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NEW RESULTS ON OPTIMAL ELLIPSOIDAL ESTIMATION FOR UNCERTAIN DYNAMICAL SYSTEMS

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Abstract: The set-membership approach to the state estimation of dynamical systems subjected to uncertain disturbances is developed. Optimal outer ellipsoidal estimates on reachable sets are considered, and various optimality criteria are discussed. Nonlinear differential equations describing the evolution of optimal estimating ellipsoids are analyzed. The asymptotic behavior of the ellipsoids is investigated in the vicinity of the initial time instant and infinity. Control problem for the systems subjected to uncertain perturbations is analyzed in the framework of the optimal ellipsoidal estimation.

Keywords: identification algorithms, uncertain dynamic systems, optimization problems, perturbations

1. INTRODUCTION

Dynamical systems subjected to unknown but bounded perturbations appear in numerous applications. The set-membership approach to such systems makes it possible to obtain outer guaranteed estimates on reachable sets and thus to evaluate all possible trajectories of the systems. In this context, the ellipsoidal estimation seems to be the most efficient technique. Among its advantages are the explicit form of approximation, smooth boundaries, invariance with respect to linear transformations, possibility of optimization, etc. The ellipsoidal technique for the approximation of reachable sets was considered by a number of authors. The earlier results were summarized in Schweppe (1973). The ellipsoidal estimates optimal in the sense of volume were first proposed by Chernousko (1980). General optimality criteria for the ellipsoidal estimates were analyzed in the books by Chernousko (1994). Various aspects of ellipsoidal estimation of reachable sets were considered in Kurzhanski and Valyi (1997), Norton (1994, 1995), Milanese (1996).

In this paper, optimal outer ellipsoidal estimates on reachable sets are investigated for various optimality criteria. Nonlinear differential equations governing the evolution of ellipsoids are analyzed, and the asymptotic behavior of ellipsoids is studied for different cases. The main attention is paid to the criterion equal to the projection of the approximating ellipsoid onto the given direction.

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2. ELLIPSOIDAL ESTIMATION

Consider a linear system of ordinary differential equations

$$\dot{x} = A(t)x + B(t)u + f(t), \quad t \geq s \qquad (1)$$

Here, $x \in R^n$ is the n-vector of state, $u \in R^m$ is the m-vector of unknown disturbances, the dot denotes the differentiation with respect to time t, A is an $n \times n$ matrix, f is an n-vector, B is an $n \times m$ matrix. The matrices $A(t)$ and $B(t)$ as well as the vector $f(t)$ are specified functions of time for $t \geq s$ where s is the initial time instant.

Denote by $E(a, Q)$ the following n-dimensional ellipsoid

$$E(a, Q) = \{x : (Q^{-1}(x-a), (x-a)) \leq 1\} \quad (2)$$

where $a \in R^n$ is the center of the ellipsoid and Q is the positive definite $n \times n$ matrix. Assume that the unknown perturbation $u(t)$ is bounded by the ellipsoid as follows

$$u(t) \in E(0, G(t)), \quad t \geq s \qquad (3)$$

where $G(t)$ is a $m \times m$ matrix specified for $t \geq s$.

The initial data for the equation (1) are also uncertain and described by the inclusion

$$x(s) \in M = E(a_0, Q_0) \qquad (4)$$

where a_0 is a given n-vector and Q_0 is a given $n \times n$ positive definite matrix.

The reachable set $D(t, s, M)$ of the system (1) for $t \geq s$ is defined as the set of all ends $x(t)$ of the state trajectories $x(\cdot)$ at the time instant t compatible with the conditions (1), (3), and (4).

The problem is to find an outer ellipsoidal approximation of the reachable set $D(t, s, M)$ for $t \geq s$ such that

$$D(t, s, M) \subset E(a(t), Q(t)) \qquad (5)$$

In other words, the n-vector $a(t)$ and the $n \times n$ positive definite matrix $Q(t)$ are to be found which determine, respectively, the center and the matrix of the approximating ellipsoid. It is natural to find the "smallest" outer ellipsoid which is, in some sense, the closest to the reachable set $D(t, s, M)$.

3. OPTIMAL ELLIPSOIDS

The ellipsoid $E(a, Q)$ will be characterized by the scalar optimality criterion, or the cost function J which is a specified function $L(Q)$ of the matrix Q of the ellipsoid:

$$J(E(a, Q)) = L(Q) \qquad (6)$$

Suppose the function $L(Q)$ in (6) is defined for all symmetric positive definite matrices Q, is smooth and monotone. The latter property means that $L(Q_1) \geq L(Q_2)$, if the difference $Q_1 - Q_2$ is a nonnegative definite matrix. In other words, if the inclusion $E(a, Q_1) \supset E(a, Q_2)$ holds, then $J(E(a, Q_1)) \geq J(E(a, Q_2))$.

