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STABILITY AND H_∞ CONTROL OF SYSTEMS WITH TIME-VARYING DELAYS

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Abstract: Three main model transformations were used in the past for delay-dependent stability. Recently a new (descriptor) model transformation has been introduced. In the present paper for systems with time-varying delays we obtain new delay-dependent stability conditions under descriptor model transformation. These conditions are written in terms of linear matrix inequalities. We also refine recent results on delay-dependent H_∞ control and extend them to the case of time-varying delays. Numerical examples illustrate the effectiveness of our method.

Keywords: time-delay systems, LMI, delay-dependent stability, H_∞ control

1. INTRODUCTION

Time-delay often appears in many control systems (such as aircraft, chemical or process control systems) and, in many cases, delay is a source of instability (see e.g. Hale and Lunel 1993). The stability issue of systems with delay is, therefore, of theoretical and practical importance.

Delay-dependent stability conditions in terms of linear matrix inequalities (LMIs) have been obtained for retarded and neutral type systems. These conditions are based on three main model transformations of the original system (see Kolmanovskii and Richard 1999). Recently a new *descriptor* model transformation was introduced for delay-dependent stability of neutral systems (Fridman 2001) and of a more general class of differential and algebraic (descriptor) system with delay (Fridman 2002). Unlike previous transformations, the descriptor model leads to a system which is equivalent to the original one, it does not depend on additional assumptions for stability of the transformed system and requires bounding of fewer cross-terms.

Two main approaches for dealing with time-varying delays have been suggested in the past. The first is based on Lyapunov-Krasovskii functionals and the second is based on Razumichin theory. Two main cases of time-varying delays have been considered:

A1 $\tau_i(t)$ are differentiable functions, satisfying for all $t \geq 0$

$$0 \leq \tau_i(t) \leq h_i, \quad \dot{\tau}_i(t) \leq d_i, \; i = 1, 2,$$

or

A2 $\tau_i(t)$ are continuous functions, satisfying for all $t \geq 0$, $0 \leq \tau_i(t) \leq h_i, \; i = 1, 2.$

To the best of our knowledge, the Razumichin's approach was the only one that was to cope with the case A2, which allows fast time-varying delays.

In the present paper, we improve the delay-dependent stability conditions of Fridman (2001), that were based on descriptor model transformation, by applying tighter bounding of the cross terms introduced in Park (1999). We extend the results of Fridman (2001) to the case of systems with time-varying delays. Our method based on Lyapunov-Krasovskii functional seems to be the first of this type for the case A2. Our results significantly improve the existing ones (see Kim 2001 and references therein). Numerical example shows that our method even for more robust case A2 leads to less restrictive results, than those of Kim (2001) which were obtained for the case A1.

A descriptor model transformation has been applied recently for H_∞ control problem Fridman and Shaked (2002). We refine results of Fridman and Shaked (2002) and extend them to the time-varying case.

Notation: Throughout the paper the superscript 'T' stands for matrix transposition, $\mathcal{R}^n$ denotes the n dimensional Euclidean space with vector norm $|\cdot|$, $\mathcal{R}^{n\times m}$ is the set of all $n\times m$ real matrices, and the notation $P>0$, for $P\in\mathcal{R}^{n\times n}$ means that P is symmetric and positive definite. We denote $x_t(\theta)=x(t+\theta)$ $(\theta\in[-h,0])$.

2. STABILITY VIA DESCRIPTOR MODEL TRANSFORMATION

Consider the following system with time-varying delays:

$$\begin{aligned}&\dot{x}(t)-\sum_{i=1}^{2}F_i\dot{x}(t-g_i)=\sum_{i=0}^{2}A_ix(t-\tau_i(t)),\\&x(t)=\phi(t),\ t\in[-h,0],\end{aligned}\tag{1}$$

where $g_i\geq 0,\ i=1,2$ and $x(t)\in\mathcal{R}^n,\ \tau_0\equiv 0$, A_i and F_i are constant $n\times n$-matrices, ϕ is a continuously differentiable initial function.

