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ON THE LOCAL SENSITIVITY OF THE DISCRETE-TIME H^∞ ESTIMATION PROBLEM

N.D. Christov [*] **M. Najim** [**] **E. Grivel** [**] **D. Henry** [***]

[*] *Department of Automatics, Technical University of Sofia, 8 Kliment Ohridski Blvd., 1000 Sofia, Bulgaria*
[**] *Equipe Signal et Image, UMR LAP, ENSEIRB, 1 avenue du Docteur Albert Schweitzer, 33402 Talence, France*
[***] *ARIA, UMR LAP, 351 cours de la libération, 33405 Talence, France*

Abstract: The paper deals with the local sensitivity analysis of the discrete-time infinite-horizon H^∞ estimation problem. Both linear and nonlinear perturbation bounds are derived for the solution of the matrix Riccati equation that determines the sensitivity of the problem.

Keywords: Estimation algorithms, Perturbation analysis, Condition numbers.

1. INTRODUCTION

In the last 15 years, considerable research has been done in the field of H^∞ estimation, see for instance (Hassibi *et al.*, 1999) and the references therein. Important theoretical results have been obtained for both continuous-time and discrete-time systems. However, the computational aspects of the H^∞ estimation have not been studied in a sufficient extent. For this reason, designing and implementing H^∞ filters often lead to serious difficulties. This is due to the use of unreliable computational tools and to the lack of sensitivity estimates for the H^∞ estimation problem.

This paper presents a local perturbation analysis of the matrix Riccati equation that determines the sensitivity of the discrete-time infinite-horizon H^∞ estimation problem. Using the approach developed in (Konstantinov *et al.*, 1987, 1999a, b), linear perturbation bounds for this equation are first derived in terms of condition numbers relative to perturbation in the data. Then, a first order homogeneous perturbation bound is obtained, which in general is tighter than the condition number based perturbation bounds.

The following notations are used later on: $\mathcal{R}^{m\times n}$ – the space of real $m\times n$ matrices; I_n – the unit $n\times n$ matrix; $A^\top = [a_{ji}]$ – the transpose of the matrix $A = [a_{ij}]$; $\mathrm{vec}(A) \in \mathcal{R}^{mn}$ – the column-wise vector representation of the matrix $A \in \mathcal{R}^{m\times n}$; $\Pi_{n^2} \in \mathcal{R}^{n^2\times n^2}$ – the vec-permutation matrix such that $\mathrm{vec}(X^\top) = \Pi\mathrm{vec}(X)$ for all $X \in \mathcal{R}^{n\times n}$; $A\otimes B = [a_{ij}B]$ – the Kronecker product of the matrices A and B; $\|\cdot\|_2$ – the spectral (or 2-) norm in $\mathcal{R}^{m\times n}$; $\|.\|_{\mathrm{F}}$ – the Frobenius (or F-) norm in $\mathcal{R}^{m\times n}$. The notation ':=' stands for 'equal by definition'.

2. PROBLEM STATEMENT

Consider the linear discrete-time system

$$\begin{cases} x_{i+1} &= Fx_i + Gu_i \\ y_i &= Hx_i + v_i, \quad i \geq 0 \\ s_i &= Lx_i \end{cases} \tag{1}$$

where $x_i \in \mathcal{R}^n$, $y_i \in \mathcal{R}^p$, $u_i \in \mathcal{R}^m$ and $v_i \in \mathcal{R}^p$ are respectively the state, observation and disturbance vectors, $s_i \in \mathcal{R}^q$ is linear combination of the state to be estimated and F, G, H, L are known constant distribution matrices. The pairs (F, H) and (F, G) are assumed to be respectively detectable and controllable on the unit circle.

Given the system (1) and a constant $\gamma > 0$, the infinite-horizon H_∞ filtering problem consists in finding a stable estimator (filter) for s_i, achieving

$$\|T(z)\|_\infty < \gamma$$

where $T(z)$ is the transfer matrix relating the disturbances $\{u_i, v_i\}$ to the estimation errors $\{s_i - \hat{s}_{i|i}\}$.

Here $\hat{s}_{i|i}$ denotes the estimation of s_i using the observations $\{y_k\}_{k=1,\dots,i}$.

