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# SLIDING BIFURCATIONS IN RELAY CONTROL SYSTEMS: AN APPLICATION TO NATURAL RESOURCES MANAGEMENT

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Abstract: This paper is devoted to the study of the long term dynamic behavior of exploited natural resources. The focus is on the case of protected resources, namely resources that can not be harvested when they are too scarce. The model is a controlled system composed of a nonlinear second order SISO system and an on-off feedback controller. The analysis is performed through the numerical continuation of the sliding bifurcations of the system. The results show that for suitable combinations of the parameters the system can have multiple attractors. *Copyright © 2002 IFAC*

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## 1. INTRODUCTION

This paper is devoted to the study of the dynamics of exploited natural resources. The most traditional approach in this sector is to introduce notions like harvesting effort (the control variable), cost of effort, benefits of harvest, risk of extinction and then formulate and solve an optimal control problem (Clark, 1976). However, the optimal solution can hardly be implemented in practice for organizational and institutional difficulties. Thus, the real management of natural resources is much more often performed by fixing quotas, costs of licenses, subsidies, lengths of fishing/hunting seasons, or similar variables. In other words, the structure of the controller is fixed and only a few parameters are selected by the control Agency.

Very often, in order to avoid high risks of extinction of the exploited population, harvesting is forbidden when the population density drops below a prescribed threshold, in the following denoted by $\alpha$. Such a rule is supported by conservation ecologists, who a priori associate an infinite cost to the loss of any population and is economically justified whenever the maximization of the benefits is performed using sufficiently low discount factors. The threshold introduces a discontinuity, so that the controlled system is, in the end, a discontinuous piecewise smooth system (also called *Filippov system*) in which sliding motions (Utkin, 1977; Filippov, 1988) are possible on the manifold separating the region where harvesting is allowed from that where it is forbidden. In the particular case examined in this paper, the controlled system is simply a second order SISO nonlinear system with an on-off feedback controller specified by two parameters: the threshold $\alpha$ and the harvesting effort $E$. Thus, when sliding, the density of the exploited population remains practically constant at the threshold value $\alpha$ and the harvesting effort switches at high frequency between 0 and $E$.

The aim of the paper is to determine all possible asymptotic modes of behavior of the system for all possible combinations of the two control parameters $\alpha$ and $E$. Technically, this is accomplished by performing the bifurcation analysis of the system with respect to $\alpha$ and $E$. Of course, some (actually many) bifurcations, called *sliding bifurcations*, critically involve some structural change in the sliding motion of the system. Since the sliding bifurcations are many, the analysis should be performed through their system-

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atic detection and numerical continuation. However, this is not possible because the complete catalogue of sliding bifurcations is not yet available even in the case of planar systems where only local bifurcations have been considered until now (Bautin and Leontovich, 1976; Filippov, 1988). Actually, the existing contributions on global sliding bifurcations refer either to specific bifurcations (Bernardo di *et al.*, 1998) or to particular classes of systems, like mechanical systems of the stick-slip type (Galvanetto *et al.*, 1995; Kunze and Küpper, 1997; Leine, 2000; Dankowitz and Nordmark, 2000) and linear systems with relay feedback controllers (Bernardo di *et al.*, 2001; Kowalczyk and di Bernardo, 2001). Without developing here the complete theory of sliding bifurcations of Filippov systems, we show through the example of resource exploitation how one can proceed in continuing these bifurcations through the use of their associated *defining equations* (Kuznetsov, 1998).

The paper is organized as follows. First, we describe the model and interpret it as a relay control system and then we study its bifurcations. For this, we start with the case in which the population is not protected (i.e. $\alpha = 0$): this is simple (and already known) because the system is continuous for $\alpha = 0$. Then, we consider the case in which the system is protected ($\alpha \neq 0$) and, interpreting bifurcations as collisions of invariant sets (including sliding segments), we determine all bifurcations with respect to $\alpha$ for a given value of $E$. Finally, after identifying their defining equations, we continue these bifurcations with respect to $\alpha$ and $E$, thus producing the entire bifurcation diagram. A short discussion of our findings and of some possible extensions close the paper.