Consider some important particular cases of the general optimality criterion (6).

1. The volume of an ellipsoid is given by

$$J(E(a, Q)) = L(Q) = c_n (\det Q)^{1/2} \qquad (7)$$

where c_n is a constant depending on n.

2. The sum of the squared semiaxes of an ellipsoid is equal to

$$J = L(Q) = \operatorname{Tr} Q \qquad (8)$$

3. The linear optimality criterion

$$J = L(Q) = \operatorname{Tr}(CQ) \qquad (9)$$

where C is a symmetric nonnegative definite $n \times n$ matrix, is a generalization of (8).

4. The following criterion

$$J = L(Q) = (Qv, v) \qquad (10)$$

where v is a given n-vector, is a particular case of the linear criterion (9). Here,

$$C = v * v, \quad C_{ij} = v_i v_j, \quad i, j = 1, \ldots, n \; (11)$$

where the symbol $*$ denotes the dyadic product of vectors.

The criterion (10) has a clear geometric interpretation: it is related to the projection $\Pi_v(E)$ of the ellipsoid $E(a, Q)$ onto the direction of the vector v. The above-mentioned projection is expressed through the support function $\rho(v, E)$ of the ellipsoid as follows

$$\Pi_v(E) = \rho(v, E) + \rho(-v, E) \qquad (12)$$

Since the support function for the ellipsoid (2) is given by

$$\rho(v, E) = (Qv, v)^{1/2} + (a, v) \qquad (13)$$

it follows from (12) and (13) that

$$\Pi_v(E) = 2(Qv, v)^{1/2}$$

Thus, minimization of the criterion (10) is equivalent to the minimization of the projection of the ellipsoid onto the direction of the vector v. Other examples of optimality criteria are given in Chernousko (1994).

The smooth family $E^*(t) = E(a(t), Q(t))$ of ellipsoids is called *locally optimal*, if for all $\tau \in [s, t]$

$$E^*(t) \supset D(t, \tau, E^*(\tau)) \qquad (14)$$

(such ellipsoids are called superreachable, see Chernousko (1994)), and, besides, for all $t \geq s$

$$\left.\frac{dL(Q(\tau))}{d\tau}\right|_{\tau=t} \to \min \tag{15}$$

where the minimum is taken over all smooth families of superreachable ellipsoids satisfying (14).

The smooth family of superreachable ellipsoids satisfying (14) is called *globally optimal* for the given $t = T$, if it solves the minimization problem $L(Q(T)) \to \min$.

As shown in Chernousko (1994), the determination of globally optimal ellipsoids is reduced to the two-point boundary value problem. In this paper, only locally optimal ellipsoids are considered.

The parameters $a(t)$ and $Q(t)$ of the locally optimal ellipsoids satisfy the following initial value problems

$$\dot{a} = A(t)a + f(t), \quad a(s) = a_0 \tag{16}$$

$$\dot{Q} = A(t)Q + QA^\top(t) + hQ + h^{-1}K(t) \tag{17}$$

$$h = \left\{ \mathrm{Tr}\left[\frac{\partial L}{\partial Q}K(t)\right] \Big/ \mathrm{Tr}\left[\frac{\partial L}{\partial Q}Q\right] \right\}^{1/2}$$

$$K(t) = B(t)G(t)B^\top(t), \quad Q(s) = Q_0$$

Here, $^\top$ denotes a transposed matrix. Note that the ellipsoids optimal in the sense of volume, i.e. for $L(Q)$ given by (7), are invariant with respect to linear transformations of the x-space. The ellipsoids optimal in the sense of linear criteria (8)–(10) lack the invariance property but can provide better approximations than the ellipsoid optimal in the sense of volume and lead to more simple equations. For the case of the linear criterion (9), equation (17) becomes

$$\dot{Q} = AQ + QA^\top + hQ + h^{-1}K, \; Q(s) = Q_0$$
$$K = BGB^\top, \; h = [\mathrm{Tr}(CK)/Tr(CQ)]^{1/2} \tag{18}$$

The vector v in the criterion (10) need not to be constant and can be assumed to be a function of time: $v = v(t)$. It can be chosen arbitrarily, e.g., its choice can be made in order to minimize the projection of the reachable set onto the desired direction. For any piecewise continuous $v(t)$, the ellipsoids $E(a(t), Q(t))$ where the center $a(t)$ and the matrix $Q(t)$ satisfy the initial value problems (16) and (17), respectively, provide the outer estimates of the reachable set. Therefore, the vector function $v(t)$ can be chosen in such a way as to simplify and/or improve the outer estimates of the reachable set.