Taking in (1) $h=max\{h_1,h_2,g_1,g_2\}$. We are looking for stability criteria, delay-independent with respect to g_i and dependent on h_i and d_i. We consider, for simplicity, two delays $g_1,\ g_2$ and $\tau_1,\ \tau_2$, but all the results are easily generalized for the case of any finite number of delays. Representing (1) in the descriptor form

$$\begin{aligned}&\dot{x}(t)=y(t),\\&y(t)-\sum_{i=0}^{2}F_iy(t-g_i)=[\sum_{i=0}^{2}A_i]x(t)\\&-\sum_{i=1}^{2}A_i\int_{t-\tau_i(t)}^{t}y(s)ds,\end{aligned}\tag{2}$$

and denoting

$$\bar{x}(t)=col\{x(t),\ y(t)\},$$

consider the following Lyapunov-Krasovskii functional

$$V(t)=\bar{x}^T(t)EP\bar{x}(t)+V_2+V_3+V_4,\tag{3}$$

where

$$\begin{aligned}&E=\begin{bmatrix}I&0\\0&0\end{bmatrix},\ P=\begin{bmatrix}P_1&0\\P_2&P_3\end{bmatrix},\\&V_2=\sum_{i=1}^{2}\int_{-h_i}^{0}\int_{t+\theta}^{t}y^T(s)A_i^TR_iA_iy(s)dsd\theta,\\&V_3=\sum_{i=1}^{2}\int_{t-g_i}^{t}y^T(s)U_iy(s)ds,\\&V_4=\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}x^T(s)S_ix(s)ds.\end{aligned}$$

Term V_4 is used in order to apply inequality of Park (1999). By arguments similar to (Fridman, 2001), (Fridman and Shaked, 2001) we obtain

Theorem 1. In the case A1 the neutral system (1) is stable if there exist $n\times n$ matrices $0<P_1,\ P_2,\ P_3,\ S_i=S_i^T,\ U_i=U_i^T,\ W_{i1},\ W_{i2}$ and $R_i=R_i^T,\ i=1,2$ that satisfy the following LMI :

$$\begin{bmatrix}\Psi_1&\Psi_2&h_1\Phi_{11}&h_2\Phi_{21}&-W_{11}^TA_1&-W_{21}^TA_2&P_2^TF_1&P_2^TF_2\\ *&\Psi_3&h_1\Phi_{12}&h_2\Phi_{22}&-W_{12}^TA_1&-W_{22}^TA_2&P_3^TF_1&P_3^TF_2\\ *&*&-h_1R_1&0&0&0&0&0\\ *&*&*&-h_2R_2&0&0&0&0\\ *&*&*&*&-S_1(1-d_1)&0&0&0\\ *&*&*&*&*&-S_2(1-d_2)&0&0\\ *&*&*&*&*&*&-U_1&0\\ *&*&*&*&*&*&*&-U_2\end{bmatrix}<0.\tag{4}$$

where

$$\Psi_1 = (\sum_{i=0}^{2} A_i^T)P_2 + P_2^T(\sum_{i=0}^{2} A_i)$$
$$+\sum_{i=1}^{2}(W_{i1}^T A_i + A_i^T W_{i1}) + \sum_{i=1}^{2} S_i,$$
$$\Psi_2 = P_1 - P_2^T + (\sum_{i=0}^{2} A_i^T)P_3 + \sum_{i=1}^{2} A_i^T W_{i1},$$
$$\Psi_3 = -P_3 - P_3^T + \sum_{i=1}^{2}(U_i + h_i A_i^T R_i A_i),$$
$$\Phi_{i1} = \left[\, W_{i1}^T + P_2^T \,\right],$$
$$\Phi_{i2} = \left[\, W_{i2}^T + P_3^T \,\right], \quad i = 1,2.$$

Similar to (Fridman 2001) we obtain

Corollary 2. Assume A2. The neutral system (1) is stable if there exist $n \times n$ matrices $0 < P_1,\ P_2,\ P_3,\ U_i = U_i^T$ and $R_i = R_i^T,\ i = 1,2$ that satisfy the following LMI :

$$\begin{bmatrix} \Psi_1 & \Psi_2 & h_1 P_2^T & h_2 P_2^T & P_2^T F_1 & P_2^T F_2 \\ * & \Psi_3 & h_1 P_3^T & h_2 P_3^T & P_3^T F_1 & P_3^T F_2 \\ * & * & -h_1 R_1 & 0 & 0 & 0 \\ * & * & * & -h_2 R_2 & 0 & 0 \\ * & * & * & * & -U_1 & 0 \\ * & * & * & * & * & -U_2 \end{bmatrix} < 0,$$

