it is clear that they can be integrated and expressed as constraints of the positions [3] $\pi(t) := \int_{t_\circ}^t \nu(t)$.

Remark 2. In most of the literature we have seen the term $\frac{1}{2}\,\nu_2^\top M_2\nu_2$, which corresponds to the gyroscopic forces between each rotor's spinning and the preceding links, is neglected. This simplification, which leads to important differences in the inertia matrix, is sometimes motivated by the assumption that the the kinetic energy contribution of the rotors due to their angular velocity, is due only to their own rotation. An interesting case where these terms are also considered can be found in (Springer *et al.* 1985). □

2.2 *Properties of the kinematics Jacobian*

We present next some important properties of the Jacobian matrix $\mathcal{J}(q)$. Basically we stress that it is

[2] Notice however that $\mathcal{T}(\nu)$ is not positive for all $\nu \in \mathbb{R}^{15n}$.
[3] One reason no to express the constraints in the form $\phi(\pi) = 0$ where $\phi : \mathbb{R}^{15n} \to \mathbb{R}^{13n}$ is that $\int_{t_\circ}^t \omega(t)$ does not always have a clear physical interpretation.

bounded and full column rank. For a more detailed discussion and proofs of the properties below, the reader is invited to see (Melhem and Loría 2001).

P1.

a) The matrices $\mathcal{J}_1(q_\ell)$ and $\mathcal{J}_3$ are full-column rank for all $q_\ell \in \mathbb{R}^n$

b) For any kinematic chain with *only* revolute or *only* prismatic joints there exist positive constants k_{j_m}, k_{j_M} such that

$$k_{j_m} \leq \|\mathcal{J}(q)\| \leq k_{j_M} \quad \forall \;\; q \in \mathbb{R}^{2n}\,. \tag{15}$$

For further development, we stress that $\mathcal{J}_1(q_\ell)$ (respectively $\mathcal{J}_3$) is of full column rank if and only if it is left invertible, i.e., if there exists $\mathcal{J}_1^\dagger(q_\ell) \in \mathbb{R}^{n\times 9n}$ (respectively $\mathcal{J}_3^\dagger \in \mathbb{R}^{n\times 3n}$) such that $\mathcal{J}_1^\dagger(q_\ell)$ $\mathcal{J}_1(q_\ell) \equiv I_{n\times n}$ (respectively $\mathcal{J}_3^\dagger \mathcal{J}_3 = I_{n\times n}$).

P2. The induced norms[4] of $\mathcal{J}_1^\dagger(q_\ell)$ and $\mathcal{J}_3^\dagger$ are bounded.

P3. $\dfrac{d}{dt}\{\mathcal{J}(q)\} =: \dot{\mathcal{J}}(q,\dot q)$ is globally Lipschitz in $\dot q$, uniformly in q, i.e., $\exists\; l_j > 0$ such that

$$\|\dot{\mathcal{J}}(q,x) - \dot{\mathcal{J}}(q,y)\| \leq l_j\|x-y\| \qquad \forall\, q \in \mathbb{R}^n\,. \tag{16}$$

Moreover, it has been shown in (Loría*et al.* 2000) for the rigid-joints model[5] that 3 implies the existence of $l'_j > 0$ such that

$$\|\dot{\mathcal{J}}^\dagger(q,x) - \dot{\mathcal{J}}^\dagger(q,y)\| \leq l'_j\|x-y\| \qquad \forall\, q \in \mathbb{R}^n\,. \tag{17}$$

Roughly speaking this property holds since $\mathcal{J}(q)$ contains trigonometric functions of q_ℓ and q_m for rotational joints manipulators and constants corresponding to prismatic joints.

Fact 3. (Melhem and Loría 2001) Given a robot manipulator with *only* prismatic or *only* revolute joints, the Jacobian matrix $\mathcal{J}(q)$ given in (12d) *always* admits a *constant* left pseudo-inverse.

Remark 4. Interestingly enough, it is not clear from the proof of this fact how whether it is valid also for kinematic structures with both prismatic and revolute joints. This is because the structural properties of the Jacobians, exploited to prove the Fact for the only-revolute or only-prismatic cases, are destroyed. □

2.3 *Derivation of the potential energy*

The potential energy stocked in the manipulator at any moment can be decomposed in three main terms:

$$\mathcal{U}(q) = \mathcal{U}_\ell(q_\ell) + \mathcal{U}_m(q_\ell) + \mathcal{U}_e(q)\,. \tag{18}$$

[4] Or any other compatible norm.