## 2. THE MODEL

We consider a community composed of two populations, namely prey and predator with densities $x_1$ and $x_2$, respectively, where the predator population is harvested only when abundant, i.e. when $x_2 > \alpha$. The dynamics of the two populations are described by the Rosenzweig-MacArthur prey-predator model which is the most frequently used model in theoretical as well as in applied ecology (Rosenzweig and MacArthur, 1963). In that model the prey population grows logistically in the absence of predator and each predator transforms the harvested prey into new bornes. More precisely, the model for $x_2 < \alpha$ is the following

$$\dot{x}_1 = r x_1 \left(1 - \frac{x_1}{K}\right) - \frac{a x_1}{b + x_1} x_2 \tag{1}$$

$$\dot{x}_2 = e \frac{a x_1}{b + x_1} x_2 - d x_2 \tag{2}$$

where $r$ and $K$ are natural per-capita growth rate and carrying capacity of the prey, $a$ is the maximum per-capita predation rate, $b$ is the half-saturation constant (i.e. the prey density at which the predation rate is half maximum), $e$ is the efficiency, namely a conversion factor specifying the number of predator new bornes for each unit of predation, and $d$ is per-capita natural death rate of the predator.

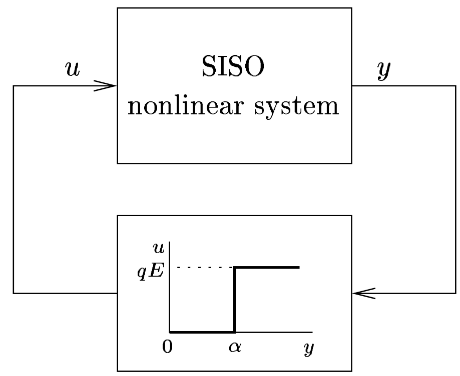


Fig. 1. The structure of the controlled system.

When the predator population is abundant ($x_2 > \alpha$) an extra mortality must be added to the second equation in order to take exploitation into account. Assuming that the resource is exploited at constant effort $E$, the second state equation for $x_2 > \alpha$ takes the form

$$\dot{x}_2 = e \frac{a x_1}{b + x_1} x_2 - d x_2 - qE x_2 \tag{3}$$

where $q$ is the catchability coefficient.

System (1-3) is composed of a nonlinear second order SISO system described by

$$\dot{x}_1 = r x_1 \left(1 - \frac{x_1}{K}\right) - \frac{a x_1}{h + x_1} x_2$$

$$\dot{x}_2 = e \frac{a x_1}{b + x_1} x_2 - d x_2 - u x_2$$

$$y = x_2$$

and of an algebraic relay system

$$u = \frac{qE}{2} \left[1 + \mathrm{sign}(y - \alpha)\right]$$

as shown in Fig. 1.

For notational convenience, we define the regions

$$S_1 = \{x : x_2 < \alpha\} \qquad S_2 = \{x : x_2 > \alpha\}$$

and we indicate by $\Sigma$ the manifold separating the two regions, i.e. $\Sigma = \{x : x_2 = \alpha\}$. Thus, the state space is the union of $S_1$, $S_2$, and $\Sigma$ and the model equations are

$$\dot{x} = f^{(i)}(x, \alpha, E) \quad x \in S_i, \quad i = 1, 2 \tag{4}$$

where $f^{(1)}$ and $f^{(2)}$ are specified by eqs. (1,2) and (1,3), respectively.

Since the first components of the vectors $f^{(i)}$ do not depend upon $\alpha$ and $E$, the nontrivial zero-isocline $\dot{x}_1 = 0$ is the same in both regions $S_i$ and is given by the parabola $x_2 = r\left(b + x_1\right)\left(1 - x_1/K\right)/a$. By contrast, the nontrivial zero-isoclines $\dot{x}_2 = 0$ are different in the two regions $S_1$ and $S_2$. More precisely, they are vertical straight lines given by $x_1 = bd/(ea - d)$ for $x \in S_1$, and $x_1 = b(d + qE)/(ea - d - qE)$ for $x \in S_2$,

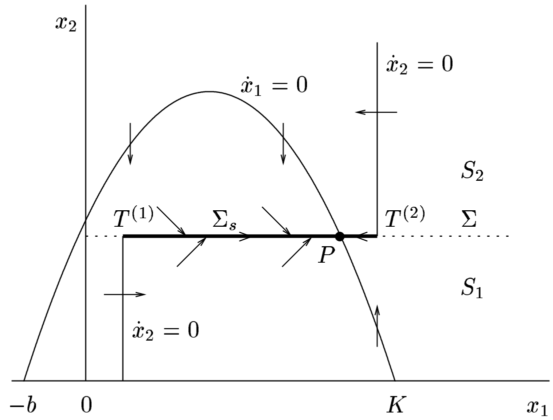


Fig. 2. Regions $S_1$ and $S_2$ separated by the manifold $\Sigma$ and the isoclines of system (4). Point $P$ is a pseudo-equilibrium. Parameter values are: $r = 5$, $K = 1$, $a = 2$, $b = 0.2$, $e = 0.8$, $d = 0.6$, $q = 1$, $\alpha = 0.45$, $E = 0.72$.