where $W_{i1} = W_{i2} = 0,\ S_i = 0,\ i = 1,2.$

Remark 1. Note that in (Fridman and Shaked 2002) application of Park inequality lead to $2n \times 2n$ matrices R_i and $W_i = R_i M_i P$. As a result, a more complicated form of LMI was derived. The latter LMI leads to conservative conditions in the case of state-feedback controller design, where it was ssumeed that $W_i = \epsilon_i P, \epsilon_i \in \mathcal{R}$.

For $W_{i1} = -P_2,\ W_{i2} = -P_3,\ R_i = \frac{\epsilon I_n}{h_i},\ i = 1,2,$ LMI (4) implies for $\epsilon \to 0^+$ the following delay-independent/ delay-derivative-dependent LMI:

$$\begin{bmatrix} \Phi & P^T \begin{bmatrix} 0 \\ A_1 \end{bmatrix} & P^T \begin{bmatrix} 0 \\ A_2 \end{bmatrix} & P^T \begin{bmatrix} 0 \\ F_1 \end{bmatrix} & P^T \begin{bmatrix} 0 \\ F_2 \end{bmatrix} \\ * & -S_1(1-d_1) & 0 & 0 & 0 \\ * & * & -S_2(1-d_2) & 0 & 0 \\ * & * & * & -U_1 & 0 \\ * & * & * & * & -U_2 \end{bmatrix} < 0,$$

where

$$\Phi = P^T \begin{bmatrix} 0 & I \\ A_1 & -I_{n_1} \end{bmatrix} + \begin{bmatrix} 0 & A_1^T \\ I & -I_{n_1} \end{bmatrix} P + \sum_{i=1}^{2} \begin{bmatrix} S_i & 0 \\ 0 & U_i \end{bmatrix}.$$

If the latter LMI is feasible then (4) is feasible for a small enough $\epsilon > 0$ and for R_i and $\bar{W}_i$ given above. Thus, Theorem 1 implies the delay-independent/ delay-derivative-dependent conditions given by the latter LMI. The conditions of Theorem 1 are feasible for all $h_i \geq 0$ if this LMI holds.

Since the LMI of (4) is affine in the system matrices, therefore Theorem 1 can be used to derive a criterion that will guarantee the stability in the case where the system matrices are not exactly known and they reside within a given polytope.

Example 1 Kim (2001). We consider

$$\dot{x}(t) = \begin{bmatrix} -2+\delta_1 & 0 \\ 0 & -1+\delta_2 \end{bmatrix} x(t)$$
$$+ \begin{bmatrix} -1+\gamma_1 & 0 \\ -1 & -1+\gamma_2 \end{bmatrix} x(t-\tau(t)),$$
$$|\delta_1| \leq 1.6,\ |\delta_2| \leq .05,\ |\gamma_1| \leq .1,\ |\gamma_2| \leq .3$$

In Kim (2001), where the case A1 (with $0 \leq \tau \leq h,\ \dot{\tau} \leq d < 1$) was treated via transformation I, the maximum values of h for which stability is secured was found as a function of the bound d on the delay rate of change. For $d = 0$ the maximum value of $h = .2412$ was reported and compared with previous results in the literature. Applying the method of Theorem 1 we obtained for $d = 0$ that the system is asymptotically stable for the maximum value of $h = 1$. Our results are favorably compared to those in Kim (2001) also for $d \neq 0$. Even in the more general case A2, which includes fast varying delays, we find $h = .77$, that is significantly better than $h \leq .24$ by Kim (2001) for the case A1.

3. H_∞ CONTROL OF SYSTEMS WITH TIME-VARYING STATE DELAYS

In this section we improve results of Fridman and Shaked (2001, 2002), based on descriptor transformation, and extend them to the case of time-varying delay.

