WELL-POSEDNESS OF THE COMPLEMENTARITY CLASS OF HYBRID SYSTEMS

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Abstract: One of the most fundamental properties of any class of dynamical systems is the study of well-posedness, i.e. the existence and uniqueness of a particular type of solution trajectories given an initial state. In case of interaction between continuous dynamics and discrete transitions this issue becomes highly non-trivial. In this survey an overview is given of the well-posedness results for complementarity systems, which form a class of hybrid systems described by the interconnection of differential equations and a specific combination of inequalities and Boolean expressions as appearing in the linear complementarity problem of mathematical programming.

Keywords: Hybrid systems, solution concepts, well-posedness, Zeno behaviour, inequalities, complementarity problems and systems.

1. INTRODUCTION

In the companion paper (Çamlıbel *et al.*, 2002*a*) the importance of well-posedness, i.e. the existence and uniqueness of solution trajectories given an initial condition, has been highlighted for hybrid dynamical systems. In the current paper we will consider this problem for a subclass encompassing a broad range of interesting discontinuous dynamical systems: unilaterally constrained mechanical systems, switched electrical circuits, piecewise linear systems, optimal control problems with inequality constraints, relay and variable structure systems, and so on (Heemels *et al.*, 1999*a*). Typically these systems are characterized by the interconnection of a smooth dynamical system and a special combination of inequalities as appearing in the linear complementarity problem (Cottle *et al.*, 1992) of mathematical programming. The sys-

tems arising in this manner are called complementarity systems and can be written in terms of a state variable x and auxiliary vectors v and z of the same length:

$$\dot{x}(t) = f(x(t), v(t)) \tag{1a}$$
$$z(t) = h(x(t), v(t)) \tag{1b}$$
$$0 \le z(t) \perp v(t) \ge 0, \tag{1c}$$

where the last line means that the components of the auxiliary variables $v(t)$ and $z(t)$ should be nonnegative, and satisfy $z^\top(t)v(t) = 0$. Note that this implies that for each index i and for each time t at least one of the two variables $v_i(t)$ and $z_i(t)$ should be equal to 0.

The aim of the current paper is to present the state-of-the-art of well-posedness results for the complementarity class of hybrid dynamical systems.

2. SOLUTION CONCEPTS

Although an extensive discussion on solution concepts has been presented in the companion paper (Çamlıbel *et al.*, 2002a), we recall here the necessary aspects to be self-contained.

The system (1) consists of a number of different dynamical regimes or "modes" that are glued together. The modes correspond to a fixed choice, for each of the indices i, between the two possibilities $v_i \geq 0$, $z_i = 0$ and $v_i = 0$, $z_i \geq 0$, so that a complementarity system in which the vectors v and z have length m has 2^m different modes. The specification (1) is in general not complete yet; one has to add a rule that describes possible jumps of the state variable x when a transition from one mode to another takes place (think of mechanical systems with impacts). However, we will first introduce notions of solutions for the case in which jumps are absent.

For complementarity systems one may develop several solution concepts, which may be similar to the notion of an execution for hybrid automata (Johansson *et al.*, 1999; Lygeros *et al.*, 1999), or to the solution concept for differential inclusions as used for differential equations with discontinuous right-hand sides (Filippov, 1988). A solution concept of the first type can for instance be formulated as follows.

Definition 2.1. A set $\mathcal{E} \subset \mathbb{R}_+$ is called an *admissible event times set*, if it is closed and countable, and $0 \in \mathcal{E}$. To each admissible event times set $\mathcal{E}$, we associate a collection of intervals between events $\tau_{\mathcal{E}} = \{(t_1, t_2) \subset \mathbb{R}_+ \mid t_1, t_2 \in \mathcal{E} \cup \{\infty\}$ and $(t_1, t_2) \cap \mathcal{E} = \emptyset\}$.

Note that both left and right accumulations[1] of event times are allowed by the above definition.

