15th Triennial World Congress, Barcelona, Spain

APPROXIMATION OF DIFFERENTIAL GAMES WITH MAXIMUM COST AND INFINITE HORIZON

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Abstract: We consider an integral form of the Isaacs equations associated to differential games with L_∞ criterion, for the characterization of their value functions. We prove that upper and lower values are the lowest super-solution and the largest element of a special set of sub-solutions, of the dynamic programming equation. This is an alternative to the viscosity solutions approach, without requiring any regularity assumption on the value functions.

For finite horizon approximations, we propose a scheme in terms of an infinitesimal operator defined over the set of Lipschitz continuous functions. The images of this operator can be characterized classically in terms of viscosity solutions.

We illustrate these results on a example, which values functions can be determined analytically.

Keywords: Differential games, Dynamic Programming, Convergence analysis.

1. INTRODUCTION

We consider dynamical system driven by two controls (ξ, η) :

$$\begin{cases} \dfrac{d}{ds} y_{t,x}^{\xi,\eta}(s) = f(y_{t,x}^{\xi,\eta}(s), \xi(s), \eta(s)), & s \geq t \\ y_{t,x}^{\xi,\eta}(t) = x \in \Omega \end{cases} \tag{1}$$

where Ω is an open domain of $\Re^n$.
The player ξ (respectively η) minimizes (respectively maximizes) a criterion involving an essential supremum of a given instantaneous cost, over an infinite horizon :

$$P(x, \xi, \eta) = \operatorname{ess} \sup_{\tau \in [t, +\infty)} h(y_{t,x}^{\xi,\eta}(\tau), \xi(\tau), \eta(\tau))$$

[1] Part of this work has been achieved while the first author was visiting France.

As usual in the theory of differential games, we consider the following sets of controls, where I is either a finite number larger than t or $+\infty$:

$$\mathcal{Z}(t, I) := \{\xi : [t, I) \mapsto Z \,|\, \xi \text{ measurable}\},$$

$$\mathcal{Y}(t, I) := \{\eta : [t, I) \mapsto Y \,|\, \eta \text{ measurable}\},$$

where Z and Y are compact subsets of $\Re^p$, and $\Re^q$ respectively, and the sets of non-anticipating strategies :

$$\begin{aligned} \Gamma(t, I) = \{&\alpha : \mathcal{Z}(t, I) \mapsto \mathcal{Y}(t, I) \text{ s.t. } \forall T \in [t, I], \\ &\xi_1|_{[t,T]} \equiv \xi_2|_{[t,T]} \Rightarrow \alpha[\xi_1]|_{[t,T]} \equiv \alpha[\xi_2]|_{[t,T]}, \\ &\forall \xi_1, \xi_2 \in \mathcal{Z}(t, I)\}, \end{aligned}$$

$$\begin{aligned} \Delta(t, I) = \{&\beta : \mathcal{Y}(t, I) \mapsto \mathcal{Z}(t, I) \text{ s.t. } \forall T \in [t, I], \\ &\eta_1|_{[t,T]} \equiv \eta_2|_{[t,T]} \Rightarrow \beta[\eta_1]|_{[t,T]} \equiv \beta[\eta_2]|_{[t,T]}, \\ &\forall \xi_1, \xi_2 \in \mathcal{Z}(t, I)\}. \end{aligned}$$

We consider then the lower and the upper value functions of the game, given respectively by

$$V^-(x) = \inf_{\beta \in \Delta(t,+\infty)} \sup_{\eta \in \mathcal{Y}(t,+\infty)} P(x, \beta[\eta], \eta)$$

$$V^+(x) = \sup_{\alpha \in \Gamma(t,+\infty)} \inf_{\xi \in \mathcal{Z}(t,+\infty)} P(x, \xi, \alpha[\xi])$$

A typical instance of such criterion occurs when the function h is chosen as the oriented distance function to the complementary of a target. In that case, the criterion P is a kind of measure of the risk of entering the target (see (Yong, 1998)) :

- When P is positive, the target is avoided at any time, and the value of P gives an idea of the "risk" of entering a dilated target.
- When P is negative, the target is entered in finite time, and the value of P gives an idea of how deep the trajectory can enter the target, and avoid a smaller target.
- The null values of P characterize the trajectories belonging to the barrier delimiting the capture and evasion domains (see the "games of kind" in (Rapaport, 1998; Bernhard *et al.*, 2001)).