4. TRANSFORMATION OF EQUATIONS

Nonlinear equation (18) for the matrix $Q(t)$ depends on three matrices: A, K, and C. This equation can be simplified by the change of variable

$$Q = VQ_*V^\top \tag{19}$$

where $V(t)$ is an invertible $n \times n$ matrix and Q_* is a new variable. Taking $V(t)$ equal to the fundamental matrix of equation (1), i.e.

$$\dot{V} = AV, \quad V(s) = I, \quad t \geq s \tag{20}$$

where I is the unit $n \times n$ matrix, one obtains from (18), (20)

$$\dot{Q}_* = h_*Q_* + h_*^{-1}K_*, \quad Q_*(s) = Q_0$$
$$h_* = [\mathrm{Tr}(C_*K_*)/\,\mathrm{Tr}(C_*Q_*)]^{1/2} \tag{21}$$
$$K_* = V^{-1}K\left(V^{-1}\right)^\top, \quad C_* = V^\top CV$$

Let the matrix K be positive definite for all $t \geq s$. Substituting $V(t) = [K(t)]^{1/2}$, one obtains from (18) and (19)

$$\dot{Q}_* = A_*Q_* + Q_*A_*^\top + h_*Q_* + h_*^{-1}I$$
$$Q_*(s) = K^{-1/2}(s)Q_0K^{-1/2}(s)$$
$$A_* = K^{-1/2}\left(AK^{1/2} - dK^{1/2}\big/dt\right) \tag{22}$$
$$C_* = K^{1/2}CK^{1/2}, \; h_* = \left[\frac{\mathrm{Tr}\,C_*}{\mathrm{Tr}(C_*Q_*)}\right]^{1/2}$$

Note that each of equations (21) and (22) for Q_* depends only on two matrices: K_* and C_*, or A_* and C_*, for equations (21) and (22), respectively. Thus, without loss of generality, one can always put either $A = 0$ or $K = I$ (in case of a positive definite matrix K) in equations (18). For the criterion (10), the formula for h in (18) becomes

$$h = [(Kv, v)/\,(Qv, v)]^{1/2} \tag{23}$$

Note that all formulas for the criterion (10) are invariant with respect to the change $v \to \lambda v$ where λ is a scalar. Therefore, one can always assume that v is a unit vector: $|v| = 1$.

5. EXAMPLE: EXACT SOLUTION

Consider a system of the second order

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = u$$
$$|u| \leq 1, \quad x_1(0) = x_2(0) = 0 \tag{24}$$

for which an exact solution of nonlinear equations (18) can be obtained and compared with the reachable set. The optimality criterion (10) is chosen with $v_1 = 0$, $v_2 = 1$ so that the rate of the projection of the outer approximating ellipse onto the axis x_2 is minimized. Equations (18) and (23) for this example give

$$\dot{Q}_{11} = 2Q_{12} + hQ_{11}, \quad \dot{Q}_{12} = Q_{22} + hQ_{12}$$
$$\dot{Q}_{22} = hQ_{22} + h^{-1}, \quad h = Q_{22}^{-1/2} \qquad (25)$$
$$Q_{11}(0) = Q_{12}(0) = Q_{22}(0) = 0$$

The nonlinear initial value problem (25) has the following exact solution

$$Q_{11} = \frac{1}{3}t^4, \; Q_{12} = \frac{1}{2}t^3, \; Q_{22} = t^2 \qquad (26)$$

Equations (16) for the center of the approximating ellipse give for the example (24): $a_1(t) = a_2(t) \equiv 0$. Thus, the center of the ellipse stays at the origin of coordinates, and its matrix is defined by (26). This ellipse E_1 is shown in Fig. 1. For comparison, the exact reachable set D and the approximating ellipse E_2 locally optimal in the sense of volume are also depicted in Fig. 1 (see Chernousko 1994). These sets are drawn in the normalized variables $x_1 t^{-2}$ and $x_2 t^{-1}$; in these variables the sets remain constant. The areas V_D of the reachable set, V_1 and V_2 of the corresponding ellipses E_1 and E_2 are

$$V_D = 2/3 \approx 0.667$$
$$V_1 = \pi(2 \cdot 3^{1/2})^{-1} \approx 0.907 \qquad (27)$$
$$V_2 = 8\pi(9 \cdot 5^{1/2})^{-1} \approx 1.25$$

It is obvious from Fig. 1 and formulas (27) that the ellipse E_1 gives much better approximation of the reachable set D than the ellipse E_2, even in the sense of volume. This example shows that the ellipsoids optimal in the sense of the criterion (10) may give a rather efficient outer approximation of reachable sets.

