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A HIGHER DIMENSIONAL GENERALIZATION OF BENDIXON'S CRITERION

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Abstract: Conditions are given that guarantee the *nonexistence of periodic orbits* lying entirely in a simply connected set. The conditions are formulated in terms of matrix inequalities involving the variational equation. For systems defined in $\mathbb{R}^2$ the conditions are equivalent to Bendixon's criterion. A connection with analytic estimates of the Hausdorff dimension of invariant compact sets is emphasized. *Copyright © IFAC 2002*

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1. INTRODUCTION

Stability analysis of nonlinear systems is usually based on Lyapunov functions (nonlocal methods) or first order approximation (local methods). Control design which is based on linearization around a desired position is far simpler in general than nonlocal design. However, there are some efficient qualitative global tools based on linearization around solutions of dynamical systems. One of these tools is the classical Bendixon criterion (or divergence test) which gives sufficient conditions for the nonexistence of periodic orbits.

Recall that for a smooth planar system

$$\dot{x} = f(x), \tag{1}$$

$x \in \mathbb{R}^2$, the Bendixon criterion reads

Theorem 1. If $\operatorname{div} f(x)$ is non-zero on some simply connected domain D then no periodic orbit can lie entirely in D.

A classical proof of this statement is based on the divergence theorem and cannot be generalized to the higher dimensional case. The main purpose of this paper is to present one of the possible generalization of the Bendixon result for the case of arbitrary dimension. There are several generalizations of this criterion, see, e.g. (Smith, 1981; Muldowney, 1990; Li and Muldowney, 1993; Li and Muldowney, 1996). The criterion due to Smith reveals a connection between the method to estimate the Hausdorff dimension of invariant compact sets and the method to prove the nonexistence of periodic orbits. A generalization of this criterion is the subject of this paper. Muldowney and Li (Muldowney, 1990; Li and Muldowney, 1993; Li and Muldowney, 1996) used an approach based on compound matrices to prove a negative Bendixon-like criterion.

In this paper we investigate this question by a method which allows to estimate the Hausdorff dimension of invariant compact sets (Pogromsky and Nijmeijer, 2000). The conditions presented in this paper are formulated in terms of inequalities involving two eigenvalues of some matrix pencil. An equivalent reformulation based on some state dependent coordinate change for the variational equation can also be derived following ideas of (Leonov, 2002).

The paper is organized as follows. First we present preliminary results on Hausdorff dimension and Hausdorff measure of invariant compact sets. Those results are utilized in Section 4 to derive a higher dimensional generalization of Bendixon's criterion. Section 4 presents an illustrative example.

In the paper we use the following notations. The Euclidean norm in $\mathbb{R}^n$ is denoted as $|\cdot|$, $|x|^2 = x^\top x$, where $^\top$ stands for the transpose. For matrices the notation $||P||$ stands for the spectral norm of P, i.e. $||P||^2$ is the largest eigenvalue of the matrix $P^\top P$. Eigenvalues of the matrix $P^\top P$ are called singular values of P.

2. HAUSDORFF DIMENSION AND HAUSDORFF MEASURE

Consider a compact subset K of a compact metric space X. Given $d \geq 0$, $\varepsilon > 0$, consider a covering of K by open spheres B_i with radii $r_i \leq \varepsilon$. Denote by

$$\mu(K, d, \varepsilon) = \inf \sum_i r_i^d \tag{2}$$

the d-measured volume of covering of the set K. Here the infimum is calculated over all ε-coverings of K. There exists a limit, which may be infinite,

$$\mu_d(K) := \sup_{\varepsilon>0} \mu(K, d, \varepsilon).$$

It can be proved that μ_d is an outer measure on X (see, e.g. Proposition 5.3.1 in (Leonov *et. al.*, 1996)).