The so-called ”central solution” of this problem can be written as (Hassibi *et al.*, 1999)

$$\begin{aligned} \hat{x}_{i+1|i} &= F\hat{x}_{i|i} + K_1(y_i - H\hat{x}_{i|i}) \\ \hat{s}_{i|i} &= L\hat{x}_{i|i} + K_2(y_i - H\hat{x}_{i|i}) \end{aligned} \tag{2}$$

where $\hat{x}_{i|i}$ denotes the estimation of x_i using the observations $\{y_k\}_{k=1,\dots,i}$. The gain matrices K_1 and K_2 are defined by

$$\begin{aligned} K_1 &= -FP_0H^\top(I_p + HP_0H^\top)^{-1} \\ K_2 &= LP_0H^\top(I_p + HP_0H^\top)^{-1} \end{aligned}$$

where $P_0 \geq 0$ is the stabilising solution of the discrete-time algebraic Riccati equation

$$P = FPF^\top + GG^\top - KRK^\top \tag{3}$$

with

$$K = FP[H^\top \; L^\top]R^{-1} \tag{4}$$

$$R = \begin{bmatrix} I_p & 0 \\ 0 & -\gamma^2 I_q \end{bmatrix} + \begin{bmatrix} H \\ L \end{bmatrix} P[H^\top \; L^\top]. \tag{5}$$

In the sequel we shall write equation (3) in the equivalent form

$$\bar{F}(P,D)PF^\top - P + GG^\top = 0 \tag{6}$$

where

$$\begin{aligned} \bar{F}(P,D) &= F - FPC^\top R^{-1}C \\ C &= [H^\top \; L^\top]^\top \\ D &= (F,C). \end{aligned} \tag{7}$$

Suppose that F, G, H, L in (6), (7) are subject to perturbations ΔF, ΔG, ΔH, ΔL. Then the perturbed equation is

$$\begin{aligned} &\bar{F}(P,D+\Delta D)P(F+\Delta F)^\top \\ &- P + (G+\Delta G)(G+\Delta G)^\top = 0 \end{aligned} \tag{8}$$

where $\Delta D = (\Delta F, \Delta C)$, $\Delta C = [\Delta H^\top \; \Delta L^\top]^\top$,

$$\begin{aligned} &\bar{F}(P,D+\Delta D) = (F+\Delta F) \\ &- (F+\Delta F)P(C+\Delta C)^\top R^{-1}(P,D+\Delta D) \\ &\quad \times (C+\Delta C) \end{aligned}$$

and

$$\begin{aligned} &R(P,D+\Delta D) = R(P,D) + \Delta CPC^\top \\ &+ CP\Delta C^\top + \Delta CP\Delta C^\top. \end{aligned}$$

Since the Fréchet derivative of the left-hand side of (6) in P at $P = P_0$ is invertible, the perturbed equation (8) has a unique solution $P = P_0 + \Delta P$ in the neighborhood of P_0.

Denote $\Delta_M = \|\Delta M\|_F$ the absolute perturbation of a matrix M and let $\Delta := [\Delta_F, \Delta_G, \Delta_C]^\top \in \mathcal{R}^3_+$.

The problem considered in this paper is to find first order local bounds of the type

$$\Delta_P \leq f(\Delta) + O(\|\Delta\|^2), \; \Delta \to 0, \tag{9}$$

for the perturbation $\Delta_P := \|\Delta P\|_{\mathrm{F}}$ in the solution of the Riccati equation (6). Local linear bounds

$$\Delta_P \leq K_F\Delta_F + K_G\Delta_G + K_C\Delta_C + \mathrm{O}(\|\Delta\|^2)$$

and

$$\Delta_P \leq \sqrt{3}K_{\mathcal{R}}\Delta_{\max} + \mathrm{O}(\|\Delta\|^2)$$

shall be first obtained, where K_F, K_G and K_C are the individual condition numbers of (6), $K_{\mathcal{R}}$ is the overall condition number of (6) and $\Delta_{\max} = \max\{\Delta_F, \Delta_G, \Delta_C\}$. Then, a tighter perturbation bound of type (9) will be derived, where f is not a linear but a first order homogeneous function of Δ.

3. CONDITION NUMBERS

Denote $\Phi(P,D)$ the left-hand side of the Riccati equation (6). Then

$$\Phi(P_0, D) = 0. \tag{10}$$

Setting $P = P_0 + \Delta P$, the perturbed equation (8) may be written as

$$\begin{aligned} &\Phi(P_0 + \Delta P, D + \Delta D) = \\ &\Phi(P_0, D) + \Phi_P(\Delta P) + \Phi_F(\Delta F) + \Phi_G(\Delta G) \\ &+ \Phi_C(\Delta C) + S(\Delta P, \Delta D) = 0 \end{aligned} \tag{11}$$

where $\Phi_P(.)$,$\Phi_F(.)$ and $\Phi_G(.)$,$\Phi_C(.)$ are the Fréchet derivatives of $\Phi(P,D)$ in the corresponding matrix arguments, evaluated for $P = P_0$, and $S(\Delta P, \Delta D)$ contains the second and higher order terms in ΔP, ΔD.