4.1. Delay-dependent Bounded Real Lemma (BRL). Given the following system:

$$\dot{x}(t) - \sum_{i=1}^{2} F_i \dot{x}(t-g_i) = \sum_{i=0}^{2} A_i x(t-\tau_i(t))$$
$$+B_1 w(t), \quad x(t) = 0 \quad t \leq 0, \tag{5}$$
$$z(t) = C x(t)$$

where $x(t) \in \mathcal{R}^n$ is the system state vector, $w(t) \in \mathcal{L}_2^q[0,\ \infty]$ is the exogenous disturbance signal and $z(t) \in \mathcal{R}^p$ is the state combination (objective function signal) to be attenuated. The time delays are defined in Section 3. The matrices $A_i,\ i=0,...,2,\ F_i,\ i=1,2,\ B_1$ and C are constant matrices of appropriate dimensions. For a prescribed scalar $\gamma>0$, we define the performance index:

$$J(w)=\int_0^\infty (z^Tz-\gamma^2 w^T w)ds. \qquad (6)$$

Using argument of the previous section we obtain the following BRL:

Lemma 3. Consider the system of (5). Assume A1. For a prescribed $\gamma>0$, the cost function (6) achieves $J(w)<0$ for all nonzero $w\in\mathcal{L}_2^q[0,\ \infty)$ and for all positive delays g_1,g_2, if there exist P,

$$W_i=\begin{bmatrix}-P_1 & 0\\ W_{i1} & W_{i2}\end{bmatrix}$$

and $n\times n$-matrices $S_i=S_i^T,\ U_i=U_i^T,\ R_i=R_i^T$ that satisfy the following LMI:

$$\begin{bmatrix}\bar\Psi & P^T\begin{bmatrix}0\\B_1\end{bmatrix} & h_1\Phi_1 & h_2\Phi_2 & -W_1^T\begin{bmatrix}0\\A_1\end{bmatrix} & -W_2^T\begin{bmatrix}0\\A_2\end{bmatrix} & P^T\begin{bmatrix}0\\F_1\end{bmatrix} & P^T\begin{bmatrix}0\\F_2\end{bmatrix} & \begin{bmatrix}C^T\\0\end{bmatrix}\\
* & -\gamma^2 I & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
* & * & -h_1R_1 & 0 & 0 & 0 & 0 & 0 & 0\\
* & * & * & -h_2R_2 & 0 & 0 & 0 & 0 & 0\\
* & * & * & * & -(1-d_1)S_1 & 0 & 0 & 0 & 0\\
* & * & * & * & * & -(1-d_2)S_2 & 0 & 0 & 0\\
* & * & * & * & * & * & -U_1 & 0 & 0\\
* & * & * & * & * & * & * & -U_2 & 0\\
* & * & * & * & * & * & * & * & -I_p\end{bmatrix}<0, \qquad (7)$$

where for $i=1,2$

$$\bar\Psi \triangleq P^T\begin{bmatrix}0 & I\\ (\sum_{i=0}^{2}A_i) & -I\end{bmatrix}+\begin{bmatrix}0 & (\sum_{i=0}^{2}A_i^T)\\ I & -I\end{bmatrix}P$$
$$+\begin{bmatrix}\sum_{i=1}^{2}S_i & 0\\ 0 & \sum_{i=1}^{2}(U_i+h_iA_i^TR_iA_i)\end{bmatrix}$$
$$+\sum_{i=1}^{2}W_i^T\begin{bmatrix}0 & 0\\ A_i & 0\end{bmatrix}+\sum_{i=1}^{2}\begin{bmatrix}0 & A_i^T\\ 0 & 0\end{bmatrix}W_i,$$
$$\Phi_i^T=[0\ I_n][W_i+P].$$

Similarly to Corollary 2, the rate-independent result is obtained

Corollary 4. Assume A2. For a prescribed $\gamma>0$, the cost function (6) achieves $J(w)<0$ for all nonzero $w\in\mathcal{L}_2^q[0,\ \infty)$ and for all positive delays g_1,g_2, if there exist P and $n\times n$-matrices $U_i=U_i^T,\ R_i=R_i^T$ that satisfy the following LMI:

$$\begin{bmatrix}\Psi & P^T\begin{bmatrix}0\\B_1\end{bmatrix} & h_1\Phi_1 & h_2\Phi_2 & P^T\begin{bmatrix}0\\F_1\end{bmatrix} & P^T\begin{bmatrix}0\\F_2\end{bmatrix} & \begin{bmatrix}C^T\\0\end{bmatrix}\\
* & -\gamma^2 I & 0 & 0 & 0 & 0 & 0\\
* & * & -h_1R_1 & 0 & 0 & 0 & 0\\
* & * & * & -h_2R_2 & 0 & 0 & 0\\
* & * & * & * & -U_1 & 0 & 0\\
* & * & * & * & * & -U_2 & 0\\
* & * & * & * & * & * & -I_p\end{bmatrix}<0,$$

where $\Phi_i=[P_2\ P_3]^T,\ i=1,2$ and

$$\Psi=P^T\begin{bmatrix}0 & I\\ (\sum_{i=0}^{2}A_i) & -I\end{bmatrix}+\begin{bmatrix}0 & (\sum_{i=0}^{2}A_i^T)\\ I & -I\end{bmatrix}P$$
$$+\begin{bmatrix}0 & 0\\ 0 & \sum_{i=1}^{2}(U_i+h_iA_i^TR_iA_i)\end{bmatrix}.$$

4.2. State-feedback H_∞ control.

Given the system

$$\begin{array}{l}\dot x(t)-\sum_{i=1}^{2}F_i\dot x(t-g_i)=\sum_{i=0}^{2}\bar A_i x(t-\tau_i(t))\\ +B_1w(t)+B_2u(t),\ z(t)=col\{\bar Cx(t),\ D_{12}u(t)\},\\ x(t)=0\ \forall t\le 0\end{array} \qquad (8)$$

where $u\in\mathcal{R}^\ell$ is the control input, $F_1,\ F_2,\ \bar A_0,\ \bar A_1,\ \bar A_2,\ B_1,\ B_2$ are constant matrices of appropriate dimension, z is the objective vector, $\bar C\in\mathcal{R}^{p\times n}$ and $D_{12}\in\mathcal{R}^{r\times\ell}$. We look for a state-feedback gain matrix K which, via the control law

$$u(t)=Kx(t), \qquad (9)$$

achieves $J(w)<0$ for all nonzero $w\in\mathcal{L}_2^q[0,\ \infty)$.

Substituting (9) into (8), we obtain the structure of (1) with

$$\begin{array}{l}A_0=\bar A_0+B_2K,\ A_1=\bar A_1,\\ C^TC=\bar C^T\bar C+K^TD_{12}^TD_{12}K\end{array} \qquad (10)$$

Applying the BRL of Section 4.1 to the above matrices, a nonlinear matrix inequality is obtained due to the terms $P_2^TB_2K$ and $P_3^TB_2K$.

In order to obtain a LMI we restrict ourselves to the case of

$$W_i=diag\{-I_n,\ \epsilon_i\}P,\ i=1,2,$$

where $\epsilon_i\in\mathcal{R}^{n\times n}$ is a diagonal matrix. Such a choice for W_i is less conservative than the one in (Fridman and Shaked 2002), where $W_i=\epsilon_iP$ for a scalar ϵ_i. For $\epsilon_i=-I_n$ (7) yields the delay-independent condition.

It is obvious from the requirement of $0 < P_1$, and the fact that in (7) $-(P_3 + P_3^T)$ must be negative definite, that P is nonsingular. Defining

$$P^{-1} = Q = \begin{bmatrix} Q_1 & 0 \\ Q_2 & Q_3 \end{bmatrix} \quad \Delta = diag\{Q,\ I\} \tag{11}$$

we multiply (7) by Δ^T and Δ, on the left and on the right, respectively. Applying Schur formula to the quadratic term in Q, and denoting $\bar{S}_i = S_i^{-1}$, $\bar{U}_i = U_i^{-1}$ and $\bar{R}_i = R_i^{-1}$, $i = 1, 2$ we obtain, similarly to (Fridman and Shaked 2002), the following