Definition 1. The measure μ_d is called the *Hausdorff d-measure.*

The properties of the measure μ_d can be summarized as follows. There exists a single value $d = d_*$, such that for all $d < d_*$, $\mu_d(K) = +\infty$ and for all $d > d_*$, $\mu_d(K) = 0$. Here

$$d_* = \inf\{d : \mu_d(K) = 0\} = \sup\{d : \mu_d(K) = +\infty\}.$$

Definition 2. The value d_* is called the *Hausdorff dimension* of the set K.

In the sequel, we will use the notation $\dim_H K$ for the Hausdorff dimension of the set K.

Now, following Douady and Oesterlé (Douady and Osterlé, 1980), we define the elliptic Hausdorff d-measure of a compact set $K \subset \mathbb{R}^n$. Let E be an open ellipsoid in $\mathbb{R}^n$. Let $a_1(E) \geq a_2(E) \geq \ldots \geq a_n(E)$ be the lengths of semiaxis of E numbered in decreasing order. Represent an arbitrary number d, $0 \leq d \leq n$ in the form $d = d_0 + s$, where $d_0 \in \mathbb{Z}_+$ and $s \in [0, 1)$ and introduce the following

$$\omega_d(E) = \prod_{i=1}^{d_0} a_i(E)(a_{d_0+1}(E))^s. \tag{3}$$

Fix a certain d and $\varepsilon > 0$ and consider all possible finite coverings of the compact K by ellipsoids E_i for which

$$[\omega_d(E_i)]^{1/d} \leq \varepsilon$$

(if $d = 0$ we put $[\omega_d(E_i)]^{1/d} = a_1(E_i)$). Similar to the definition of Hausdorff d-measure we denote

$$\tilde{\mu}_d(K, d, \varepsilon) = \inf \sum_i \omega_d(E_i),$$

where the infimum is calculated over all coverings $\{E_i\}$ of K.

Definition 3. The value

$$\tilde{\mu}_d(K) = \sup_{\varepsilon>0} \tilde{\mu}(K, d, \varepsilon)$$

is called the *Hausdorff elliptical d-measure* of the compact set K.

It was proven in (Douady and Osterlé, 1980; Témam, 1988) that the elliptical and spherical Hausdorff d-measures are equivalent and therefore, using extremal properties of μ_d, the values of the Hausdorff dimensions determined by means of spherical and elliptic coverings are equal.

Let $\{\varphi^t\}$ be a one-parameter semigroup of diffeomorphisms $\Omega \to \Omega$, $\Omega \subset \mathbb{R}^n$, $t \in \mathbb{I}$. We will only consider the case $\mathbb{I} = \mathbb{R}_+$, the case $\mathbb{I} = \mathbb{Z}_+$ can be treated in the same fashion. The subset $\gamma(x_0)$ of $\mathbb{R}^n$ of the form $\gamma(x_0) = \{x \ : \ \exists t \in \mathbb{R}_+ \ x = \varphi^t(x_0)\}$ is called the *trajectory, or orbit,* of the point x_0. A trajectory $\gamma(x_0)$ is called periodic if $x(t, x_0)$ is a periodic solution with a nontrivial period.

By $T_x\varphi^t$ we denote the derivative of φ^t with respect to x (Jacobian) at the point $x \in \mathbb{R}^n$, that is, $T_x\varphi^t$ is a linear operator $\mathbb{R}^n \to \mathbb{R}^n$. For a linear operator $L : \mathbb{R}^n \to \mathbb{R}^n$ denote by $a_1(L) \geq a_2(L) \geq \ldots \geq a_n(L)$ its singular values. For arbitrary $k \in \mathbb{Z}_+$, $0 \leq k \leq n$ we denote

$$\omega_k(L) = \begin{cases} \prod_{i=1}^{k} a_i(L), \ k > 0 \\ 1, \ k = 0 \end{cases}$$

For arbitrary $d \in [0, n]$ we put $d = d_0 + s$, where $d_0 \in \mathbb{Z}_+$ and $s \in [0, 1)$ and introduce the following definition

$$\omega_d(L) = \omega_{d_0}^{1-s}(L)\omega_{d_0+1}^{s}(L).$$

Consider an open set $\tilde{K}$ such that $K \subset \tilde{K}$, $\varphi^t(K) \subset \tilde{K}$.