It can be shown that

$$\begin{aligned} \Phi_P(Z) &= \bar{F}_0Z\bar{F}_0^\top - Z \\ \Phi_F(Z) &= \bar{F}_0P_0Z + Z^\top P_0\bar{F}_0^\top \\ \Phi_G(Z) &= GZ + Z^\top G^\top \\ \Phi_C(Z) &= -\bar{F}_0P_0ZR_0^{-1}CP_0F^\top \\ &\quad -FP_0^\top R_0^{-1}Z^\top P_0\bar{F}_0^\top \end{aligned} \tag{12}$$

where

$$\bar{F}_0 = \bar{F}(P_0, D), \quad R_0 = R(P_0, D).$$

It follows from (10), (11)

$$\begin{aligned} \Phi_P(\Delta P) = &-\Phi_F(\Delta F) - \Phi_G(\Delta G) - \Phi_C(\Delta C) \\ &- S(\Delta P, \Delta D). \end{aligned} \tag{13}$$

Since $\bar{F}_0$ is stable, the operator $\Phi_P(.)$ is invertible and (13) yields

$$\Delta P = -\Phi_P^{-1}\circ\Phi_F(\Delta F) - \Phi_P^{-1}\circ\Phi_G(\Delta G) \quad (14)$$
$$-\Phi_P^{-1}\circ\Phi_C(\Delta C) - \Phi_P^{-1}(S(\Delta P,\Delta D)).$$

From relation (14) we obtain

$$\Delta_P \le K_F\Delta_F + K_G\Delta_G + K_C\Delta_C + \mathrm{O}(\|\Delta\|^2) \quad (15)$$

where

$$K_F = \|\Phi_P^{-1}\circ\Phi_F\|, \quad K_G = \|\Phi_P^{-1}\circ\Phi_G\|$$
$$K_C = \|\Phi_P^{-1}\circ\Phi_C\|. \quad (16)$$

Here $\|\,.\,\|$ is the induced norm in the corresponding space of linear operators.

Denote by $M_P \in \mathcal{R}^{n^2.n^2}$, $M_F \in \mathcal{R}^{n^2.n^2}$, $M_G \in \mathcal{R}^{n^2.n^2}$ and $M_C \in \mathcal{R}^{n^2.n^2}$ the matrix representations of the operators $\Phi_P(.)$, $\Phi_F(.)$, $\Phi_G(.)$ and $\Phi_C(.)$:

$$M_P = \bar F_0\otimes\bar F_0 - I_{n^2}$$
$$M_F = I_n\otimes\bar F_0P_0 + (\bar F_0P_0\otimes I_n)\Pi \quad (17)$$
$$M_G = I_n\otimes G + (G\otimes I_n)\Pi$$
$$M_C = -FP_0C^\top R_0^{-1}\otimes\bar F_0P_0$$
$$-(\bar F_0P_0\otimes FP_0C^\top R_0^{-1})\Pi$$

Here $\Pi \in \mathcal{R}^{n^2.n^2}$ denotes the permutation matrix such that $\mathrm{vec}(M^\top) = \Pi\mathrm{vec}(M)$ for each $M \in \mathcal{R}^{n\times n}$. Thus

$$K_F = \|M_P^{-1}M_F\|_2, \quad K_G = \|M_P^{-1}M_G\|_2$$
$$K_C = \|M_P^{-1}M_C\|_2. \quad (18)$$

From relation (14), we can also deduce that

$$\Delta_P \le \sqrt{3}\,K_{\mathcal{R}}\,\Delta_{\max} + \mathrm{O}(\|\Delta\|^2),\ \Delta\to 0 \quad (19)$$

where

$$K_{\mathcal{R}} = \|\,M_P^{-1}\,[\,M_F, M_G, M_C\,]\,\|_2 \quad (20)$$

4. FIRST ORDER HOMOGENEOUS ESTIMATE

The linear perturbation bounds (15) and (19) may eventually produce pessimistic results. At the same time it is possible to derive a local first order homogeneous bound which is tighter in general.

The operator equation (14) may be written in a vector form as

$$\mathrm{vec}(\Delta P) = N_1\mathrm{vec}(\Delta F) + N_2\mathrm{vec}(\Delta G) \quad (21)$$
$$+ N_3\mathrm{vec}(\Delta C) - M_P^{-1}\mathrm{vec}(S(\Delta P,\Delta D))$$

where

$$N_1 := -M_P^{-1}M_F,\ N_2 := -M_P^{-1}M_G,$$
$$N_3 := -M_P^{-1}M_C.$$

The linear bound (15), (18) is a corollary of (21):

$$\Delta_P = \|\Delta P\|_{\mathrm{F}} = \|\mathrm{vec}(\Delta P)\|_2$$
$$\le \mathrm{est}_1(\Delta,N) + \mathrm{O}(\|\Delta\|^2) \quad (22)$$
$$:= \|N_1\|_2\Delta_F + \|N_2\|_2\Delta_G + \|N_3\|_2\Delta_C$$
$$+ \mathrm{O}(\|\Delta\|^2)$$
$$= K_F\Delta_F + K_G\Delta_G + K_C\Delta_C$$
$$+ \mathrm{O}(\|\Delta\|^2), \quad \Delta\to 0$$

where $N := [N_1, N_2, N_3]$.

