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SLIDING MODE SIMPLEX CONTROL METHODS FOR MECHANICAL SYSTEMS

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Abstract: We consider the control of mechanical systems based on sliding mode control techniques. Recently developed simplex control methods are shown to converge in a finite time when applied to nonlinear systems under bounded deterministic uncertainty. Applications are considered to the control of mechanical systems in which the control action is provided by monodirectional devices.

Keywords: Control, sliding mode control, control of nonlinear uncertain systems, control of mechanical systems.

1. INTRODUCTION AND PROBLEM STATEMENT

In this note a class of nonlinear uncertain controlled objects, sufficiently wide to include the representation of most of the robotic structures, is considered. The control problem consists in forcing this class of objects to fulfill the control aim with the constraint that the control vector u belongs to a finite set of vectors $\left\{u_i \in \mathrm{R}^K : \ i = 1, \ldots, p\right\}$. The control design consists in the choice of the vectors u_i, the integer value p, the set of events together with the suitable commutation criterion in order to guarantee the achievement of a prespecified control objective through a suitable sliding condition. The formulation of the above control problem, in many cases, implies a representation of the controlled system in terms of discontinuous differential equations.

The basic problem is the following. We consider a fixed known nominal control system

$$\dot{x} = f(t, x, u), \quad t \geq 0 \tag{1}$$

with control constraint

$$u \in U. \tag{2}$$

The system is subject to unknown additive uncertain perturbations of deterministic nature. Thus every uncertain system's state y evolves according to

$$\dot{y} = f(t, y, u) + \varphi(t, y, u), \quad t \geq 0 \tag{3}$$

where φ represents some uncertainty acting on the nominal system (1). Here the control vector $u \in \mathrm{R}^K$ and the state variables $x, y \in \mathrm{R}^N$.

A given sliding manifold

$$s(t, x) = 0 \tag{4}$$

is fixed in order to fulfill prescribed control aims, where $s(t, x) \in \mathrm{R}^M$. The sliding manifold is designed in such a way that, when the trajectories of system (3) belong to it, then the desired control objectives, e.g. stabilization or model tracking, are satisfied.

The control aim is to select an admissible feedback control law $u = u^*(t, x) \in U$ such that the corresponding state y through (3), issued from a given initial position at time $t = 0$, reaches in a finite time t^* some point fulfilling (4) (*attainability*) and remains on the sliding manifold (4) for all $t \geq t^*$ (*sliding*).

A known constant $L > 0$ is fixed. We want to control those uncertain state variables $y = y(t)$ of the control system (3) which fulfill the condition

$$|y(t)| \leq L, \quad t \geq 0 \tag{5}$$

in order to guarantee the sliding property

$$s\,[t, y(t)] = 0 \tag{6}$$

for every t sufficiently large. We assume that $s\,[t, y(t)]$ is available to the controller for feedback purposes for each $t \geq 0$ and every uncertain state y.

We consider plants (3) affected by noise in such a way that the statistical informations needed to employ stochastic control techniques are unavailable to the controller. Instead we rely only on the nominal system (1), on available bounds about uncertainties and some qualitative features of the dynamics.

At least three methods are known to control a dynamical system by sliding mode techniques, i.e. in order to fulfill (4).

The first two, namely component-wise sliding control and unit control, have been widely investigated, see (Utkin, 1992) for a survey.

In the following we focus on the less known simplex methods (originated by (Bajda and Izosimov, 1985)) which we now describe in a more general and new setting.

2. MOVING SIMPLEX CONTROL UNDER UNCERTAINTY

Let Ω be an open set of R^N containing the closed ball of center 0 and radius L. We assume that $N \geq M$, U is a closed set in R^K, and

$$s : [0, +\infty) \times \Omega \to \mathrm{R}^M$$

is a smooth mapping;

$$f, \varphi : [0, +\infty) \times \Omega \times U \to \mathrm{R}^N$$

are Carathéodory maps, for every uncertain dynamics φ. Denote by $Ds(t, x)$ the $M \times N$ jacobian matrix $\frac{\partial s_i(t,x)}{\partial x_h}$, $i = 1, \ldots, M$, $h = 1, \ldots, N$ and assume that $Ds(t, x)$ has maximum rank everywhere.