Theorem 2. Assume that there exists $d \in [0, n]$ such that for any $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that for all $t \geq t_\varepsilon$

$$\sup_{x\in\tilde{K}} \omega_d(T_x\varphi^t) \leq \varepsilon. \tag{4}$$

Then

$$\mu_d(K) < \infty \Longrightarrow \lim_{t\to\infty} \mu_d(\varphi^t(K)) = 0$$

Additionally, if the compact set K is invariant (i.e. $\varphi^t(K) = K, \ \forall t \in \mathbb{R}_+$) then $\dim_H K \leq d$.

Basically, this theorem is a reformulation of the well known Douady-Oesterlé theorem (Douady and Osterlé, 1980), an analog of this statement for arbitrary Hilbert spaces is proved in (Témam, 1988). Leonov (Leonov, 1991) (see also Theorem 5.4.1 in (Leonov *et. al.*, 1996) and Theorem 8.1.2 in (Leonov *et. al.*, 1996)) proved a generalization of the Douady-Oesterlé theorem: instead of (4) it is sufficient to require that

$$\sup_{x\in\tilde{K}} \left[\frac{p(\varphi^t(x))}{p(x)}\omega_d(T_x\varphi^t)\right] \leq \varepsilon \tag{5}$$

where $p : \tilde{K} \to (0, \infty)$ is a scalar positive continuous function. This approach turns out to be useful for estimates of the Hausdorff dimension in terms of auxillary (Lyapunov) functions satisfying certain partial differential inequalities. We will further develop this idea in the sequel.

Compared with the statement due to Leonov (Leonov, 1991) it is supposed here that $\tilde{K}$ is not necessarily bounded.

Consider the system

$$\dot{x} = f(x), \tag{6}$$

where $x \in \Omega \subset \mathbb{R}^n$ and $f : \Omega \to \Omega$, is a smooth vector field on $\Omega \subset \mathbb{R}^n$. Let $\varphi^t(x_0) : x_0 \mapsto x(t, x_0)$ and $K \subset \tilde{K} \subset \Omega$.

Along with the system (6) consider the first order approximation

$$\dot{y} = J(x(t, x_0))y, \tag{7}$$

where $y \in \mathbb{R}^n$ and

$$J(x(t, x_0)) = \frac{\partial f}{\partial x}(x(t, x_0)).$$

Consider an arbitrary matrix function $G : \tilde{K} \to \mathbb{R}^{n\times n}$ which is smooth and invertible in $\tilde{K}$. For any $t \in \mathbb{R}_+$ and $x \in \tilde{K}$, $G(\varphi^t(x))$ defines a linear operator $\mathbb{R}^n \to \mathbb{R}^n$. For any $x \in \tilde{K}$ the singular values of $G(\varphi^t(x))$ are bounded from above and below. Given an arbitrary nonsingular $n \times n$ matrix $S(t)$ which is bounded from below and above for all $t \in \mathbb{R}_+$, we have for the singular values $\sigma_i'(t)$ of the matrix $X(t)S(t)$ the following simple estimate $\xi_{\min}\sigma_i(t) \leq \sigma_i'(t) \leq \xi_{\max}\sigma_i(t)$ where $\xi_{\min}$ and $\xi_{\max}$ are the lower and upper bounds for the singular values of the matrix $S(t)$ and $\sigma_i(t)$ are the singular values of the matrix $X(t)$. Therefore, using Theorem 2, we arrive at the following result.