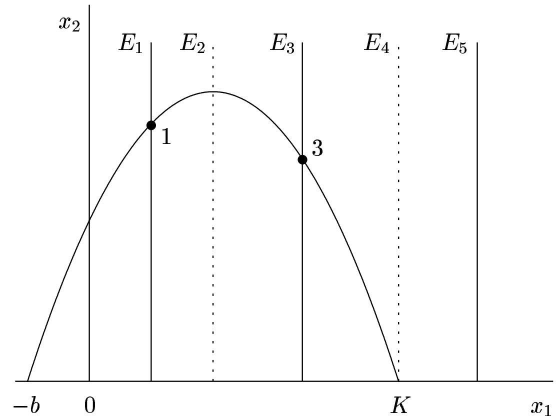


Fig. 3. Isoclines of system (4) in the case $\alpha = 0$ (unprotected predator population). The vertical isoclines $\dot{x}_2 = 0$ are drawn for increasing values of harvesting effort: $E_1 = 0.2$, $E_2 = 0.47$, $E_3 = 0.64$, $E_4 = 0.73$, $E_5 = 0.78$. Other parameters as in Fig. 2.

as shown in Fig. 2 for the parameter values specified in the caption.

Sliding occurs on the segment $\Sigma_s$ delimited by the two intersections $T^{(1)}$ and $T^{(2)}$ of $\Sigma$ with the two zero-isoclines $\dot{x}_2 = 0$ (see Fig. 2). As first pointed out by Filippov, these points, at which one of the two vectors $f^{(i)}$ is tangent to $\Sigma$, are strategically important for bifurcation analysis. Sliding motions on $\Sigma_s$ obey the smooth scalar differential equation

$$\dot{x} = g(x, \alpha, E) \quad x \in \Sigma_s \tag{5}$$

where $g$ is the unique convex combination of $f^{(1)}(x, \alpha, E)$ and $f^{(2)}(x, \alpha, E)$ parallel to $\Sigma_s$ (Filippov, 1988). Points $P$ for which $g(x, \alpha, E) = 0$, are called pseudo-equilibria (Gatto *et al.*, 1973) and correspond to a stationary sliding solution. Pseudo-equilibria are generically interior points of the sliding segment $\Sigma_s$ (as in Fig. 2), but for special combinations of the parameters they can also collide with the boundary points of the sliding segment.

## 3. THE CASE OF UNPROTECTED POPULATIONS ($\alpha = 0$)

If the harvested population is not protected when scarce, i.e. if $\alpha = 0$, the bifurcation analysis with respect to $E$ is easy, because the system is continuous. The results are summarized in Fig. 3, where the vertical isocline $\dot{x}_2 = 0$ is drawn for five different values $E_1 < E_2 < \ldots < E_5$ of the effort. The values $E_2$ and $E_4$ are the critical values corresponding to two different bifurcations. For $E < E_2$ the vertical isocline is on the left of the vertex of the parabola and the analysis of the Jacobian matrix evaluated at the positive equilibrium (point 1 of Fig. 3) reveals that such an equilibrium is unstable. For obvious topological reasons (Poincaré theory) such an equilibrium must be surrounded by at least one limit cycle. More detailed analyses show that the limit cycle is unique (Cheng, 1981) and that it shrinks for $E$ approaching $E_2$ and, finally, disappears through a supercritical Hopf bifurcation for $E = E_2$ (Kuznetsov, 1998). The same analysis, performed for $E_2 < E < E_4$, shows that the positive equilibrium (point 3 of Fig. 3) is globally stable in the first quadrant. Finally, for $E = E_4$ the equilibrium 3 collides with the trivial equilibrium $(K, 0)$ (transcritical bifurcation), and for $E > E_4$ the global attractor is the trivial equilibrium $(K, 0)$ characterized by the absence of the predator population. The bifurcation values $E_2$ and $E_4$ can be easily computed and are given by $E_2 = [ea(K-b)/(K+b) - d]/q$ and $E_4 = [eaK/(K+b) - d]/q$.