For economical modeling, the target might correspond to the set of crises. The particularities of this criterion is to belong to a purely deterministic framework over an unbounded horizon, without the consideration of any discount factor. The price to pay for the study of such games is their lack of regularity.

Such games have been already analyzed, using viscosity solutions, but in finite horizon (Barron, 1990). An extension to infinite horizon has been proposed, but under additional hypotheses requiring the semi-continuity of the values functions V^-, V^+ (Rapaport, 1998). The optimal control problem (i.e. with only one player) over infinite horizon has already been studied in (DiMarco and Gonzalez, 1998), using a different approach without requiring any regularity assumption on the regularity of the value function. In this work, we study how to extend these last results to the two players case, without requiring any a priori regularity of the value functions. We discuss also their finite time approximations.

Because of the poor regularity presented by the value functions (they might even be non semi-continuous, see examples with one player in (DiMarco and Gonzalez, 1998)), the viscosity solutions framework cannot provide a unique characterization of the value functions (see (Rapaport, 1998)). As an alternative to this approach, we consider the integral form of the Isaacs equations associated to the problem.

We assume the following hypotheses along this work :

(A1) $f \in BUC(\Omega \times Z \times Y, \Re^n)$, $\exists M_f, L_f$ s.t. $\|f(y,\xi,\eta)\| \leq M_f$, $\|f(y_1,\xi,\eta) - f(y_2,\xi,\eta)\| \leq L_f \|y_1 - y_2\|$, $\forall y, y_1, y_2, \xi, \eta$.

(A2) $h \in BUC(\Omega \times Z \times Y, \Re)$, $\exists M_h, L_h$ s.t. $|h(y,\xi,\eta)| \leq M_h$, $|h(y_1,\xi,\eta) - h(y_2,\xi,\eta)| \leq L_h \|y_1 - y_2\|$, $\forall y, y_1, y_2, \xi, \eta$.

(A3) Ω is invariant by the dynamics (1), for any $(\xi,\eta) \in \mathcal{Z}(t,+\infty) \times \mathcal{Y}(t,+\infty)$.

2. THE OPERATORS M_T

Define the payoff over finite time intervals :

$$P_{t,T}(x,\xi,\eta) = \operatorname{ess}\sup_{\tau \in [t,T)} h(y_{t,x}^{\xi,\eta}(\tau), \xi(\tau), \eta(\tau)).$$

and consider the operator $M_t^+ : B(\Omega) \mapsto B(\Omega)$:

$$M_t^+ w(x) = \sup_{\alpha \in \Gamma(0,t)} \inf_{\xi \in \mathcal{Z}(0,t)} \max\left\{ P_{0,t}(x,\xi,\alpha[\xi]),\, w(y_{0,x}^{\xi,\alpha[\xi]}(t)) \right\}. \tag{2}$$

Analogously, we consider the lower operator M_t^- : $B(\Omega) \mapsto B(\Omega)$:

$$M_t^- w(x) = \inf_{\beta \in \Delta(0,t)} \sup_{\eta \in \mathcal{Y}(0,t)} \max\left\{ P_{0,t}(x,\beta[\eta],\eta),\, w(y_{0,x}^{\beta[\eta],\eta}(t)) \right\}. \tag{3}$$

The dynamic programming principle provides the following result :

Proposition 1. The functions V^+ and V^- are bounded fixed points of M_t^+ and M_t^- respectively, for any $t \in [0,\infty)$.