Definition 2.2. A quadruple $(\mathcal{E}, v, x, z)$ where $\mathcal{E}$ is an admissible event times set, and $(v, x, z) : \mathbb{R}_+ \mapsto \mathbb{R}^{m+n+m}$ is said to be a *hybrid solution* of (1) with initial state x_0, if $x(0) = x_0$, x is continuous on $\mathbb{R}_+$ and the following conditions hold for each $\tau \in \tau_{\mathcal{E}}$:

(1) The triple $(v, x, z)|_\tau$ is real-analytic.
(2) For all $t \in \tau$, it holds that
$$\dot{x}(t) = f(x(t), v(t))$$
$$z(t) = h(x(t), v(t))$$
$$0 \leq v(t) \perp z(t) \geq 0$$

Without loss of generality , we assume that a hybrid solution $(\mathcal{E}, v, x, z)$ is *nonredundant*, i.e. there does not exist a $t \in \mathcal{E}$ and t', t'' with $t' < t < t''$ such that (v, x, z) is analytic on (t', t'').

[1] An element t of a set $\mathcal{E}$ is said to be a *left (right) accumulation point* if for all $t' > t$ ($t' < t$) $(t, t') \cap \mathcal{E}$ $((t', t) \cap \mathcal{E})$ is not empty.

Definition 2.3. A triple (v, x, z) of vector functions is said to be a *forward solution* of the system (1) on the interval $[a, b]$ if x is continuous and there exists a sequence of time points $(t_0, t_1, \dots)$ with $t_0 = a$, $t_{j+1} > t_j$ for all j, and either $t_N = b$ or $\lim_{j \to \infty} t_j = b$, as well as for each $j = 0, 1, \dots$ an index set I_j, such that for all j the restrictions of $x(\cdot)$, $v(\cdot)$, and $z(\cdot)$ to (t_j, t_{j+1}) are real-analytic, and for all $t \in (t_j, t_{j+1})$ the following holds:

$$\dot{x}(t) = f(x(t), v(t)), \ z(t) = h(x(t), v(t))$$
$$z_i(t) = 0 \text{ for } i \in I_j, \ v_i(t) = 0 \text{ for } i \notin I_j$$
$$z_i(t) \geq 0 \text{ for } i \notin I_j, \ v_i(t) \geq 0 \text{ for } i \in I_j.$$

Both definitions require that the state x of a solution trajectory is continuous across events. For so-called "high-index" systems (e.g. constrained mechanical systems), this requirement is too strong and one has to add jump rules that connect continuous states before and after an event has taken place. Under suitable conditions (specifically, in the case of linear complementarity systems and in the case of Hamiltonian complementarity systems), a general jump rule may be given; see (Heemels *et al.*, 2000; van der Schaft and Schumacher, 1998) and Section 5 below. Another possibly restrictive aspect of the forward solution concept lies in the fact that it assumes that the set of event times is well-ordered[2] by the usual order of the reals, but not necessarily by the reverse order; in other words, event times may accumulate to the right, but not to the left. Hence, a forward solution is a hybrid solution with a particular type of event times set $\mathcal{E}$.

Definition 2.4. An admissible event times set $\mathcal{E}$ is said to be *left (right) Zeno free* if it does not contain any left (right) accumulation points. A hybrid solution is said to be *left (right) Zeno free* if the corresponding event times set is left (right) Zeno free. It is said to be *left (right) Zeno* if it is not left (right) Zeno free, and *non-Zeno* if it is both left and right Zeno free.

A forward solution is a left Zeno free hybrid solution, but not vice versa as continuation beyond a right-accumulation is not possible in Def. 2.3 (although it might be extended).

An alternative concept that foregoes explicit mention of events is the following one, which turns out to be convenient for complementarity systems that satisfy a certain passivity condition.