[5] The proof is long but straightforward and can be established along the same lines for the flexible-joints model.

$\mathcal{U}_\ell$ is the contribution of the links and is given by

$$\mathcal{U}_\ell(q_\ell) = -\sum_{i=1}^{n} m_{\ell_i} g_o^\top p_{\ell_i}(q_\ell) \tag{19}$$

where we recall that $p_{\ell_i}(q_\ell)$ is the vector of Cartesian coordinates of the i-th link's center of mass. $\mathcal{U}_m$ is the contribution of the rotors' centers of mass and under assumption 1, $\mathcal{U}_m$ does not depend on q_m. More precisely,

$$\mathcal{U}_m(q_\ell) = -\sum_{i=1}^{n} m_{m_i} g_o^\top p_{m_i}(q_\ell) \tag{20}$$

where $g_o \in \mathbb{R}^3$ is the vector of gravity acceleration expressed in the base frame. The third term $\mathcal{U}_e$ corresponds to the contribution due to the elasticity in the transmissions and is given by the well known relation

$$\mathcal{U}_e(q) = \frac{1}{2}(q_\ell - q_m)^\top K (q_\ell - q_m) \tag{21}$$

where $K = \mathrm{diag}\{k_1, \cdots, k_n\} > 0$ is the stiffness matrix of the articulations. For notational simplicity we will write $\mathcal{U}_e$ as

$$\mathcal{U}_e(q) = \frac{1}{2} q^\top K_e\, q \tag{22}$$

where $q = \mathrm{col}[q_\ell, q_m]$ and

$$K_e = \begin{pmatrix} K & -K \\ -K & K \end{pmatrix}. \tag{23}$$

2.4 *A redundant Lagrangian model*

We present here the Lagrangian dynamics which we will derive from the kinetic and potential energies proposed above. To that end we will need to introduce temporarily the coordinates $\pi = \mathrm{col}[p, \phi]$ such that $\dot\pi := \nu$. As mentioned before, while in certain cases the coordinates ϕ may correspond to Euler angles or any other set of angles to describe a rotation, in general, do not have a clear physical interpretation however, since the gravitational energy of the bodies (links and actuators) does not depend on their orientation but only on their Cartesian positions, we will use

$$\widetilde{\mathcal{U}}_m(\pi) = -\sum_{i=1}^{n} m_{m_i} g_o^\top p_{m_i}\,, \tag{24}$$

$$\widetilde{\mathcal{U}}_\ell(\pi) = -\sum_{i=1}^{n} m_{\ell_i} g_o^\top p_{\ell_i}\,, \tag{25}$$

$$\widetilde{\mathcal{U}}_e(\pi) = \frac{1}{2}\eta(\pi)^\top K_e\, \eta(\pi) \tag{26}$$

where $\eta : \pi \mapsto q$ is not one to one. In other words, regarding the generalized coordinates as function of the Cartesian ones, one can find more than one value for π corresponding to the *same* q. However, the right hand side of (26) equals always $\mathcal{U}_e(q)$ given in (22).

Under these considerations we will derive the dynamic model using the Lagrange's equations

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \nu} - \frac{\partial \mathcal{L}}{\partial \pi} = Q \tag{27}$$

where $\mathcal{L}(\pi, \nu) = \mathcal{T}(\nu) - \widetilde{\mathcal{U}}(\pi(q))$ is the Lagrangian function and $Q \in \mathbb{R}^{15n}$ is the vector of reaction and external forces compatible with the constraints (13).