Theorem 3. Assume that there exists $d \in [0, n]$ such that

$$\sup_{x \in \tilde{K}} \omega_d \left[G(\varphi^t(x)) T_x \varphi^t\right] \to 0 \quad \text{as} \quad t \to \infty. \tag{8}$$

Then

$$\mu_d(K) < \infty \Longrightarrow \lim_{t\to\infty} \mu_d(\varphi^t(K)) = 0,$$

if $\varphi^t(K) = K, \quad \forall t \in \mathbb{R}_+$ then $\dim_H K \leq d$,

Consider some symmetric positive definite matrix $P(x)$ which is continuously differentiable in $\tilde{K}$, and which therefore is bounded from above and below in $\tilde{K}$ and the symmetric matrix $Q(x(t, x_0))$ defined by

$$Q = \dot{P} + PJ + J^\top P.$$

Here $\dot{P}(x(t, x_0)) = \frac{d}{dt} P(x(t, x_0))$ stands for the matrix with entries equal to

$$\left(\frac{\partial p_{ij}(x(t, x_0))}{\partial x} f(x(t, x_0)) \right)_{ij}.$$

Consider the equation

$$\det[Q(x) - \lambda(x) P(x)] = 0. \tag{9}$$

For any $x \in \tilde{K}$ the equation (9) has n *real* solutions $\lambda_i(x)$ since the matrix Q is symmetric and P is positive definite. Indeed, (9) can be rewritten as

$$\det[G(x)^\top (G(x)^{-\top} Q(x) G(x)^{-1} - \lambda(x) I_n) G(x)] = 0$$

or, equivalently,

$$\det[G(x)^{-\top} Q(x) G(x)^{-1} - \lambda(x) I_n] = 0,$$

where

$$P(x) = G(x)^\top G(x)$$

and the matrix $G(x)^{-\top} Q(x) G(x)^{-1}$ is symmetric. Order the solutions of (9) in the decreasing order for all x: $\lambda_1(x) \geq \lambda_2(x) \geq \ldots \geq \lambda_n(x)$.

Theorem 4. (Pogromsky and Nijmeijer, 2000) Suppose that for some $P(x)$ satisfying the above assumptions there exist numbers $d_0 \in \mathbb{Z}_+$, $s \in [0, 1)$ such that

$$\limsup_{\tau\to\infty} \frac{1}{\tau} \int_0^\tau \quad [\lambda_1(x(t, x_0)) + \ldots + \lambda_{d_0}(x(t, x_0)) + s\lambda_{d_0+1}(x(t, x_0))] dt < 0 \tag{10}$$

for any $x_0 \in \tilde{K}$. Then $\mu_d(K) < \infty \Longrightarrow \lim_{t\to\infty} \mu_d(\varphi^t(K)) = 0$, additionally, if K is invariant then $\dim_H K \leq d_0 + s$

Corollary 5. (Leonov) (Leonov, 1991; Leonov *et. al.*, 1996)) Let $\lambda_i(x)$, $i = 1, \ldots, n$ be the eigenvalues of the matrix $(J(x) + J(x)^\top)/2$ ordered in decreasing order. Suppose there exist numbers $d_0 \in \mathbb{Z}_+$, $s \in [0, 1)$, and a continuously differentiable in $\tilde{K}$ function $v : \tilde{K} \to \mathbb{R}$ such that

$$\lambda_1(x) + \ldots + \lambda_{d_0}(x) + s\lambda_{d_0+1}(x) + \frac{\partial v}{\partial x} f(x) < 0 \tag{11}$$

for any $x_0 \in \tilde{K}$. Then

$$\mu_d(K) < \infty \Longrightarrow \lim_{t\to\infty} \mu_d(\varphi^t(K)) = 0.$$

Additionally, if K is invariant then $\dim_H K \leq d_0 + s$.