PROOF. See (Barron, 1990; Rapaport, 1998). □

Remark 2. Since the dynamic programming equation has not unique solution, it is not possible to use standard methods to characterize the values functions of the game. For this reason we consider some special sets that allow us to do it.

Definition 3. Consider the following sets :

$$\begin{aligned} S_t^{\pm} &= \left\{ w \in B(\Omega) | M_t^{\pm} w \leq w \right\}, \\ W_t^{\pm} &= \left\{ w \in B(\Omega) | M_t^{\pm} w \geq w \right\}, \end{aligned} \tag{4}$$

and their intersections :

$$S^{\pm} = \bigcap_{t \in [0,\infty)} S_t^{\pm} \quad , \quad W^{\pm} = \bigcap_{t \in [0,\infty)} W_t^{\pm}.$$

Remark 4. Since now, we will work with the upper value of the game. Similar results can be derived for the lower value.

Remark 5. The sets $S^+$and W^+ are non empty ($V^+ \in S^+ \cap W^+$). Moreover, the function $w \equiv$

$-M_h$ belongs to W^+, and S^+ is lower bounded by $-M_h$.

Since M_t^+ is a non-decreasing monotonic operator ($w_1 \le w_2$, then $M_t^+ w_1 \le M_t^+ w_2$) and has the semi-group property : $t_1 < t_2 \Rightarrow M_{t_1}^+ \cdot M_{t_2-t_1}^+ w = M_{t_1}^+ (M_{t_2-t_1}^+ w) = M_{t_2}^+ w$ (which is easy to check), it can be proved that S^+ and W^+ are closed by application of $M_t^+, \forall t > 0$. Moreover, for all $\delta > 0$, for all $s \in S^+$, $M_{t+\delta}^+ s \le M_t^+ s$ and for all $w \in W^+$, $M_{t+\delta}^+ w \ge M_t^+ w$.

All these properties allow us to define for each $v \in S^+ \cup W^+$ and each $x \in \Omega$,

$$\lim_{t\to\infty} M_t^+ v(x) = M^+ v(x) \tag{5}$$

Lemma 6.

- $M^+ s \le s$, for all $s \in S^+$ and $M^+ w \ge w$, for all $w \in W^+$.
- M^+ is monotonic on S^+ and on W^+.
- $M^+ S^+ \subseteq S^+$and $M^+ W^+ \subseteq W^+$.

Definition 7. Let $\underline{S} = \inf\{s : s \in S^+\}$ (which exists by Remark 5) and $\overline{W} = \sup\{w : w \in W_m^+\}$ where W_m^+ is the minimum class of the family $\mathcal{W}^+$:

$$\mathcal{W}^+ = \{W_*^+ \subseteq W^+, s.t. \text{(C1)}, \text{(C2)}, \text{(C3)}\}$$

(C1) $-M_h \in W_*^+$.
(C2) $M^+ W_*^+ \subseteq W_*^+$.
(C3) Let $\{w_p\}_{p\in I} \subseteq W_*^+$ then $\sup\{w_p : p \in I\} \in W_*^+$.

Remark 8. The set W^+ is not necessarily upper bounded, then we take the supremum of a smaller set of sub-solution which possess these properties. In fact, $W_m^+ = \bigcap_{W_*^+ \in \mathcal{W}^+} W_*^+$ which is nonempty and $\{w \in W^+ : w \le \underline{S}\}$ are in $\mathcal{W}^+$.

3. CHARACTERIZATION OF THE VALUES

The main result is the double characterization :

Theorem 9. $V^+ = \underline{S} = \overline{W}$.

Similarly, V^- can be characterized using the sets S^- and W^-.

PROOF. We first show that $M^+ \underline{S} = \underline{S}$.