We consider that the system's coordinates are $\pi = \operatorname{col}[p, \phi] \in \mathbb{R}^{15n}$ and $\nu = \operatorname{col}[\dot{p}, \omega] \in \mathbb{R}^{15n}$ which are given by (8). Considering the kinetic energy (14) the Lagrange's equations (27) become

$$\mathcal{M}\dot{\nu} + \frac{\partial \widetilde{\mathcal{U}}_\ell}{\partial \pi} + \frac{\partial \widetilde{\mathcal{U}}_m}{\partial \pi} + \frac{\partial \widetilde{\mathcal{U}}_e}{\partial \pi} = Q \tag{28}$$

where it is clear from (24) and (25) that

$$\frac{\partial \widetilde{\mathcal{U}}_\ell}{\partial \pi} = \operatorname{col}[-m_{\ell_1} g_o, \cdots, -m_{\ell_n} g_o, 0_{12n\times 1}] \tag{29}$$

$$\frac{\partial \widetilde{\mathcal{U}}_m}{\partial \pi} = \operatorname{col}[0_{3n\times 1}, -m_{m_1} g_o, \cdots, -m_{m_n} g_o, 0_{9n\times 1}]\,. \tag{30}$$

Also, using the left invertibility of $\mathcal{J}(q)$ it can be shown (see (Melhem and Loría 2001)) that the solutions of $\mathcal{J}^\top(q)\dfrac{\partial \widetilde{\mathcal{U}}_e}{\partial \pi} = K_e q$ are characterized by

$$\frac{\partial \widetilde{\mathcal{U}}_e}{\partial \pi} = \mathcal{J}^\dagger(q)^\top K_e q \tag{31}$$

where $\mathcal{J}^\dagger(\cdot) \in \mathbb{R}^{2n\times 15n}$ is such that $\mathcal{J}^\dagger(q)\mathcal{J}(q) \equiv I_{n\times n}$. Thus, using (31), (29), (30) in (28) we obtain

$$\mathcal{M}\dot{\nu} + v + \mathcal{J}^\dagger(q)^\top K_e q = Q \tag{32}$$

where

$$v := \operatorname{col}[-m_{\ell_1} g_o, \cdots, -m_{\ell_n} g_o, -m_{m_1} g_o, \cdots, -m_{m_n} g_o,\ 0_{9n\times 1}]\,.$$

Thus, the redundant model is given by the dynamics equation (32) and the kinematics equation (13). In words, the equation (32) represents the dynamics of $2n$ rigid bodies (n links and n rotors) as if they could move freely in the space hence possesing each 6 degrees of freedom. The kinematics equation (13) expresses the $13n$ *holonomic* constraints owed to the fact that the bodies are mechanically connected together. With this in mind one might think that $12n$ coordinates is a more natural choice to express the dimension of the redundant model. Notice that the $3n$ "extra" coordinates are generated by the fact that we do not neglect the angular velocities of the rotors relative to the links which carry them, i.e., the coordinates ω_{i-1,m_i}.

3. DISCUSSION

The following remarks are in order.

1) We stress that the Jacobian $\mathcal{J}_2$ depends on the link *and* motor coordinates while it can be proven that the matrix $\mathcal{J}_3$ is constant since $R_{m_i}^\top(q)\mathbf{J}_O^{(m_i)}(q_\ell)$ is constant for any $i \leq n$ and any $q \in \mathbb{R}^n$.

2) Under assumption 1, the linear velocity of each rotor is independent of the motor variables q_m. This is reflected in the fact that $\mathcal{J}_1(\cdot)$ depends only on q_ℓ for both the full and reduced order model.

3) For the full model, considering (6), the contributions of the motor angular velocities to the kinetic energy is given by the following 3 terms

$$\begin{aligned}\frac{1}{2}\omega_{m_i}^\top R_{m_i} I_{m_i}^{m_i} R_{m_i}^\top \omega_{m_i} = \frac{1}{2}\omega_{i-1}^\top R_{m_i} I_{m_i}^{m_i} R_{m_i}^\top \omega_{i-1} + \\ \omega_{i-1}^\top R_{m_i} I_{m_i}^{m_i} R_{m_i}^\top \omega_{i-1,m_i} + \\ \frac{1}{2}\omega_{i-1,m_i}^\top R_{m_i} I_{m_i}^{m_i} R_{m_i}^\top \omega_{i-1,m_i}\end{aligned}$$

where

- The sum of the 3rd term for all rotors, using (6), (11) and (12c), yields

$$\frac{1}{2}\sum_{i=1}^{n}(\omega_{i-1,m_i}^\top R_{m_i} I_{m_i}^{m_i} R_{m_i}^\top \omega_{i-1,m_i}) = \frac{1}{2}\dot{q}_m^\top \mathcal{J}_3^\top M_2 \mathcal{J}_3 \dot{q}_m$$

where $H_3 := \mathcal{J}_3^\top M_2 \mathcal{J}_3$ is a *constant* diagonal matrix which depends on the constant inertia tensors and the gear ratios.