6. ASYMPTOTIC BEHAVIOR OF ELLIPSOIDS NEAR THE INITIAL POINT

Consider an important special case, where the initial set M in (4) degenerates into a point (as in (24)). In this case $x(s) = a_0$, $Q_0 = 0$ in (4), and the right-hand side of the matrix equation (18) has a singularity at $t = s$. Thus, the straightforward numerical integration near $t = s$ in this case is impossible.

The analysis of the arising singularity will be carried out for the case where the matrix K in (18) is positive definite. Therefore, equation (18) can be replaced by equation (22). Consider the latter equation under zero initial conditions

$$\dot{Q}_* = A_* Q_* + Q_* A_*^\top + h_* Q_* + h_*^{-1} I,$$
$$h_* = [\mathrm{Tr}\, C_* / \mathrm{Tr}(C_* Q_*)]^{1/2}, \; Q_*(s) = 0 \qquad (28)$$

The matrices $A_*(t)$ and $C_*(t)$ are supposed to be smooth functions of time, so that the following expansions are true

$$A_*(t) = A_0 + \theta A_1 + O(\theta^2), \; \theta = t - s \geq 0$$
$$C_*(t) = C_0 + \theta C_1 + O(\theta^2) \qquad (29)$$

Here, A_0, A_1, C_0, and C_1 are constant matrices. The solution of the initial value problem (28) is sought as a power series satisfying the initial condition $Q_*(s) = 0$ in (28)

$$Q_*(t) = \theta Q_1 + \theta^2 Q_2 + \theta^3 Q_3 + \theta^4 Q_4 + O(\theta^5) \qquad (30)$$

Here, $Q_1, Q_2, \ldots$ are constant symmetric matrices as yet unknown. By substituting expansions (29) and (30) into equation (28) and equating coefficients of the obtained expansions in the both sides of the obtained equations, one can find the unknown coefficients in expansion (30). After rather lengthy but straightforward calculations, one obtains

$$Q_1 = 0, \; Q_2 = I, \; Q_3 = D_0$$
$$Q_4 = \frac{2}{3}D_0^2 + \frac{2}{3}D_1 + \frac{[\mathrm{Tr}(C_0 D_0)]^2}{12\,(\mathrm{Tr}\, C_0)^2} I - \frac{\mathrm{Tr}(C_0 D_0)}{6\,\mathrm{Tr}\, C_0} D_0 \qquad (31)$$

where the following denotations are used

$$D_0 = \left(A_0 + A_0^\top\right)/2, \quad D_1 = \left(A_1 + A_1^\top\right)/2$$

Note that the matrix C_1 from (29) does not appear in the coefficients (31).

Consider two particular cases.

1) Let $C_0 = I$; this equality holds in the case (8). Then

$$Q_4 = \frac{2}{3}D_0^2 + \frac{2}{3}D_1 + \frac{(\mathrm{Tr}\, D_0)^2}{12n^2} I - \frac{\mathrm{Tr}\, D_0}{6n} D_0 \qquad (32)$$

Here, the expansions (31), (32) coincide with those given in Chernousko (1994) for the ellipsoids optimal in the sense of volume. Thus, the approximating ellipsoids optimal in the sense of the sum of squared semiaxes coincide with ellipsoids locally optimal in the sense of volume, up to the terms of order of $O(\theta^5)$.