Theorem 5. Assume A1. Consider the system of (8) and the cost function of (6). For a prescribed $0 < \gamma$, the state-feedback law of (9) achieves $J(w) < 0$ for all nonzero $w \in \mathcal{L}_2^q[0,\ \infty)$ if for some diagonal matrices $\epsilon_1,\ \epsilon_2 \in \mathcal{R}^{n \times n}$, there exist $Q_1 > 0,\ \bar{S}_1,\ \bar{S}_2,\ \bar{U}_1,\ \bar{U}_2,\ ,\ Q_2,\ Q_3,\ \bar{R}_1,\ \bar{R}_2 \in \mathcal{R}^{n \times n}$ and $Y \in \mathcal{R}^{\ell \times n}$ that satisfy the following LMI:

$$\left[\begin{array}{ccccc}
Q_2 + Q_2^T & \Xi & 0 & 0 & 0 \\
* & -Q_3 - Q_3^T & B_1 & h_1(\epsilon_1 + I_n)\bar{R}_1 & h_2(\epsilon_2 + I_n)\bar{R}_2 \\
* & * & -\gamma^2 I_q & 0 & 0 \\
* & * & * & -h_1\bar{R}_1 & 0 \\
* & * & * & * & -h_2\bar{R}_2 \\
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & * \\
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* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & *
\end{array}\right.$$

$$\begin{array}{cccccc}
0 & 0 & Q_1 & Q_1 & Q_1\bar{C}^T & Y^T D_{12}^T \\
\epsilon_1 A_1 \bar{S}_1 & \epsilon_2 A_2 \bar{S}_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
-(1-d_1)\bar{S}_1 & 0 & 0 & 0 & 0 & 0 \\
* & -(1-d_2)\bar{S}_2 & 0 & 0 & 0 & \\
* & * & -\bar{S}_1 & 0 & 0 & 0 \\
* & * & * & -\bar{S}_2 & 0 & 0 \\
* & * & * & * & -I & 0 \\
* & * & * & * & * & -I \\
* & * & * & * & * & * \\
* & * & * & * & * & * \\
* & * & * & * & * & * \\
* & * & * & * & * & * \\
* & * & * & * & * & * \\
* & * & * & * & * & *
\end{array}$$

$$\left.\begin{array}{cccccc}
0 & Q_2^T & 0 & Q_2^T & h_1 Q_2^T A_1^T & h_2 Q_2^T A_2^T \\
F_1\bar{U}_1 & Q_3^T & F_2\bar{U}_2 & Q_3^T & h_1 Q_3^T A_1^T & h_2 Q_3^T A_2^T \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
-\bar{U}_1 & 0 & 0 & 0 & 0 & 0 \\
* & -\bar{U}_2 & 0 & 0 & 0 & \\
* & * & -\bar{U}_2 & 0 & 0 & \\
* & * & * & -h_1\bar{R}_1 & 0 & \\
* & * & * & * & -h_2\bar{R}_2 &
\end{array}\right] < 0,$$

where

$$\Xi = Q_3 - Q_2^T + Q_1\left(\sum_{i=0}^{2} A_i^T + \sum_{i=1}^{2} A_i^T \epsilon_i\right) + Y^T B_2^T.$$

The state-feedback gain is then given by

$$K = Y Q_1^{-1}. \tag{12}$$

Choosing $\epsilon_i = 0$, we obtain the counterpart of the Theorem 5 for the case A2:

Corollary 6. Assume A2. Consider the system of (8) and the cost function of (6). For a prescribed $0 < \gamma$, the state-feedback law of (9) achieves $J(w) < 0$ for all nonzero $w \in \mathcal{L}_2^q[0,\ \infty)$ if there exist $Q_1 > 0,\ \bar{U}_1,\ \bar{U}_2,\ ,\ Q_2,\ Q_3,\ \bar{R}_1,\ \bar{R}_2 \in \mathcal{R}^{n \times n}$ and $Y \in \mathcal{R}^{\ell \times n}$ that satisfy the following LMI:

$$\left[\begin{array}{ccccccc}
Q_2 + Q_2^T & \Xi & 0 & 0 & 0 & Q_1\bar{C}^T & Y^T D_{12}^T \\
* & -Q_3 - Q_3^T & B_1 & h_1\bar{R}_1 & h_2\bar{R}_2 & 0 & 0 \\
* & * & -\gamma^2 I_q & 0 & 0 & 0 & 0 \\
* & * & * & -h_1\bar{R}_1 & 0 & 0 & 0 \\
* & * & * & * & -h_2\bar{R}_2 & 0 & 0 \\
* & * & * & * & * & -I & 0 \\
* & * & * & * & * & * & -I \\
* & * & * & * & * & * & * \\
* & * & * & * & * & * & * \\
* & * & * & * & * & * & * \\
* & * & * & * & * & * & * \\
* & * & * & * & * & * & * \\
* & * & * & * & * & * & *
\end{array}\right.$$