The basic assumption is the following. There exist Carathéodory functions

$$u_1(t, x), \ldots, u_{M+1}(t, x)$$

taking values in U, such that the vectors

$$g_i = g_i(t, x) = Ds(x) f[t, x, u_i(t, x)]$$

fulfill

$$0 < a \leq |g_i| \quad \text{and} \quad g_i^T g_h \leq -c^2\, |g_i|\, |g_h| \quad \text{if} \;\; i \neq h \tag{7}$$

for every (t, x) and some constants $a, c \neq 0$.

If (7) holds, then (for any given (t, x)) the vectors g_i, $i = 1, \ldots, M+1$ define a simplex in R^M in the following sense. R^M is partitioned in $M + 1$ cones

$$Q_h = \operatorname{cone}(g_i : \; i \neq h) = = \left\{ \sum_{i \neq h} \alpha_i g_i : \alpha_i \geq 0 \quad \text{if} \;\; i \neq h \right\}, \quad h = 1, \ldots, M+1 \tag{8}$$

with pairwise disjoint interiors. Thus for every $t \geq 0$ and x with $s(t, x) \neq 0$ there exist coefficients $\alpha_i = \alpha_i(t, x) \geq 0$ such that

$$s(t, x) = \sum_{i \neq h} \alpha_i g_i(t, x) \tag{9}$$

with the smallest possible index $h = h(t, x)$. Then the *moving* simplex control law is defined by

$$u^*(t, x) = u_h(t, x). \tag{10}$$

The cones Q_h given by (8) cover R^M, hence u^* given by (10) is well defined.

By (10), the control law u^* undergoes discontinuities as a function of x. By injecting u^* into (3) we consider the system

$$\dot{y}(t) = = f\,(t, y(t), u^*\,[t, y(t)]) + \varphi\,(t, y(t), u^*\,[t, y(t)]) \tag{11}$$

where the dynamics $f\,[t, x, u^*(t, x)]$, $\varphi\,[t, x, u^*(t, x)]$ are now discontinuous functions of x. Based on the Filippov notion of solution, it is possible to build a rigorous theory, see (Filippov, 1988) and (Utkin, 1992), in good agreement with the observed behaviour of some real control systems, see (Utkin, 1978).

In the following, states of (3) corresponding to u^* will be understood as Filippov solutions of (11).

We emphasize that no information beyond those available about the nominal system (1), (2), (4) are needed in order to obtain the moving simplex control law u^* in (10).

Under explicit conditions involving the known nominal system, the geometry of the simplex with obtuse angles made by $g_1, \ldots, g_{M+1}$, and an estimate of the maximal amount of uncertainty, the feedback u^* guarantees the sliding condition (6) for every uncertain state fulfilling (5).

Suppose that there exist constants A^*, H such that

$$|f[t, x, u_i(t, x)]| \le A^* \quad \text{and}$$
$$|\varphi[t, x, u_i(t, x)]| \le H \tag{12}$$

for all t, x, $i = 1, \ldots, M+1$, H explicitly known to the controller. Finally assume that for some constants W_0, W we have

$$\left|\frac{\partial s(t,x)}{\partial t}\right| \le W_0 \quad \text{and} \quad |Ds(t,x)| \le W \tag{13}$$

everywhere.

Theorem 1

Suppose that (7), (12), (13) hold and

$$ac^2 > (W_0 + WH)\, E \tag{14}$$

where

$$E = \max\left\{ \frac{\sum_{i\in I} \alpha_i\, |g_i|}{\left|\sum_{i\in I} \alpha_i g_i\right|} : \quad \alpha_i \ge 0, \quad |\alpha| = 1 \right\},$$

I of M elements.

Then every uncertain state fulfilling (5) verifies the sliding condition (6) for every t sufficiently large.

Given the maximal amount H of uncertainty, condition (14) requires sufficient control authority about the nominal system in order to fix $a > 0$ sufficiently large, because of (7).

We sketch the proof of Theorem 1 in the (very particular) case when $s = s(x)$ only, and no uncertainty acts on the system (hence $W_0 = 0 = H$).

Let $s(y) = s[y(t)]$ fulfill (6) in a given time interval. If y corresponds to u^* we formally compute

$$s^T(y)\dot{s}(y) = s^T(y) Ds(y)\dot{y} = \sum_{i\ne h} \alpha_i g_i^T g_h \tag{15}$$
$$\le -c^2 \sum_{i\ne h} \alpha_i\, |g_i|\, |g_h| \le -ac^2\, |s(y)|$$

by (7). As well known, the differential inequality (15) yields $s[y(t)] = 0$ for all sufficiently large t. The flaw is that a Filippov solution is not necessarily an almost everywhere solution, and a more involved proof is needed, see (Bartolini *et al.*, n.d.).