It is clear that $\underline{S} \le s$, for any $s \in S^+$. By the monotonic property of M^+, we have $M^+ \underline{S} \le M^+ s \le s$, for any $s \in S^+$. In particular,

$$M^+ \underline{S} \le \underline{S}.$$

Then, $\underline{S} \in S^+$. Moreover, $M^+ \underline{S} \in S^+$ and in consequence, since $\underline{S}$ is the infimum in S^+,

$$M^+ \underline{S} \ge \underline{S}.$$

Since V^+ is a fixed point of M^+, it is clear that $V^+ \in S^+$ and that $V^+ \ge \underline{S}$. On the other hand, let $s \in S^+$, then by definition of S^+, given $t > 0$, we have $M_t^+ s \le s$.

Hence, for each $\alpha \in \Gamma(0,t)$,

$$\inf_{\xi\in\mathcal{Z}(0,t)} \max\left\{P_{0,t}(x,\xi,\alpha[\xi]), s(y_{0,x}^{\xi,\alpha[\xi]}(t))\right\} \le s(x). \tag{6}$$

This implies that given $\varepsilon > 0$, there exists $\xi_\varepsilon \in \mathcal{Z}(0,t)$ such that

$$\max\left\{P_{0,t}(x,\xi_\varepsilon,\alpha[\xi_\varepsilon]), s(y_{0,x}^{\xi_\varepsilon,\alpha[\xi_\varepsilon]}(t))\right\} \le s(x) + \varepsilon \tag{7}$$

Consider a partition of $[0,\infty)$:

$$[0,+\infty) = \bigcup_{\nu\in N} [t_\nu, t_{\nu+1}) \quad \text{where } t_\nu = t\nu.$$

Let $\varepsilon_\nu = \varepsilon/2^\nu$. Repeating the arguments in (6) and (7), we have for each $\alpha_\nu \in \Gamma(t_\nu, t_{\nu+1})$ the existence of $\xi_\nu \in \mathcal{Z}(t_\nu, t_{\nu+1})$ such that

$$\max\Big\{P_{t_\nu,t_\nu+1}(y_{0,x}^{\xi,\alpha[\xi]}(t_\nu),\xi_\nu,\alpha_\nu[\xi_\nu]), s(y_{0,x}^{\xi,\alpha[\xi]}(t_\nu+1))\Big\} \le s(y_{o,x}^{\xi,\alpha[\xi]}(t_\nu)) + \varepsilon_\nu.$$

We define recursively $\overline{\xi} \in \mathcal{Z}(0,\infty)$ by :

$$\begin{cases} \overline{\xi}(\tau) = \xi_\nu(\tau) \\ \alpha_\nu[\xi|_{[t_\nu,t_{\nu+1})}](\tau) = \alpha[\xi](\tau) \end{cases} \quad \tau \in [t_\nu, t_{\nu+1})$$

and write :

$$x_\nu = y_{0,x}^{\overline{\xi},\alpha[\overline{\xi}]}(t_\nu).$$

Then, it is clear that the trajectory generated by $(\overline{\xi}, \alpha[\overline{\xi}])$ verifies

$$x_{\nu+1} = x_\nu + \int_{t_\nu}^{t_{\nu+1}} f(y_{0,x}^{\overline{\xi},\alpha[\overline{\xi}]}(\tau),\xi_\nu(\tau),\alpha_\nu[\xi_\nu](\tau)\, d\tau$$

By induction it is easy to check that :

$$\max\left\{P_{0,t_\nu}(x,\overline{\xi},\alpha[\overline{\xi}]), s(x_\nu)\right\} \le s(x) + \sum_{n=0}^{\nu} \varepsilon_n.$$

Then,

$$P_{0,t_\nu}(x,\overline{\xi},\alpha[\overline{\xi}]) \le s(x) + \sum_{n=0}^{\nu} \varepsilon_n.$$

Taking the limit when ν goes towards infinity,

$$P_{0,t_\nu}(x,\overline{\xi},\alpha[\overline{\xi}]) \leq s(x)+2\varepsilon.$$

It follows that

$$\inf_{\xi\in\mathcal{Z}(0,\infty)} P(x,\xi,\alpha[\xi]) \leq s(x)+2\varepsilon$$