Proof: The result directly follows from Theorem 4 if one takes $P(x) = p^2(x) I_n$ where $p(x) > 0$ is a scalar differentiable function bounded from below and above in $\tilde{K}$ and denote $v(x) = (\log p(x))/(d_0 + s)$. In this case (9) is equivalent to the equation

$$\det[J(x)^\top + J(x) + \frac{2\dot{p}}{p} I_n - \lambda I_n] = 0$$

Since

$$2\frac{\dot{p}}{p(d_0 + s)} = 2\frac{dv}{dt} = 2\frac{\partial v}{\partial x} f(x)$$

the result follows from Theorem 4. □

3. A HIGHER DIMENSIONAL GENERALIZATION OF BENDIXON'S CRITERION

The results obtained in the previous section allow to find conditions based on the first order approximation ensuring the nonexistence of closed orbits.

Consider again system (6). Let $P(x)$ be a continuously differentiable positive definite matrix function defined on some simply connected set D. As before, let λ_i be the roots of the equation (9) ordered in decreasing order for all $x \in D$.

Theorem 6. Let D be a simply connected set. Suppose that for some $P(x)$ satisfying the above assumptions

$$\lambda_1(x) + \lambda_2(x) < 0 \tag{12}$$

or

$$\lambda_{n-1}(x) + \lambda_n(x) > 0 \tag{13}$$

for any $x \in D$. Then no periodic orbit can lie entirely in D.

Proof: The proof of Theorem 6 follows an idea used in the proof of the Leonov theorem (Theorem 8.3.1 in (Leonov *et. al.*, 1996)). Suppose (12) holds but there is a periodic orbit γ which lies entirely in D. We put on γ some smooth two-dimensional surface $K \subset D$ having finite area. The existence of such a surface for a smooth curve follows from the fact that D is simply connected, see, for example, (Courant, 1950). As before, we denote by φ^t the flow of system (6). Let $\mu(S)$ be the Hausdorff 2-measure of a smooth 2-dimensional surface S. Since γ is invariant under φ^t and $K \subset D$ for any $t \geq 0$ we have

$$\inf_{t \geq 0} \mu\left(\varphi^t(K)\right) > 0. \tag{14}$$

At the same time, using (12) from Theorem 4, it follows that

$$\lim_{t \to \infty} \mu\left(\varphi^t(K)\right) = 0, \tag{15}$$

which contradicts (14). Therefore, (12) ensures the absence of periodic trajectories lying in D. To prove that (13) is also sufficient for the nonexistence of periodic orbits, one can consider the system (6) in inverse time. □

Remark 7. For planar systems, if we take $P(x) = I_2$, conditions (12), (13) are equivalent to Bendixon's criterion

As an immediate consequence of the previous theorem we formulate the following result.

Corollary 8. Suppose (12) or (13) is satisfied in a simply connected region D for some positive definite matrix $P(x)$ and there are only transverse intersections of the system trajectories and boundary of D. Then, any bounded solution lying entirely in D, if it exists, tends to a set consisting of equilibrium points. Additionally, if the set of equilibria in D consists of isolated points, then any bounded solution lying entirely in D tends to an equilibrium.

Remark 9. The transversalty condition in the previous result can be relaxed. It is sufficient to impose an assumption that the ω-limit set of any solution lying entirely in D does not have points in common with the boundary of D.

The arguments used above allows us to prove the existence, orbital stability and basin of attraction of periodic solutions. Suppose for example that system (6) is defined on $\mathbb{R}^3$ and there is a positively invariant open torus D such that each trajectory intersects D transversely. Assume that (12) is satisfied everywhere in cl$D$ and there are no equilibria in cl$D$. Then there is an orbitally stable 1-period solution lying in D and its trajectory attracts all solutions originating in cl$D$. To prove this statement it is sufficient to build the standard Poincare map $\mathcal{P}$, then to prove the existence of a periodic solution (Schauder principle) and finally to prove that (12) implies that 1-Hausdorff measure of any set is vanishing with time under $\mathcal{P}$ as soon as this measure is initially bounded.