Theorem 1 generalizes the convergence result of (Bartolini *et al.*, 1999) to control systems subject to uncertainty and time-dependent sliding manifold.

3. FIXED SIMPLEX CONTROL

A different but related control method deals with a *fixed* simplex, i.e. the edges do not depend of t, x and the dynamics, as follows. Suppose that, as in most mechanical systems, the nominal dynamics (1) are affine in the control variables. Moreover let the nominal system be autonomous, namely

$$\dot{x} = A(x) + B(x)u \tag{16}$$

with control constraint

$$|u| \le \rho \tag{17}$$

$K = M$, and sliding manifold

$$s(t, x) = Cx + d(t) = 0. \tag{18}$$

Let the control system be described by Lagrangian coordinates q, so $x = (q^T, \dot{q}^T)^T$. Assume that $B = B(q)$ only and that deterministic uncertainty acts on the nominal system (16) as

$$\dot{y} = A + \Delta A + (B + \Delta B)u.$$

Constants A_0, B_0 are known such that the unknown dynamics fulfill

$$|\Delta A(t, x, u)| \le A_0 \text{ and } |\Delta B(t, x, u)| \le B_0$$

for all t, x, u. Fix points $u_1, \ldots, u_{M+1} \in \mathrm{R}^M$ such that

$$|u_i| = \rho, \quad i = 1, \ldots, M+1 \text{ and}$$
$$u_i^T u_h \le -c^2\, |u_i|\, |u_h|\,, \quad i \ne h$$

for some constant $c \ne 0$. Then for every $t \ge 0$, $|x| \le L$ with $s(t, x) \ne 0$ we have

$$s(t, x) \in \mathrm{cone}(u_i : \; i \ne h)$$

with the least possible $h = h(t, x)$. Then define the *fixed* simplex control law as

$$u_*(x) = u_h.$$

Accordingly, the fixed simplex control law u_* is constant in every region of $[0, +\infty) \times \mathrm{R}^N$ where $s(t, x)$ is interior to any $\mathrm{cone}(u_i : \; i \ne h)$.

We assume that $\dot{d}$ in (18) is bounded, and that

$$G(q) = [CB(q)]^{-1} \quad \text{is positive definite} \tag{19}$$

for all q, $|q| \le L$.

Suppose that A, B are continuous and ΔA, ΔB are Carathéodory functions. Then constants C_1, C_2 are available to the controller such that for every t, x, u

$$C_1 \geq \left| GC\left(A + \Delta A - \frac{1}{2}\frac{\partial B}{\partial q}\dot{q}Gs\right) + G\dot{d}\right|,$$

$$C_2 \geq |\Delta B|,$$

for each uncertainty ΔA, ΔB.

Theorem 2

Every uncertain state y fulfilling (5) verifies the sliding condition (4) for every t sufficiently large, provided that (19) and the following hold

$$C_1 + C_2\rho < mc^2\rho^2 \qquad (20)$$

where

$$m = \min\left\{ \frac{\sum_{i\in I}\alpha_i}{\left|\sum_{i\in I}\alpha_i u_i\right|} : \quad \alpha_i \geq 0, \quad |\alpha| = 1 \right\},$$

I of M elements.

Sketch of the proof.

Let

$$K_0(x) = [CB(x)]^{-1} \text{ and}$$

$$V(x) = s(x)^T K_0(x) s(x).$$

Then (formally) if y is any uncertain state corresponding to u_*, we have

$$\dot{V} = s^T(2K_0\dot{s} - K_0C\dot{B}K_0s) =$$

$$= 2s^TK_0C\left[A + \Delta A + (B + \Delta B)u_* + \right.$$

$$\left. -\frac{1}{2}\dot{B}K_0s\right] + 2s^TK_0\dot{d}$$

and by (20) it turns out that

$$\dot{V} \leq -\epsilon\sqrt{V}$$

for some $\epsilon > 0$, hence $V(t) = 0$, whence $s(t) = 0$ for all t sufficiently large. The rigorous proof is given in (Bartolini *et al.*, n.d.).

4. APPLICATIONS

We consider an important and specific area of mechanical systems in which the control action is provided by monodirectional devices such as jet thrusters, tendons and contact forces. The simplex control theory can be suitably applied to this kind of systems and the control problem consists in relating the generalized force of the lagrangian representation of the mechanical system to a vector of external forces and torques with either positive or null components.