- The sum over all rotors of the 2nd term, using (4), (11), (12b) and (12c), takes the form

$$\sum_{i=1}^{n}(\omega_{i-1}^\top R_{m_i} I_{m_i}^{m_i} R_{m_i}^\top \omega_{i-1,m_i}) = \frac{1}{2}\dot{q}_\ell^\top \mathcal{J}_2(q)^\top M_2 \mathcal{J}_3 \dot{q}_m + \frac{1}{2}\dot{q}_m^\top \mathcal{J}_3^\top M_2 \mathcal{J}_2(q)\dot{q}_\ell$$

where under 1 (which implies that $I_{m_i}^{m_i}$ is diagonal) the matrix $H_2(q_\ell) = \mathcal{J}_2(q)^\top M_2 \mathcal{J}_3$ is independent of the rotor generalized positions q_m.

- The sum of the first term over all rotors, using (4), (11) and (12b), is given by

$$\frac{1}{2}\sum_{i=1}^{n}(\omega_{i-1}^\top R_{m_i} I_{m_i}^{m_i} R_{m_i}^\top \omega_{i-1}) = \frac{1}{2}\dot{q}_\ell^\top \mathcal{J}_2(q)^\top M_2 \mathcal{J}_2(q)\dot{q}_\ell$$

where the matrix $\mathcal{J}_2(q)^\top M_2 \mathcal{J}_2(q)$ depends in general, on both the rotor and links positions q_m, q_ℓ and determines the gyroscopic terms.

Thus owing to these observations one can derive the $(2n \times 2n)$-inertia matrix as a function of q_ℓ and q_m, explicitly one has that

$$H(q) = \begin{bmatrix} H_1(q) & H_2(q_\ell) \\ H_2(q_\ell)^\top & H_3 \end{bmatrix} \tag{33}$$

where the matrix $H_1(q)$ contains the inertia contributions of the links and that of the gyroscopic terms

$\mathcal{J}_2(q)^\top M_2 \mathcal{J}_2(q)$. It is clear that if the latter are neglected $H_1(\cdot)$ depends only on q_ℓ. The matrix $H_2(q_\ell)$ has a triangular form and is the same that one finds in the inertia matrix of the model used in (Nicosia and Tomei 1990) and succeeding references.

4) Indeed, it is straight forward to verify that using the kinematics equation (13) in (32) and premultiplying on both sides of (32) one obtains the model of flexible-joint robots with a *non* block-diagonal inertia matrix used in (Nicosia and Tomei 1990) and related references of Nicosia's coauthors. If furthermore, one neglects the second line and second column of the matrix in (10) and the second line of the Jacobian matrix in (12d) one obtains the model of (Spong 1987).

5) The model (32) is the flexible-joints counterpart of the model presented in (Loría*et al.* 2001) for rigid-joint manipulators.

6) Conversely, one can obtain the model (32), (13) starting from the Lagrangian equations [6]

$$H(q)\ddot{q} + C(q,\dot{q})\dot{q} + K_e q + g(q) = u$$

using (13), observing that $H = \mathcal{J}(q)^\top \mathcal{M}\mathcal{J}(q)$, $C(q\dot{q}) = \mathcal{J}(q)^\top \mathcal{M}\dot{\widehat{\mathcal{J}(q)}}$, $g(q) = \mathcal{J}(q)^\top v$ and restricting the exrternal forces $Q \in \mathbb{R}^{15n}$ to satisfy

$$\mathcal{J}(q)^\top Q \;=\; u \;=\; \begin{bmatrix} 0_n \\ u_m \end{bmatrix}$$

where u_m corresponds to the vector of n generalized torques delivered by the motors.

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4. CONCLUSIONS

We presented a new model for open kinematic chains which is a non minimal realization of the system dynamics and is composed by a set of dynamic and kinematic equations. It is expressed in the Cartesian velocities and generalized positions.

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[6] Where $C(q,\dot{q})$ corresponds to the Coriolis and centrifugal forces matrix and $g(q)$ corresponds to the gravitational forces