Now, for the same ε, choose $\alpha_\varepsilon \in \Gamma(0,\infty)$ such that

$$V^+(x) \leq \inf_{\xi\in\mathcal{Z}(0,\infty)} P(x,\xi,\alpha_\varepsilon)+\varepsilon.$$

As a consequence, we have :

$$V^+(x) \leq s(x)+3\varepsilon.$$

Taking finally the limit when ε goes to zero, we obtain :

$$V^+(x) \leq s(x).$$

These arguments are valid for any $s \in S^+$, and in particular for $\underline{S}$:

$$V^+(x) \leq \underline{S}(x), \quad \forall x \in \Omega.$$

Since $W^+ \in \mathcal{W}^+$, $\mathcal{W}^+ \neq \emptyset$. Let $w \in \{w \in W^+ \,|\, w \leq \underline{S}\}$. By Remark 5, it follows that $-M_h \leq \underline{S}$. By the monotonic property of M^+ (see Lemma 6), $M^+w \leq M^+\underline{S}$ and we have that $M^+w \leq M^+\underline{S} = \underline{S}$.
From this and the fact that $M^+w \in W^+$, it follows that :

$$M^+w \in \{w \in W^+ \,|\, w \leq \underline{S}\}.$$

Let $\{w_p\}_{p\in I} \subseteq \{w \in W^+ \,|\, w \leq \underline{S}\}$ then for any $p \in I$, it is valid that $w_p \leq \underline{S}$. In consequence, $\sup_{p\in I} w_p \leq \underline{S}$ and therefore,

$$\sup_{p\in I} w_p \in \{w \in W^+ \,|\, w \leq \underline{S}\}.$$

Thus, $\{w \in W^+ \,|\, w \leq \underline{S}\} \in \mathcal{W}^+$ and it is easy to check that $W_m^+ \in \mathcal{W}^+$. By condition (C3), $\overline{W} \in W_m^+$ and $\overline{W} \leq \underline{S}$. On the other hand, $\overline{W} \leq M^+\overline{W}$ and $M^+\overline{W} \in W_m^+$. Then, by definition of $\overline{W}$, we get $\overline{W} \geq M^+\overline{W}$. Therefore, $\overline{W} \in S^+$. By definition of $\underline{S}$, it results that $\overline{W} \geq \underline{S}$.
So, we conclude that $V^+ = \underline{S} = \overline{W}$. □

4. FINITE TIME APPROXIMATIONS

For $(t,x) \in [0,T]\times\Omega$, define the value functions :

$$V_T^+(t,x) = \sup_{\alpha\in\Gamma(t,T)} \inf_{\xi\in\mathcal{Z}(t,T)} P_{t,T}(x,\xi,\alpha[\xi])$$
$$V_T^-(t,x) = \inf_{\beta\in\Delta(t,T)} \sup_{\eta\in\mathcal{Y}(t,T)} P_{t,T}(x,\beta[\eta],\eta)$$

It is clear that $\{V_T^\pm\}_{T>0}$ are bounded non decreasing sequences which admit limits when $T \to +\infty$:

$$\overline{V}^\pm(x) = \lim_{T\to+\infty} V_T^\pm(0,x) \leq V^\pm(x).$$

Remark 10. As it can be seen in the example of section 5, $\overline{V}^\pm$, which is clearly lower semi-continuous, can be different from $V^\pm$ (which is not necessarily semi-continuous, see examples in (DiMarco and Gonzalez, 1998)).