2) For the case (10) we have $C_0 = v * v$. Here we obtain from (31)

$$Q_4 = \frac{2}{3}D_0^2 + \frac{2}{3}D_1 + \frac{(D_0 v, v)^2}{12} I - \frac{(D_0 v, v)}{6} D_0$$

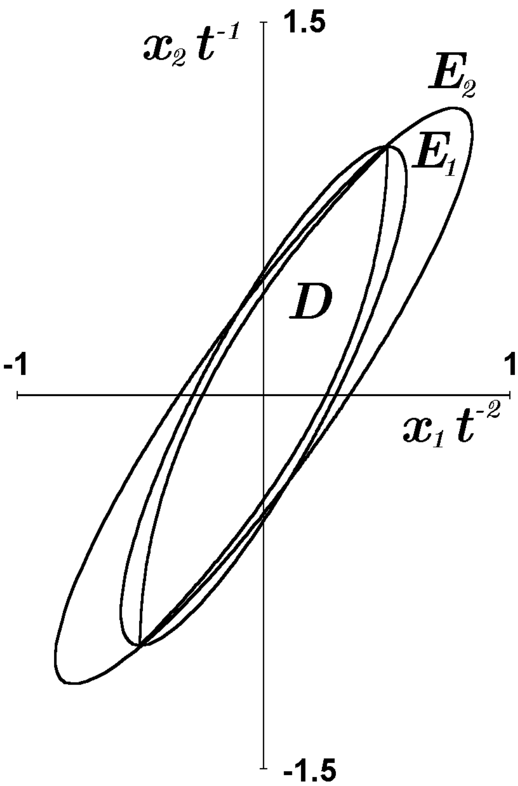

Fig. 1. Reachable set approximations by ellipsoids

The obtained expansions can be used for starting the numerical integration of equations (18) near the initial point $t = s$ in case of $Q(s) = 0$.

7. ASYMPTOTIC BEHAVIOR OF ELLIPSOIDS AT INFINITY

Here, the asymptotic behavior of approximating ellipsoids locally optimal in the sense of criterion (8) is studied at infinity, i.e. as $t \to \infty$. Suppose for simplicity that the matrix K in equations (18) is positive definite, so that equations of ellipsoids can be taken in the form (22) which corresponds to the case $K = I$. Let the matrix C_* in (22) have the form (11), where v is a constant unit vector, $|v| = 1$, and the matrix A_* be constant and diagonal:

$$A_* = \mathrm{diag}(\alpha_1, \ldots, \alpha_n),\ \alpha_1 \le \alpha_2 \le \ldots \le \alpha_n \quad (33)$$

Then the solution of the matrix equation (22) is a diagonal matrix, its diagonal elements $y_i(t)$ being positive and equal to the squares of semiaxes of the approximating ellipsoid:

$$Q_*(t) = \mathrm{diag}(y_1(t), \ldots, y_n(t)) \quad (34)$$

Equation (22) under the conditions (33) and (34) takes the form

$$\dot{y}_i = 2\alpha_i y_i + h_* y_i + h_*^{-1}, \quad i = 1, \ldots, n \quad (35)$$

Here, h_* can be presented in the form (23) with $K = I$, i.e.

$$h_* = \left(\sum_{i=1}^{n} v_i^2 y_i \right)^{-1/2} \quad (36)$$

Suppose at least one of α_i is nonnegative: $\alpha_n \ge 0$. Then, since $y_n(t) \ge 0$, the right-hand side of the nth equation (35) is positive for all t, and $y_n(t)$ grows monotonically with t. The supposition that there exists a bounded limit $y_n(t) \to y_n^0 > 0$ as $t \to \infty$ leads to the contradiction with the nth equation (35): its left-hand side tends to zero, whereas the right-hand side is nonzero as $t \to \infty$. Therefore, $y_n(t) \to \infty$ as $t \to \infty$. Then, according to (36), $h_* \to 0$ and $h_*^{-1} \to \infty$ as $t \to \infty$. Thus, the right-hand sides of all equations (35) tend to infinity as $t \to \infty$, and all $y_i \to \infty$ as $t \to \infty$.

Consider now the case where all α_i are negative and denote

$$\alpha_i = -\beta_i, \quad \beta_1 \ge \beta_2 \ge \ldots \ge \beta_n > 0 \quad (37)$$

Equations (35) can be rewritten as follows

$$\dot{y}_i = -2\beta_i y_i + h_* y_i + h_*^{-1}, \quad i = 1, \ldots, n \quad (38)$$

Setting the right-hand sides of the system (38) equal to zero, one can find the stationary points of this system:

$$y_i^0 = \frac{1}{h_0(2\beta_i - h_0)},\ h_0 = \left(\sum_{i=1}^{n} v_i^2 y_i^0 \right)^{-1/2} \quad (39)$$