Thus, in conclusion, if $\alpha = 0$ there are two bifurcations with respect to $E$: a Hopf bifurcation for $E = E_2$ and a transcritical bifurcation for $E = E_4$. Moreover, for any value of the effort, the system has a unique global attractor in the positive quadrant: a cycle for $E < E_2$, a positive equilibrium for $E_2 < E < E_4$ and a trivial equilibrium for $E > E_4$.

## 4. THE CASE OF PROTECTED POPULATIONS ($\alpha \neq 0$)

As already said, the analysis of the bifurcations with respect to $E$ and $\alpha$ is performed in two steps. First, we fix $E$ at a specified value and we look for bifurcations with respect to $\alpha$. In the present case we will identify four bifurcations, all involving structural changes of the sliding motion. Then, we determine the so-called defining equations (Kuznetsov, 1998) of these sliding bifurcations and we use them to produce numerically, through continuation techniques, the corresponding bifurcation curves in the space $(\alpha, E)$. This allows one to detect all qualitatively different modes of behavior of the system. Each mode of behavior is described by a state portrait, which is characterized by a set of specific equilibria, cycles, pseudo-equilibria, sliding

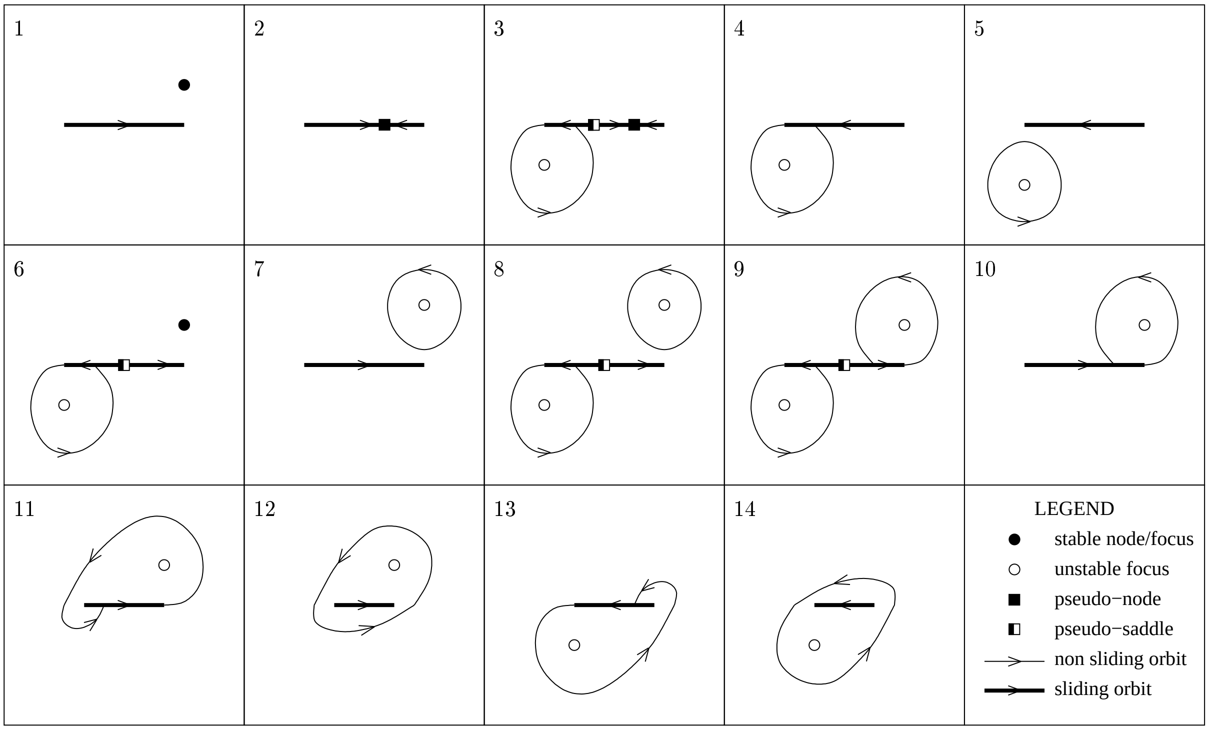


Fig. 4. The basic elements (see legend) of all the state portraits identified through bifurcation analysis.

cycles, and sliding segments. All these sets of the state portraits are collected in Fig. 4. The first five of them are identified in the first step of the analysis while the remaining ones are obtained in the second step.