To provide a method to approximate $\overline{V}^+$, we consider now the particular case where the instantaneous cost does not depend explicitly on the controls. Fix an integer $N > 0$ and define the operators :

$$\widetilde{M}_{t,N}^+ w(x) = \sup_{\alpha\in\Gamma(0,t)} \inf_{\xi\in\mathcal{Z}(0,t)} \max\left\{ \max_{i=1,\dots,N} h(y_{0,x}^{\xi,\alpha[\xi]}(it/N)),\ w(y_{0,x}^{\xi,\alpha[\xi]}(t)) \right\}$$
$$\widetilde{M}_{t/N}^+ w(x) = \sup_{\alpha\in\Gamma(0,t/N)} \inf_{\xi\in\mathcal{Z}(0,t/N)} \max\left\{ h(y_{0,x}^{\xi,\alpha[\xi]}(t/N)),\ w(y_{0,x}^{\xi,\alpha[\xi]}(t/N)) \right\}.$$

Then, it is straightforward to prove the

Lemma 11. For any $w \in B(\Omega)$, we have :

$$\left|\widetilde{M}_{t,N}^+ w(x) - M_t^+ w(x)\right| \leq L_h M_f \frac{t}{N},\ \forall x \in \Omega,$$

$$\text{and}\ \left(\widetilde{M}_{t/N}^+\right)^N w = \widetilde{M}_{t,N}^+ w.$$

The main interest in considering this infinitesimal operator is that we have the following approximation :

Theorem 12.

$$\lim_{N\to\infty} \left(\widetilde{M}_{1/N}^+\right)^{N^2} h = \overline{V}^+.$$

PROOF. We have, for any $x \in \Omega$,

$$\left[\left(\widetilde{M}_{1/N}^+\right)^{N^2} h\right](x) = \sup_{\alpha\in\Gamma(0,N)} \inf_{\xi\in\mathcal{Z}(0,N)} \max_{i=1,\dots,N} h(y_{0,x}^{\xi,\alpha[\xi]}(t_i))$$
$$\leq \sup_{\alpha\in\Gamma(0,N)} \inf_{\xi\in\mathcal{Z}(0,N)} \sup_{\tau\in[0,N)} h(y_{0,x}^{\xi,\alpha[\xi]}(\tau)) + \frac{L_h M_f}{N}$$
$$= V_N^+(x) + L_h M_f/N$$

Taking the limit, we obtain

$$\lim_{N\to\infty} \left[\left(\widetilde{M}_{1/N}^+\right)^{N^2} h\right](x) = \overline{V}^+(x). \qquad \square$$

This leads to the following algorithm :

$$w_0(.) = h(.)$$
$$w_n(.) = U_N w_{n-1}(0,.), \quad n \in [1, N^2]$$

where $U_N w$ is the unique viscosity solution in $BUC([0,1/N] \times \Omega)$ of the Isaacs equation :

$$\begin{cases} \dfrac{\partial U_N w}{\partial t}(t,x) + \min_{z \in Z} \max_{\eta \in Y} \nabla_x U_N w . f(x,z,\eta) = 0, \\ U_N w(1/N, x) = \max\{h(x), w(x)\}, \end{cases} \tag{8}$$

when $w \in BUC([0,1/N] \times \Omega)$. Then, for large N

$$w_{N^2} = \left(\widetilde{M}_{1/N}\right)^{N^2} h$$

provides an approximation scheme of the (semi-continuous) function $\overline{V}^+$.

Remark 13. The iterations of the operator $\widetilde{M}_{1/N}$ can be computed using known numerical schemes for Lipschitz solutions of standard Isaacs equations (see for instance Appendix A of (Bardi and Capuzzo-Dolcetta, 1997)).

5. AN EXAMPLE

We study an example, for which the value function and the iterations of the operator $\widetilde{M}_{1/N}$ can be determined analytically.