Consider the standard description of a mechanical system

$$\mathcal{M}(q)\ddot{q} = \mathcal{C}(q,\dot{q})\dot{q} + \mathcal{G}(q) + \tau$$

where $q \in \mathrm{R}^n$, $\mathcal{M}(q)$ is the positive definite inertial matrix, $\mathcal{C}(q,\dot{q})\dot{q}$ collects the centrifugal and Coriolis terms, $\mathcal{G}(q)$ is the gravitational term, and τ, the vector of the generalized forces, can be expressed as

$$\tau = \mathcal{H}\mathcal{F}, \quad \mathcal{F} = [\mathcal{F}_1, \ldots, \mathcal{F}_k]^T,$$

$$\mathcal{F}_i \geq 0, \quad i = 1, \ldots, k,$$

where $\mathcal{H} = [\mathcal{H}_1, \ldots, \mathcal{H}_k]$ is the matrix which kinematically relates the external control action $\mathcal{F}_i$ to the generalized force vector τ. The values $\mathcal{F}_i$ and the matrix $\mathcal{H}$ must be designed in order to implement the simplex control methodology.

After choosing the desired trajectories $q_d(t)$, define the sliding surface

$$s = [\dot{q}(t) - \dot{q}_d(t)] + \Gamma[q(t) - q_d(t)] = 0, \quad (21)$$

$$\Gamma = \mathrm{diag}(\gamma_i), \quad \gamma_i > 0, \quad i = 1, \ldots, n,$$

and choose a set of vectors $\tau_1, \ldots, \tau_{n+1}$ of suitable modulus which form a simplex in R^n and satisfy the assumptions of Theorem 2.

If the columns of $\mathcal{H}$ for $k = n + 1$ are designed to form a simplex in R^n then, for any τ_i, there exists an index h, $1 \leq h \leq n + 1$, such that

$$\tau_i = \sum_{j=1 \;\; j\neq h}^{n+1} \lambda_j^i \mathcal{H}_j, \quad \text{for suitable} \quad \lambda_j^i \geq 0.$$

To generate τ_i it is sufficient to choose $\mathcal{F}^i = \left[\lambda_1^i, \ldots, \lambda_{(h-1)}^i, 0, \lambda_{(h+1)}^i, \ldots, \lambda_{(n+1)}^i\right]^T$.

If the columns of the kinematic matrix $\mathcal{H}$ form a simplex satisfying the obtuse angle condition, then we can choose

$$\tau_i = \lambda_i\mathcal{H}_i, \quad \lambda_i > 0;$$

this means that $\mathcal{F}$ has just one component different from zero and the strategy is that of activating just one actuator at a time.

A significant example of this kind of systems is constituted by the AMADEUS gripper (AMADEUS Project. European Commission Directorate General XII MAST III Contract: *MAS3-CT95-0024*), see (Bartolini *et al.*, 2000) for a detailed description.

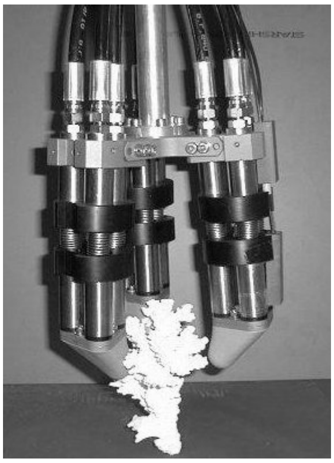

Fig. 1. The AMADEUS gripper

The gripper is an underwater robotic manipulator in which three fingers are arranged to form one hand, fig. 1. Each finger is formed by three bellows located at the vertices of an equilateral triangle and the three bellows are connected to two plates. The centers of the plates are connected by a rigid link articulated by a cardan joint. The finger-tip can span a portion of a spherical surface.

Any finger bellow is hydraulically connected to a control bellow in a constant volume hydraulic circuit. Any length variation of the control bellow results in an opposite length variation of the finger bellow. The actuator (an high bandwindth linear motor of the voice coil type) acts on the control bellow and remotely affects the finger motion with negligible friction and no backlash.

The finger can be regarded as a cardanic joint actuated by three bellows which act as tendons (tendon-based actuation system).