Substituting y_i^0 into the formula (39) for h_0, one obtains the equation for h_0:

$$\frac{1}{h_0} = \sum_{i=1}^{n} \frac{v_i^2}{2\beta_i - h_0} \quad (40)$$

Since $y_i^0 \ge 0$ and $\beta_n \le \beta_i$ for all $i = 1, \ldots, n$, see (37), only those values of h_0 are to be taken into account for which $0 \le h_0 < 2\beta_n$. When h_0 changes from 0 to $2\beta_n$, the left-hand side of equation (40) decreases monotonically from ∞ to $(2\beta_n)^{-1}$, whereas the right-hand side of this equation increases from some positive value to ∞ (note that $|v| = 1$). Therefore, there exists a unique positive root $h_0 \in (0, 2\beta_n)$ of equation (40). Substituting this root into equations (39), one obtains the unique stationary point y_i^0, $i = 1, \ldots, n$ of system (38).

Numerical investigation of system (38) shows that, in a wide range of parameter variation, the stationary point is asymptotically stable and attracts all solutions of system (38) in the domain $y_i \ge 0$, $i = 1, \ldots, n$.

8. CONTROL IN THE PRESENCE OF PERTURBATIONS

Consider a system subjected to the control w and perturbation u

$$\dot{x} = A(t)x + B(t)u + W(t)w + f(t), \ t \geq s \quad (41)$$

Here, $w(t)$ is a k-vector of control, $W(t)$ is a given $n \times k$ matrix, the other denotations are the same as in (1). Suppose the perturbation u is caused by the imperfections of the control implementation, and the possible magnitude of u grows with the magnitude of the control w. More exactly, we assume that the matrices B in (41) and G in (3) depend on w in such a way that the matrix K from (17) is presented as follows

$$K = BGB^{\top} = |w|^4 R(t) \quad (42)$$

where $R(t)$ is a given positive definite $n \times n$ matrix. Using the transformation (19), (20) and the criterion (10), we write equations (16), (17) in the form (21). Taking into account equations (42) and (23), we obtain

$$\begin{aligned} \dot{a} &= Ww + f \\ \dot{Q} &= |w|^2 \left[(r/q)^{1/2} Q + (q/r)^{1/2} R \right] \end{aligned} \quad (43)$$

Here, subscripts $_*$ are omitted, and the following denotations are used

$$q = (Qv, v), \quad r = (Rv, v) \quad (44)$$

Let us find the control $w(t)$ which minimizes the functional

$$J^0 = [q(T)]^{1/2} + (a(T), v) + b \int_s^T |w|^2 \, dt \quad (45)$$

for system (43). This functional includes the support function for the approximating ellipsoid (see (13)) and the quadratic integral control cost. Here, T is a fixed terminal time instant, and b is a positive constant. Thus, we seek for the control $w(t)$ which is, in a certain sense, optimal for the whole ensemble of possible trajectories of system (41).

According to (43) and (44), we have

$$\dot{q} = 2|w|^2 (qr)^{1/2} \quad (46)$$

Instead of matrix Q, it is sufficient to consider a scalar variable q. Thus, our optimal control problem is considerably simplified. We introduce the vector and scalar adjoint variables ψ and φ corresponding to a and q, respectively, and write down the Hamiltonian

$$H = (\psi, Ww + f) + 2|w|^2 (qr)^{1/2} \varphi - b|w|^2 \quad (47)$$

According to the maximum principle

$$w = W^{\top} \psi \Big/ \left[2b - 4(qr)^{1/2} \varphi \right] \quad (48)$$

The adjoint equations and transversality conditions for our problem are

$$\begin{aligned} &\dot{\psi} = 0, \quad \dot{\varphi} = -|w|^2 (r/q)^{1/2} \varphi \\ &\psi(T) = -v, \quad \varphi(T) = -[q(T)]^{-1/2} \Big/ 2 \end{aligned} \quad (49)$$

It follows from equations (49) and (46) that ψ and $q\varphi^2$ are constant. Taking into account the boundary conditions (49), we obtain

$$\psi(t) = -v, \quad \varphi(t) = -q^{-1/2}/2 \quad (50)$$

Inserting formulas (50) into (48), we obtain

$$w(t) = -W^{\top}(t) v \Big/ \left[2b + 2(R(t)v, v)^{1/2} \right]$$

Thus, for the specific case considered above, we obtain the control in an explicit form. Substituting it into equations (43), one can easily find state variables.

9. CONCLUSION

New criterion for optimal ellipsoids approximating reachable sets which has certain advantages is proposed. It is equal to the projection of the ellipsoid onto the given direction. The properties of the approximating ellipsoids are investigated. A specific optimal control problem for the ensemble of trajectories is considered.

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