### 4.1 *First step: bifurcation analysis with respect to $\alpha$ for a given $E$*

In this phase the effort $E$ is frozen at a particular value $E^*$ (equal to 0.8 in our case) and $\alpha$ is varied in the range of interest (see dotted line in Fig. 5). For small $\alpha$ the state portrait is characterized by a positive equilibrium in $S_2$ and by the sliding segment $\Sigma_s$, as shown in Fig. 4.1. Then, $\alpha$ is varied step by step and the system is simulated for increasing values of $\alpha$ in order to check if the state portrait remains topologically equivalent. If for two subsequent values of $\alpha$, say $\alpha'$ and $\alpha''$, the state portraits are not topologically equivalent, other simulations are performed in the interval $[\alpha', \alpha'']$ in order to detect the first bifurcation $\alpha_1$ with satisfactory approximation. Obviously, $\alpha' < \alpha_1 < \alpha''$. Then, the process is repeated starting from $\alpha''$, until the second bifurcation $\alpha_2$ is found, and so on. In the present case four bifurcations are detected for $\alpha_1 = 0.89$, $\alpha_2 = 1.08$, $\alpha_3 = 1.23$, $\alpha_4 = 2.18$. The state portraits corresponding to the open intervals $(0, \alpha_1)$, $(\alpha_1, \alpha_2)$,..., $(\alpha_4, \infty)$ are qualitatively sketched in Fig. 4.1-4.5.

The bifurcation $\alpha_1$ is called *boundary node* because the transition from Fig. 4.1 to Fig. 4.2 is characterized by the collision of a node in $S_2$ with the boundary point $T^{(2)}$ of the sliding segment $\Sigma_s$. Similarly, the second bifurcation $\alpha_2$ is called *boundary focus*, because the transition from Fig. 4.3 to Fig. 4.2 involves the collision of a focus in $S_1$ with the sliding segment $\Sigma_s$. Notice that the boundary focus bifurcation is characterized by the appearance/disappearance of three invariant sets: a pseudo-saddle, a focus and a sliding cycle, which collide for $\alpha = \alpha_2$. The third bifurcation $\alpha_3$ (transition from Fig. 4.3 to Fig. 4.4) is a *pseudo-saddle-node* bifurcation, namely the collision (and disappearance) of a pseudo-saddle with a pseudo-node. Finally, the fourth bifurcation $\alpha_4$, called *touching bifurcation*, occurs when the limit cycle located in $S_1$ (see Fig. 4.5) touches the sliding segment $\Sigma_s$ at $T^{(1)}$. Notice that the first three bifurcations are local, while the fourth one is global.

### 4.2 *Second step: continuation with respect to $E$ and $\alpha$*

With the aim of obtaining the entire bifurcation diagram, the four bifurcations $(\alpha_i, E^*)$, $i = 1, \ldots, 4$ can be numerically continued in the space $(\alpha, E)$. For this one must first determine the defining equations of each bifurcation, and then use these equations to find all pairs $(\alpha, E)$ giving rise to the same bifurcation. For example, the boundary node bifurcation (collision of an equilibrium with $\Sigma_s$ at $T^{(2)}$) is simply identified by the following equations

$$\begin{cases} f^{(2)}(x, \alpha, E) = 0 \\ x_2 = \alpha \end{cases} \qquad (6)$$

These are three equations in four unknowns $(\alpha, E, x_1, x_2)$. A particular solution $(\alpha = \alpha_1, E = E^*, x_1 = b(d + E^*)/(ea - d - E^*), x_2 = \alpha_1)$ has been obtained in the first step, when detecting the first bifurcation $(\alpha_1, E^*)$. Hence, this solution can be used as initial condition of a continuation algorithm (Doedel and Kernevez, 1986; Doedel *et al.*, 1997) that produces automatically the entire boundary node bifurcation curve.