Definition 2.5. A triple $(x, v, z) \in L_2^{n+2m}$ is said to be an *L_2-solution* of (1) on the interval $[0, T]$ with initial condition x_0 if for almost all $t \in [a, b]$ the following conditions hold:

[2] An ordered set S is said to be well-ordered if each nonempty subset of S has a least element.

$$x(t) = x_0 + \int_0^t f(x(s), v(s)) \, ds$$

$$z(t) = h(x(t), v(t))$$

$$0 \leq z(t) \perp v(t) \geq 0.$$

3. LINEAR COMPLEMENTARITY SYSTEMS

As the interconnection of a continuous, time-invariant, linear system and complementarity conditions, a *linear complementarity system* (LCS) can be given by

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad (2a)$$
$$y(t) = Cx(t) + Du(t) \qquad (2b)$$
$$0 \leq u(t) \perp y(t) \geq 0, \qquad (2c)$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^m$, and A, B, C and D are matrices with appropriate sizes. We denote (2a)-(2b) by $\Sigma(A, B, C, D)$ and (2) by $\text{LCS}(A, B, C, D)$.

One may look at LCS as a dynamical extension of the linear complementarity problem.

Problem 3.1. LCP(q, M): Given an m-vector q and $m \times m$ matrix M find an m-vector z such that

$$0 \leq q + Mz \perp z \geq 0. \qquad (3)$$

We say z *solves* (or is a *solution* of) LCP(q, M), if z satisfies (3). The set of solutions of LCP(q, M) is denoted by SOL(q, M).

Definition 3.2. A matrix $M \in \mathbb{R}^{m \times m}$ is called

- a *P-matrix* if all its principal minors $\det M_{II}$ for $I \subseteq \{1, \ldots, m\}$ are positive.
- *positive (nonnegative) definite* [3] if $x^T M x > 0$ (≥ 0) for all $0 \neq x \in \mathbb{R}^m$.

Note that every positive definite matrix is a *P*-matrix, but the converse is not true. However, every symmetric P-matrix is also positive definite.

Definition 3.3. The dual cone of a given nonempty set $\mathcal{S} \subset \mathbb{R}^m$, denoted by $\mathcal{S}^*$, is given by $\{v \in \mathbb{R}^m \mid v^T w \geq 0 \text{ for all } w \in \mathcal{S}\}$.

The final ingredient of our preparation is the "*index*" of a rational matrix.

Definition 3.4. A rational matrix $H(s) \in \mathbb{R}^{l \times l}(s)$ is said to be *of index k*, if it is invertible as a rational matrix and $s^{-k} H^{-1}(s)$ is proper. It is said to be *totally of index k*, if all its principal submatrices are of index k.

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[3] Note that the matrix is not assumed to be symmetric.

With a slight abuse of terminology, we say that a linear system $\Sigma(A, B, C, D)$ is (totally) of index k, if its transfer matrix $G(s) := C(sI - A)^{-1}B + D$ is (totally) of index k.

3.1 *Linear complementarity systems with index 1*

The following theorem provides sufficient conditions for well-posedness in the sense of existence and uniqueness of left Zeno free hybrid solutions to LCS with index 1.

Theorem 3.5. (Çamlıbel, 2001) *Consider a* $LCS(A, B, C, D)$ *with* $\Sigma(A, B, C, D)$ *totally of index 1. Suppose that* $D + C(\sigma I - A)^{-1}B$ *is a P-matrix for all sufficiently large* $\sigma \in \mathbb{R}$. *There exists a left Zeno free hybrid solution of* $LCS(A, B, C, D)$ *with the initial state* x_0 *if and only if* $LCP(Cx_0, D)$ *is solvable. Moreover, if such a solution exists it is left Zeno free unique, i.e. there is no other left Zeno free solution.*

3.2 *Linear passive complementarity systems*

When the underlying system $\Sigma(A, B, C, D)$ is passive (in the sense of (Willems, 1972)) we call the overall system (2) a *linear passive complementarity system* (LPCS). As shown in (Çamlıbel, 2001, Lemma 3.8.5), the passivity of the system (under some extra assumptions) implies that it is of index 1. Hence, Theorem 3.5 is applicable to LPCS. Additionally, it can be shown that there are no left Zeno solutions for LPCS as formulated in the following theorem (hence, a particular type of Zeno behaviour is excluded).