Relation (21) also gives

$$\Delta_P \le \mathrm{est}_2(\Delta,N) + \mathrm{O}(\|\Delta\|^2) \quad (23)$$
$$:= \|N\|_2\|\Delta\|_2 + \mathrm{O}(\|\Delta\|^2),\ \ \Delta\to 0.$$

The bounds $\mathrm{est}_1(\Delta,N)$ and $\mathrm{est}_2(\Delta,N)$ are alternative, i.e. which one is less depends on the particular value of Δ.

There is also a third bound, which is always less than or equal to $\mathrm{est}_1(\Delta,N)$. We have

$$\Delta_P \le \mathrm{est}_3(\Delta,N)\ +\ \mathrm{O}(\|\Delta\|^2) \quad (24)$$
$$:= \sqrt{\Delta^\top U(N)\Delta}\ +\ \mathrm{O}(\|\Delta\|^2),\ \ \Delta\to 0$$

where $U(N)$ is the 3×3 matrix with elements

$$u_{ij}(N) = \|N_i^\top N_j\|_2.$$

Since

$$\left\|N_i^\top N_j\right\|_2 \le \|N_i\|_2\|N_j\|_2$$

we obtain

$$\mathrm{est}_3(\Delta,N) \le \mathrm{est}_1(\Delta,N).$$

Hence we have the overall estimate

$$\Delta_P \le \mathrm{est}(\Delta,N) + \mathrm{O}(\|\Delta\|^2),\ \Delta\to 0 \quad (25)$$

where

$$\mathrm{est}(\Delta,N) := \min\{\mathrm{est}_2(\Delta,N), \mathrm{est}_3(\Delta,N)\}. \quad (26)$$

The local bound $\mathrm{est}(\Delta,N)$ in (25), (26) is a nonlinear first order homogeneous and piece-wise real analytic function in Δ.

5. NUMERICAL EXAMPLE

Consider a third order system of type (1) with matrices

$$F = VF_dV,\ \ G = VG_d,\ \ C = [H^\top\ L^\top]^\top = C_dV$$

and $p = 2$, $q = 1$, where

$$V = I_3 - 2vv^\top/3, \quad v = [1,1,1]^\top$$
$$F_d = \mathrm{diag}(-2,1,0), \quad G_d = C_d = \mathrm{diag}(1,1,1).$$

The perturbations considered in the data satisfy

$$\Delta F = V\Delta F_dV,\ \Delta G = V\Delta G_d,\ \Delta C = \Delta C_dV$$

where

$$\Delta F_d = \text{diag}(1,1,1) \times 10^{-i}$$

$$\Delta G_d = \text{diag}(10,10,1) \times 10^{-i}$$

$$\Delta C_d = \text{diag}(10,1,0) \times 10^{-i}$$

for $i = 10, 9, \ldots, 2$.

Note that for this problem the unperturbed and perturbed Riccati equations (6) and (8) have closed form solutions P_0 and $P_0 + \Delta P$.

The relative perturbations $\Delta_P / \|P_0\|_F$ in the solution are estimated by the linear bound (19) and the nonlinear homogeneous bound (25). The results obtained for $\gamma = 1.1$ and different values of i are shown in Table 1. The actual relative changes in the solution are closed to the quantities predicted by the local sensitivity analysis.

Table 1

i	$\Delta_P / \|P_0\|_F$	Est.(19)	Est.(25)
10	1.0×10^{-9}	2.0×10^{-8}	4.0×10^{-9}
9	1.0×10^{-8}	2.0×10^{-7}	4.0×10^{-8}
8	1.0×10^{-7}	2.0×10^{-6}	4.0×10^{-7}
7	1.0×10^{-6}	2.0×10^{-5}	4.0×10^{-6}
6	1.0×10^{-5}	2.0×10^{-4}	4.0×10^{-5}
5	1.0×10^{-4}	2.0×10^{-3}	4.0×10^{-4}
4	1.0×10^{-3}	2.0×10^{-2}	4.0×10^{-3}
3	1.0×10^{-2}	2.0×10^{-1}	4.0×10^{-2}
2	0.9×10^{-1}	2.0×10^{0}	4.0×10^{-1}

6. CONCLUSION

In this paper the local sensitivity of the discrete-time infinite-horizon H^∞ estimation problem is studied. Local linear and nonlinear perturbation bounds are obtained for the Riccari equation that determines the sensitivity of the problem. The linear bounds are derived in terms of condition numbers relative to data perturbations. The nonlinear perturbation bound is obtained by using a first order homogeneous function and is tighter than its linear counterparts.

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