One can proceed in a similar way for the second and third bifurcations which are local bifurcations involving collisions of equilibria, boundary points, pseudo-equilibria and shrinking sliding cycles. By contrast, the case of the touching bifurcation $(\alpha_4, E^*)$ is rather different because this bifurcation is a global one. The defining equations of the touching bifurcation say that the trajectory starting from point $T^{(1)}$ develops entirely in $S_1$ and comes back to $T^{(1)}$ in finite time $(\tau)$. Thus, the defining equations involve differential equations and take the form of the following two-boundary-value problem

$$\begin{cases} \dot{x} = f^{(1)}(x, \alpha, E) \\ x(\tau) = x(0) \\ x_2(0) = \alpha \\ x_1(0) = bd/(ea-d) \end{cases} \tag{7}$$

Since a solution to this two-boundary-value problem has been obtained when detecting the bifurcation $(\alpha_4, E^*)$, one can produce all touching bifurcations using standard continuation techniques (Doedel and Kernevez, 1986; Doedel *et al.*, 1997).

### 4.3 *Results*

The result of all these continuations (and the analysis of some codimension-2 bifurcations) is the bifurcation diagram shown in Fig. 5, containing ten different bifurcations (four local and six global) and fourteen regions 1,..., 14 (see Fig. 4 for the corresponding state portraits). This means that after detecting four bifurcations in the first step, six other bifurcations have been found when performing the second step. The corresponding bifurcation curves originate at codimension-2 bifurcation points in the space $(\alpha, E)$ where various bifurcation curves merge. The discovery of these strategically important codimension-2 points is absolutely not easy and is not described in this paper.

One important feature to remark is that for suitable values of the control parameters (see regions 3, 6, 8, and 9 in Fig. 5 and corresponding state portraits in Fig. 4) the system has two alternative attractors, which can even be two sliding cycles (see region 9). This was rather unexpected since the Rosenzweig-MacArthur model has a unique attractor when the exploited resource is not protected (see Sect. 3).

## 5. CONCLUDING REMARKS

We have studied in this paper the asymptotic behavior of a prey-predator community in which the predator population is harvested at constant effort $E$ whenever its density is above a specified threshold $\alpha$. This model fits pretty well the common practice of the management of some renewable resources (e.g. whales feeding on krill, ungulates feeding on plants, tuna feeding on sardines). The analysis has been carried out by performing a bifurcation analysis with respect to the two parameters $\alpha$ and $E$. One result (not even pointed out in the text) was expected: sufficiently high thresholds guarantee that the attractors are strictly positive (i.e. the exploited population survives). However, another result was unexpected: for suitable combinations of the control parameters there are multiple attractors. Thus, the conclusion is that one can, indeed, protect the exploited population from extinction but with the risk of generating a system with ambiguous long term behavior.

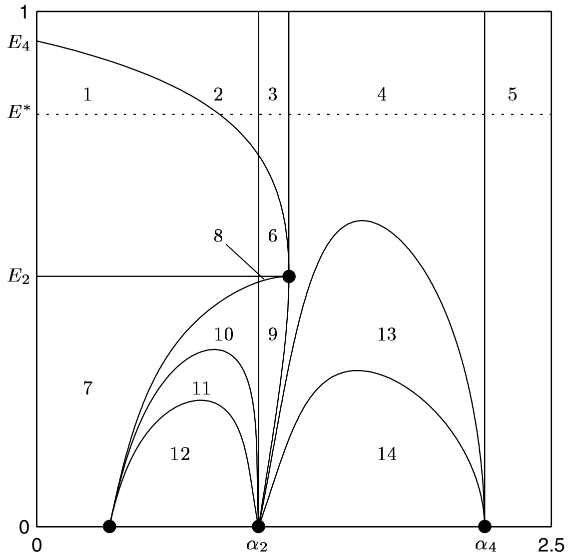


Fig. 5. Bifurcation diagram (see Fig. 4 for the corresponding state portraits). The marked points are codimension-2 bifurcation points.

There are other simple prototype models that fit the common practice of the management of other renewable resources. For example, sometimes exploitation is not restricted but support is given to the predator population whenever it drops below the threshold value $\alpha$ by stocking young predators into the system or by enriching the habitat of the prey. Both these cases could be studied through a sliding bifurcation analysis similar to that performed in this paper.

This study has proved that the number of bifurcations involved in these simple models is surprisingly high. Although we have shown how one can proceed to obtain the entire bifurcation diagram, it is obvious that the knowledge of the complete catalogue of codimension-one sliding bifurcations in planar Filippov systems would be helpful. Extending this catalogue to codimension-two bifurcations and/or to three dimensional Filippov systems would also be of great value.

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