Theorem 3.6. (Çamlıbel, 2001) *Consider a* $LCS(A, B, C, D)$ *with* $\Sigma(A, B, C, D)$ *being passive,* (A, B, C) *being minimal and* $\text{col}(B, D + D^T) := \begin{pmatrix} B \\ D + D^T \end{pmatrix}$ *of full column rank. Let* $\mathcal{Q}_D = \{z \mid z$ *solves* $LCP(0, D)\}$. *There exists a hybrid solution of* $LCS(A, B, C, D)$ *with the initial state* x_0 *if and only if* $Cx_0 \in \mathcal{Q}_D^*$. *Moreover, if a solution exists it is unique* [4] *and left Zeno free.*

An important observation is the following. If $(\mathcal{E}, u, x, y)$ is a solution of LCS(A, B, C, D) then $(\mathcal{E}, t \mapsto e^{\rho t} u(t), t \mapsto e^{\rho t} x(t), t \mapsto e^{\rho t} y(t))$ is a solution of LCS$(A + \rho I, B, C, D)$. This correspondence makes it possible to apply the above theorem to a class of nonpassive systems. Indeed, even if $\Sigma(A, B, C, D)$ is not passive $\Sigma(A + \rho I, B, C, D)$ may be passive for some ρ. In this case, we say that $\Sigma(A, B, C, D)$ is *passifiable by pole shifting* (PPS). By using the necessary and sufficient conditions for PPS property in (Çamlıbel, 2001, Thm. 3.4.3), we can state the following extension of Theorem 3.6.

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[4] It can also be shown that this solution is unique in L_2.

Theorem 3.7. (Çamlıbel, 2001) *Consider a $LCS(A, B, C, D)$ with $\mathrm{col}(B, D + D^T)$ full column rank and (A, B, C) minimal. Let E be such that $\ker E = \{0\}$ and $\mathrm{im}\, E = \ker (D + D^T)$. Suppose that D is nonnegative definite and $E^T CBE$ is symmetric positive definite. There exists a hybrid solution of $LCS(A, B, C, D)$ with the initial state x_0 if and only if $Cx_0 \in \mathcal{Q}_D^*$. Moreover, if a solution exists it is unique[4] and left Zeno free.*

3.3 Piecewise linear systems

As is well-known (see e.g. (Eaves and Lemke, 1981)), piecewise linear relations may be described in terms of linear complementarity problems. An immediate consequence is that several piecewise linear systems can be recast as linear complementarity systems. In this paper, we will focus, for the sake of simplicity, on a specific type of piecewise linear systems, namely linear saturation systems, which are of the form

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{4a}$$
$$y(t) = Cx(t) + Du(t) \tag{4b}$$
$$(u(t), y(t)) \in \text{saturation}_i, \tag{4c}$$

where saturation$_i$ is a charateristic of the form depicted in Figure 1 with $e_2^i - e_1^i > 0$ and $f_1^i \geq f_2^i$. We denote the overall system (4) by $SAT(A, B, C, D)$. Note that relay characteristics can be obtained from saturation characteristics by setting $f_1^i = f_2^i$. We adopt the solution concept defined for LCS to saturation systems as follows.

Definition 3.8. A quadruple $(\mathcal{E}, u, x, y)$ where $\mathcal{E}$ is an admissible event times set, and (u, x, y) : $\mathbb{R}_+ \mapsto \mathbb{R}^{m+n+m}$ is said to be a *hybrid solution* of $SAT(A, B, C, D)$ with the initial state x_0 if $x(0) = x_0$ and the following conditions hold for each $\tau \in \tau_{\mathcal{E}}$:

(1) The triple $(u, x, y)|_\tau$ is analytic.
(2) For all $t \in \tau$ and all $i \in \{1, \ldots, m\}$, it holds that

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$
$$(u_i(t), y_i(t)) \in \text{saturation}_i$$

One may argue that the saturation characteristic is a Lipschitz continuous function (provided that $f_1^i - f_2^i > 0$) and hence existence and uniqueness of solutions follow from the theory of ordinary differential equations. The following example shows that this is not correct in general if the feedthrough term D is nonzero.