Consider the robust control of a second order servo-mechanism (see the study in (Bernhard *et al.*, 2001)) :

$$\begin{cases} \dot{x} = f(x,u,v) = \begin{pmatrix} x_2 - \alpha u \\ \beta v \end{pmatrix} & \begin{matrix} |u| \le 1 \\ |v| \le 1 \end{matrix} \\ x(0) = x_0 \end{cases}$$

where the criterion is :

$$J(x_0, u(.), v(.)) = \inf_{t \ge 0} h(x(t))$$

with $h(x) = \gamma - |x_1|$. The parameters α, β and γ are positive numbers. The Hamiltonian of the system is :

$$\begin{aligned} H(x,\lambda) &= \min_u \max_v \lambda f(x,u,v) \\ &= \max_v \min_u \lambda f(x,u,v) \\ &= -\alpha|\lambda_1| + \beta|\lambda_2| + \lambda_1 x_2. \end{aligned}$$

It has been proved in (Bernhard *et al.*, 2001) that this game admits a value :

$$V(x) = V^-(x) = V^+(x) =$$

$$\left| \begin{array}{ll} \min(\gamma + x_1, P^+(x)) & \text{when } x_2 \ge \alpha \\ \min(\gamma - x_1, P^-(x)) & \text{when } x_2 \le -\alpha \\ \min(\gamma - \alpha^2/\beta, P^+(x), P^-(x)) & \text{when } |x_2| \le \alpha \end{array} \right.$$

$$\text{with } \begin{cases} P^+(x) = \gamma - x_1 - \dfrac{(x_2+\alpha)^2}{2\beta} \\ P^-(x) = \gamma + x_1 - \dfrac{(x_2-\alpha)^2}{2\beta} \end{cases}$$

which is a continuous viscosity solution of the variationnal inequality

$$\min\left(h(x) - V(x)\,,\, H(x, \nabla V(x)) \right) = 0 \tag{9}$$

Notice that the uniqueness of continuous viscosity solutions is not guaranteed for such p.d.e. (see (Rapaport, 1998) for counter-examples).
It can be easily shown recursively that the unique solution $U_N w_n$ of the Isaacs equation (8) is :

$$U_N w_n(t,x) = \min\left(\gamma - \frac{\alpha^2}{2\beta}\,,\, \min_{i=1\cdots n} \phi\left(\frac{i}{N}, t, x\right)\right)$$

where $\phi(\tau, t, x) =$

$$\gamma - |x_1 + x_2(\tau - t)| - \alpha(\tau - t) + \frac{\beta}{2}(\tau - t)^2.$$

After some calculations, we obtain :

$$\begin{aligned} w_{N^2}(x) = \min\Bigg(& \gamma - \frac{\alpha^2}{2\beta}\,, \\ & \min_{i^+=1\cdots N^2} P^+(x) + \frac{(\beta i^+/N - \alpha - x_2)^2}{2\beta}\,, \\ & \min_{i^-=1\cdots N^2} P^-(x) + \frac{(\beta i^-/N - \alpha + x_2)^2}{2\beta} \Bigg). \end{aligned}$$

and then

$$\overline{V}(x) = \lim_{N \to +\infty} w_{N^2}(x) =$$
$$\left| \begin{array}{l} \min(\gamma + x_1, P^+(x)) \text{ when } x_2 \ge \alpha \\ \min(\gamma - x_1, P^-(x)) \text{ when } x_2 \le -\alpha \\ \min(P^+(x), P^-(x)) \text{ when } |x_2| \le \alpha \end{array} \right.$$

We see on this example that $\overline{V}$ does not coincide with V. Although $\overline{V}$ is continuous, it can be also checked that it is not a viscosity solution of (9).

6. CONCLUSION

According to the example of section 5, we conclude that, in general, the infinite horizon game problem cannot be approximated by a sequence of finite horizon game problems. This leads to the following open questions :

(1) Under which conditions we have $\overline{V}^+ = V^+$ and $\overline{V}^- = V^-$?
(2) The uniqueness of (generalized) solution of the Isaacs equation is a classic tool to prove that $V^+ \ge V^-$, in differential games (see for instance (Bardi and Capuzzo-Dolcetta, 1997)). Can we prove it directly for maximum

cost differential games, or does there exist an example, for which this inequality is not true (in infinite horizon) ?

(3) Under which conditions we have in the infinite horizon case $V^+ = V^-$? Can we find an example where the Isaacs condition is satisfied but with $V^+ \neq V^-$?

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