$$\left.\begin{array}{cccccc}
0 & Q_2^T & 0 & Q_2^T & h_1 Q_2^T A_1^T & h_2 Q_2^T A_2^T \\
F_1\bar{U}_1 & Q_3^T & F_2\bar{U}_2 & Q_3^T & h_1 Q_3^T A_1^T & h_2 Q_3^T A_2^T \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
-\bar{U}_1 & 0 & 0 & 0 & 0 & 0 \\
* & -\bar{U}_1 & 0 & 0 & 0 & 0 \\
* & * & -\bar{U}_2 & 0 & 0 & 0 \\
* & * & * & -\bar{U}_2 & 0 & 0 \\
* & * & * & * & -h_1\bar{R}_1 & 0 \\
* & * & * & * & * & -h_2\bar{R}_2
\end{array}\right] < 0,$$

where

$$\Xi = Q_3 - Q_2^T + Q_1\left(\sum_{i=0}^{2} A_i^T\right) + Y^T B_2^T.$$

The state-feedback gain is then given by (12).

The results of this section may be adapted to the case of systems with polytopic uncertainties similarly to Section 3.2. The case of output-feedback H_∞ control for systems with time-varying delays can be treated similarly to (Fridman and Shaked 2002) with corresponding modification of the first phase (state-feedback) as above.

Example 2 de Souza and Li (1999). We consider the system:

$$\begin{array}{l}\dot{x}(t) = \bar{A}_0 x(t) + \bar{A}_1 x(t-\tau) + B_1 w(t) \\ +B_2 u(t), \;\; z(t) = col\{\bar{C}x(t), \;\; D_{12}u(t)\},\end{array} \quad (13)$$

where

$$\bar{A}_0 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \; \bar{A}_1 = \begin{bmatrix} -1 & -1 \\ 0 & -.9 \end{bmatrix}, \; B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$
$$B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; \bar{C} = \begin{bmatrix} 0 & 1 \end{bmatrix}, \;\; D_{12} = 0.1.$$

Applying method of de Souza and Li (1999) based on transformation I (Corollary 3.2 there) it was found that, for $\dot{\tau} \equiv 0$, the system is stabilizable for all $\tau < 1$. For, say, $\tau = .999$ a minimum value of $\gamma = 1.8822$ results for $K = -\begin{bmatrix} .10452 & 749058 \end{bmatrix}$. Using the method of Fridman and Shaked (2001) (descriptor transformation with conservative bounding of cross terms) for $\dot{\tau} \equiv 0$, a minimum value of $\gamma = .22844$ was obtained for the same value of τ with a state-feedback gain of $K = \begin{bmatrix} 0 & -182194 \end{bmatrix}$. By Corollary 6, the same γ and K are achieved in the case A2 of time-varying delay $\tau(t) \leq 0.999$.

Consider now the case A1 with $0 \leq \tau \leq h, \; \dot{\tau} \leq d < 1$. Applying, for $\dot{\tau} \equiv 0$ and $\epsilon = -.3$, the method of Fridman and Shaked (2002) (Theorem 3.1 there), a maximum value of $h = 1.28$ was obtained for which a state-feedback controller stabilizes the system. The corresponding feedback gain was $K = \begin{bmatrix} 0 & -1.2091 \times 10^6 \end{bmatrix}$. Using Theorem 5 of the present paper we obtain for $d = 0$ a maximum value of $h = 1.408$. for which there exists a state-feedback gain that stabilizes the system. The maximum values of h that still allow stabilizabilty via state-feedback are found as a function of d.

4. CONCLUSIONS

A delay-dependent LMI solution is proposed for the problems of stability and H_∞ control of linear systems with time-varying delays. This solution is based on the descriptor model transformation and a Park's inequality for bounding of cross terms. Two types of results for systems with time-varying delays have been derived: delay-dependent/rate-dependent and delay-dependent/rate-independent. Our results for the second case, which includes fast-varying delays, seem to be the first results based on Lyapunov-Krasovskii functionals.

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