Example 3.9. Consider the SISO system
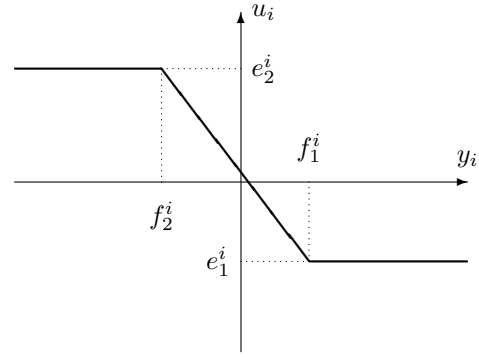
$$\dot{x} = u \tag{5}$$
$$y = x - 2u, \tag{6}$$

Fig. 1. Saturation characteristic

where u and y restricted by a saturation characteristic with $e_1 = -f_1 = -e_2 = f_2 = \frac{1}{2}$. Let the periodic function $\tilde{u} : \mathbb{R}_+ \to \mathbb{R}$ be defined by

$$\tilde{u}(t) = \begin{cases} 1/2 & \text{if } 0 \leq t < 1 \\ -1/2 & \text{if } 1 \leq t < 3 \\ 1/2 & \text{if } 3 \leq t < 4 \end{cases}$$

and $\tilde{u}(t - 4) = \tilde{u}(t)$ whenever $t \geq 4$. By using this function define $\tilde{x}(t) = \int_0^t \tilde{u}(s)\, ds$, and $\tilde{y} = \tilde{x} - 2\tilde{u}$. It can be verified that $(-\tilde{u}, -\tilde{x}, -\tilde{y})$, $(0, 0, 0)$ and $(\tilde{u}, \tilde{x}, \tilde{y})$ are all solutions of $SAT(0, 1, 1, -2)$ with the zero initial state.

As illustrated in the example, the Lipschitz continuity argument does not work in general for the case $f_1^i > f_2^i$. Also if $f_1^i = f_2^i$ (the relay case) this reasoning does not apply.

Theorem 3.10. (Çamlıbel, 2001) *Consider $SAT(A, B, C, D)$. Let R and S be the diagonal matrices with $e_2^i - e_1^i$ and $f_2^i - f_1^i$, respectively, on the diagonal. Suppose that $G(\sigma)R - S$ is a P-matrix for all sufficiently large σ. Then, there exists a unique left Zeno free hybrid solution of $SAT(A, B, C, D)$ for all initial states.*

4. NONLINEAR COMPLEMENTARITY SYSTEMS

The previous sections are concerned with *linear* complementarity systems. Results for (1) without a linearity assumption on (1b)-(1c) are limited. However, for *forward* solutions an extension can be presented of Theorem 3.2 in (van der Schaft and Schumacher, 1998) for the following systems

$$\dot{x}(t) = f(x(t)) + g(x(t))v(t) \tag{7a}$$
$$z(t) = h(x(t)) \tag{7b}$$

with complementarity conditions on v and z.

For $x_0 \in \mathbb{R}^n$ we define the i-th leading row coefficient $\rho_i(x_0)$ as

$$\rho_i(x_0) := \inf\{j \in \mathbb{N} \setminus \{0\} \mid L_g L_f^{j-1} h_i(x_0) \neq 0\} \tag{8}$$

and the index set $J(x_0)$ as

$$J(x_0) := \{j \in \bar{k} \mid (h_j(x_0), \ldots, L_f^{\rho_j(x_0)-1} h_j(x_0)) = 0\},$$
(9)

where L denotes the "Lie-derivative" (see, for instance, (Nijmeijer and van der Schaft, 1990)) and $\bar{k}$ denotes the set $\{1, \ldots, k\}$.