4. ILLUSTRATIVE EXAMPLE: THE RÖSSLER SYSTEM

Consider the Rössler system

$$\begin{aligned} \dot{x} &= -y - z \\ \dot{y} &= x + ay \\ \dot{z} &= c + z(x - b) \end{aligned} \tag{16}$$

where $a, b, c > 0$. First, notice that $z_0 \geq 0$ implies that $z(t, z_0) > 0$ for all $t > 0$ (it follows from the third equation since $c > 0$). We will consider only solutions for which $z_0 > 0$. Denoting $q = (x, y, z)^\top$, the Jacobian of (16) has the form

$$J(q) = \begin{pmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ z & 0 & x-b \end{pmatrix} \quad (17)$$

Choose the matrix $P(q)$ as

$$P(q) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z^{-1} \end{pmatrix} \quad (18)$$

Since $z(t, z_0) > 0$ this matrix is well defined together with its inverse. As one can easily check the equation (9) has the following solutions

$$\lambda_1 = 2a, \quad \lambda_2 = 0, \quad \lambda_3 = 2(x-b) - \frac{\dot{z}}{z} = x - b - \frac{c}{z},$$

Therefore, applying Theorem 6 one has the following result

Proposition 10. There are no periodic solutions of (16) lying entirely in the domain

$$D = \left\{ x, y, z \;:\; x - b > \frac{c}{z},\; z > 0 \right\}$$

The utilization of the criterion presented in (Muldowney, 1990) (for the Lozhinskii matrix norm related to the Euclidean norm) gives the following estimate for the set where no periodic orbits can lie entirely:

$$D = \left\{ x - b > \frac{1}{4a}(1-z)^2 - a,\; x - b > 0,\; z > 0 \right\}$$

5. CONCLUSION

In the paper we presented sufficient conditions guaranteeing that there are no periodic solutions lying in some simply connected set. The conditions are based on the properties of some matrix pencil associated with the first order approximation system. An interesting connection of the approach used in this paper with the problem of estimation of the Hausdorff dimension of invariant compact sets is emphasized.

6. REFERENCES

Courant, R. (1950) *Dirichlet's Principle, Conformal Mappings and Minimal Surfaces*, (Interscience, New York).

Douady A., and J. Osterlé (1980) "Dimension de Hausdorff des attracteurs", *C.R. Acad. Sci. Paris. Sér. A.*, vol. **290**(24), pp. 1135–1138.

Leonov, G.A. (1991) "On estimations of the Hausdorff dimension of attractors", *Vestnik Leningradskogo Gos. Univ., Ser. 1*, vol. **3**, pp. 41–44, in Russian.

Leonov, G.A. (2002) "Lyapunov dimension formulas for Henon and Lorenz attractors", *St. Petersburg Math. J.*, vol.**13**(3).

Leonov, G.A., I.M. Burkin and A.I. Shepeljavyi (1996) *Frequency methods in Oscillation Theory*, (Kluwer Academic Publishers, Dordrecht).

Leonov, G.A., D.V. Ponomarenko and V.B. Smirnova (1996) *Frequency Methods for Nonlinear Analysis. Theory and Applications*, (World Scientific, Singapore).

Li, Yi and J.S. Muldowney (1993) "On Bendixon's criterion", *J. Diff. Equations* , vol.**106**, pp. 27–39.

Li, Yi and J.S. Muldowney (1996) "Phase asymptotic semiflows, Poincare's condition, and the existence of stable limit cycles", *J. Diff. Equations* , vol.**124**, pp. 425–448.

Muldowney, J. (1990) "Compound matrices and ordinary differential equations", *Rocky Mountain J. of Mathematics*, vol.**20**(4), pp. 857–871.

Pogromsky A.Yu., and H. Nijmeijer (2000) "On estimates of Hausdorff dimension of invariant compact sets", *Nonlinearity*, vol. **13**, pp. 927–945.

Smith, R.A. (1981) "An index theorem and Bendixon's negative criterion for certain differential equations of higher dimension", *Proc. Roy. Soc. Edinburgh*, vol.**91A**, pp. 63–77.

Témam, R. (1988) *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, (Springer-Verlag, New York, Berlin).