Due to the design of each finger in the gripper, it results that the forces $\mathcal{F}$, exerted by the voice coil motors, are directed along the axis of the system and their application points form a simplex of vectors. Hence, according to the previously described procedure, the vectors τ of the torques applied to the finger can be realized so to form a simplex of vectors.

A mathematical model of the finger which describes the behaviour of the system with sufficient precision within the working space, has the form

$$\mathcal{J}^T(q)\mathcal{A}\mathcal{J}(q)\ddot{q} = \quad (22)$$
$$= \mathcal{N}(q,\dot{q}) + \mu(q) + \mathcal{J}^T(q)\ [\tau + \eta(q)]$$

where $q \in \mathrm{R}^2$ is the vector of the rotation angles, $\mathcal{A} > 0$ is the diagonal inertia matrix of the finger, and $\mathcal{J}(q)$ is the jacobian matrix of the system which can be easily found from the rotation matrix of the cardan joint (Bartolini *et al.*, 2000). The term $\mathcal{N}(q,\dot{q})$ accounts for standard gravitational Coriolis and centripetal torques while the uncertainty terms $\mu(q)$ and $\eta(q)$ are both associated to the nonlinear elasticity phenomena which characterize the use of the bellows as actuation system. In particular $\eta(q)$ which is bounded in the chosen working space is due to the torque associated to the normal stress of any bellow and is a nonlinear function of the rotation angles. The term $\mu(q)$ is the bending torque of each bellow which is a nonlinear uncertain function of the rotation angles, due to the residual buckling phenomenon located in the free portion of any bellow. Constant upperbounds can be easily found in the working space by simple experiments.

Given the desired trajectories $q_d(t)$, let us define the sliding surface (21) such that, on $s = 0$, the reduced system (the zero-dynamics) is arbitrarily exponentially stable.

Since the Jacobian matrix $\mathcal{J}(q)$ is known and not singular in the working space, the following transformation of the sliding output s of (21) can be performed:

$$\mathcal{S}(t,q,\dot{q}) = \mathcal{J}(q)s(t,q,\dot{q}).$$

Differentiating and remembering (22) it can be found that

$$\dot{\mathcal{S}}(t,q,\dot{q},\tau) = \dot{\mathcal{J}}s + \mathcal{J}\dot{s} = \left[\xi(t,q,\dot{q}) + \mathcal{A}^{-1}\tau\right]$$

in which $CB = \mathcal{A}^{-1}$ is positive definite.

Consider three constant vectors τ_1, τ_2, τ_3 which form a simplex in R^2 so that

$$\sum_{i=1}^{3}\mu_i\tau_i = 0, \quad \mu_i > 0, \quad \sum_{i=1}^{3}\mu_i = 1.$$

Furthermore they verify

$$|\tau_i| > |\xi| + \zeta^2, \quad \forall i = 1,2,3,$$

where ζ^2 is a constant depending on the chosen working space, and the obtuse angle condition

$$\tau_i^T\tau_j < -c^2\,|\tau_i|\,|\tau_j|\,, \quad \forall i,j = 1,2,3.$$

According to the previously introduced control methodology, the simplex control switching logic guarantees the achievement of the control goals.

In fact the simplex vectors τ_i partition the plane in three non-overlapping cones Q_i such that

$$\mathcal{S} \in Q_i \quad \text{if} \quad \mathcal{S} = \lambda_j\tau_j + \lambda_k\tau_k,$$
$$\lambda_j, \lambda_k \geq 0, \quad j,k \neq i,$$

and the control vector is chosen according to the following switching logic:

$$\text{if } \mathcal{S} \in Q_i \text{ set } \tau^* = \tau_i$$

that is, if $\mathcal{S}(t)$ belongs to a cone Q_i positively spanned by the two vectors τ_j and τ_k, with $j, k \neq i$, then τ_i is chosen as control action.

The control vector τ is given by $\tau = \mathcal{H}\mathcal{F}$, where, due to the design of the system, the matrix $\mathcal{H}$ turns out to be

$$\mathcal{H} = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}$$

and the vector $\mathcal{F} = [\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3]^T$ collects the forces $\mathcal{F}_i$ exerted by each one of the three control bellows.

The constant vectors τ_i, $i = 1, 2, 3$, of the desired simplex can be obtained in a natural way, it is sufficient to activate just one voice coil motor at a time on the basis of the knowledge of the region to which $\mathcal{S} = \mathcal{J}\, s$ belongs.

Future work regarding the application of the proposed control methodology to a wider class of mechanical and robotic systems is in progress.

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