Theorem 4.1. Consider the complementarity system (7) with f, g and h real-analytic. Consider $x_0 \in \mathbb{R}^n$ such that the matrix

$$(L_{g_{\bullet j}} L_f^{\rho_i(x_0)-1} h_i(x_0))_{i,j \in J(x_0)}$$
(10)

has only positive principal minors. There exists an $\varepsilon > 0$ such that a unique forward solution exists on $[0, \varepsilon)$ if and only if $(h_i(x_0), \ldots, L_f^{\rho_i(x_0)-1} h_i(x_0))$ is lexicographically nonnegative[5] for all $i \in \bar{k}$.

Note that the above result only deals with *smooth* continuations and does not incorporate the possibility of re-initializations.

5. GENERALIZATIONS INCLUDING JUMPS

Up to this point, we have presented well-posedness results for complementarity systems in which the x-part of the solutions is continuous. In this subsection, the available generalizations will be mentioned including the possibility of re-initializations (state jumps). In such studies the issue of irregular initial states had to be tackled, i.e., the initial states for which there is no solution in the senses defined so far for complementarity systems (e.g. in case of the systems and solution concept considered in Theorem 3.5 all initial states x_0 for which $\text{LCP}(Cx_0, D)$ is not solvable). A distributional framework was used to obtain a new solution concept for LCS (Heemels *et al.*, 2000). In principle, this framework is based on so-called *Bohl distributions* of the form $u(t) = \sum_{i=0}^{l} u^{-i} \delta^{(i)} + u_{reg}(t)$, where δ is the delta or Dirac distribution (supported at 0), $\delta^{(i)}$ is the i-th derivative of δ and u_{reg} is a Bohl function. These distributions can equivalently be characterized by the inverse Laplace transforms of rational functions. A Bohl distribution (u, x, y) is called an *initial solution* for initial state x_0, if it satisfies $\dot{x} = Ax + Bu + x_0\delta$; $y = Cx + Du$ as equalities of distributions, there exists an $I \subseteq \{1, \ldots, m\}$ with $y_i = 0$, $i \in I$ and $u_i = 0, i \notin I$ and finally, the Laplace transforms satisfy $\hat{u}(\sigma) \geq 0$ and $\hat{y}(\sigma) \geq 0$ for all sufficiently large σ. In case $(u(t), x(t), y(t))$ is an ordinary function these conditions mean that the system's equations (2) are satisfied on an interval of the form $[0, \varepsilon)$ for some $\varepsilon > 0$. In case the initial solution is not a function, the impulsive part of $u(t)$ will result in

a state jump from x_0 to $x^+ := x_0 + \sum_i A^i B u^{-i}$ (see (Hautus and Silverman, 1983)). Particularly, in (Heemels *et al.*, 2000) it is shown that the above re-initialization procedure corresponds for linear mechanical systems with unilateral constraints to the *inelastic* impact case. Moreover, in some cases the jump of the state variable can be made more explicit in terms of the linear projection operator onto the consistent subspace of the new mode along a jump space (Heemels *et al.*, 2000).

Depending on the interval on which solutions exist, we can now distinguish between three types of well-posedness; *global* well-posedness means existence and uniqueness of solutions on the interval $\mathbb{R}_+ = [0, \infty)$, *local* well-posedness on $[0, \varepsilon)$ for some $\varepsilon > 0$ and *initial* well-posedness means the existence and uniqueness of an initial solution given arbitrary initial condition $x(0) = x_0$. In the terminology of hybrid automata (Lygeros *et al.*, 1999; Johansson *et al.*, 1999), initial well-posedness is equivalent to the LCS being non-blocking and deterministic.

For the $\text{LCS}(A, B, C, D)$ the rational matrices $G(s)$ and $Q(s)$ are defined by $C(s\mathcal{I} - A)^{-1} B + D$ and $Q(s) = C(s\mathcal{I} - A)^{-1}$.

Theorem 5.1. (Heemels et al., 1999b)
$\text{LCS}(A, B, C, D)$ is initially well-posed if and only if for all x_0 $\text{LCP}(Q(\sigma)x_0, G(\sigma))$ is uniquely solvable for sufficiently large $\sigma \in \mathbb{R}$.

The strength of this theorem is that dynamical properties of an LCS are coupled to properties of families of static LCPs, for which a wealth of existence and uniqueness are available (Cottle *et al.*, 1992). For instance, a sufficient condition for initial well-posedness is $G(\sigma)$ being a P-matrix for sufficiently large σ.

Clearly, initial well-posedness does not imply local existence of solutions as in principle, an infinite number of re-initializations (jumps) may occur on one time-instance without "time progressing." This phenomenon is sometimes called "live-lock." However, sufficient conditions for *local* well-posedness have been provided for LCS (van der Schaft and Schumacher, 1996; Heemels *et al.*, 2000), as presented next. Consider the $\text{LCS}(A, B, C, D)$ with Markov parameters $H^0 = D$ and $H^i = CA^{i-1}B$, $i = 1, 2, \ldots$ and define the leading row and column indices by

$$\eta_j = \inf\{i \mid H^i_{\bullet j} \neq 0\}, \quad \rho_j = \inf\{i \mid H^i_{j \bullet} \neq 0\},$$
(11)

where $j \in \{1, \ldots, k\}$ and $\inf \emptyset := \infty$. The *leading row coefficient matrix* $\mathcal{M}$ and *leading column coefficient matrix* $\mathcal{N}$ are then given for *finite* leading row and column indices by

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[5] A sequence of real numbers is called lexicographically nonnegative, if the sequence is either the zero sequence or the first non-vanishing term is positive.

$$\mathcal{M} := \begin{pmatrix} H_{1\bullet}^{\rho_1} \\ \vdots \\ H_{k\bullet}^{\rho_k} \end{pmatrix} \text{ and } \mathcal{N} := (H_{\bullet 1}^{\eta_1} \ldots H_{\bullet k}^{\eta_k})$$

Theorem 5.2. (Heemels *et al.*, 2000) *If the leading column coefficient matrix $\mathcal{N}$ and the leading row coefficient matrix $\mathcal{M}$ are both defined and P-matrices, then $LCS(A, B, C, D)$ has a unique local left Zeno free solution on an interval of the form $[0, \varepsilon)$ for some $\varepsilon > 0$. Moreover, live-lock does not occur; after at most one jump a smooth continuation exists.*

Besides these results including irregular states and corresponding re-initializations, also the Theorems 3.5, 3.6 and 3.7 can be extended to include all initial states x_0. The details can be found in (Çamlıbel, 2001; Çamlıbel *et al.*, 2002b), but "roughly speaking" these results state that at the initial time $t = 0$ there is at most one jump to the set of regular states (i.e. satisfying the conditions of the Theorems 3.5, 3.6 and 3.7) specified by the unique initial solution after which a left-Zeno free solution exists from the re-initialized state on $\mathbb{R}_+$. Several equivalent characterizations of the jump rule can also be found in (Çamlıbel, 2001; Çamlıbel *et al.*, 2002b).

First steps in the direction of getting global well-posedness results for LCS *with external inputs* can be found in (Çamlıbel *et al.*, 2002b) for LPCS and (Çamlıbel *et al.*, 2000), where the underlying linear system is of index 1.

6. CONCLUSIONS

The purpose of this paper was to give an overview of the existing well-posedness results for the complementarity class of hybrid dynamical systems. Under varying conditions, statements on initial, local and global existence and uniqueness of particular (initial, hybrid, forward or L_2) types of solutions have been presented. In certain cases phenomena like left-accumulation points of event times or live-lock have been excluded. The exclusion of Zeno behaviour is important to go from initial to local existence (e.g. by ruling out live-lock) or from local to global (no right-accumulations of events) and for uniqueness of hybrid or L_2-solutions (see e.g. (Pogromsky *et al.*, 2001)). Hence, Zeno behaviour plays a crucial role in the analysis of well-posedness and deserves further attention as is also pointed out in, for instance, (Johansson *et al.*, 1999). Also in the simulation and the analysis of the behaviour of hybrid systems the absence of Zenoness is preferable. Although the absence is often assumed, conditions to verify this are rare. Some initial work in this direction for linear complementarity systems can be found in (Çamlıbel and Schumacher, 2001)

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