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ELLIPSOIDAL ESTIMATION UNDER MODEL UNCERTAINTY

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Abstract: Ellipsoidal outer-bounding under model uncertainty is a natural extension of state estimation for models with unknown-but-bounded errors. The technique described in this paper applies to linear discrete-time dynamic systems. Many difficulties arise because of the non-convexity of feasible sets. Analytical optimal or suboptimal solutions are presented, which are counterparts in this context of uncertainty to classical approximations of the sum and intersection of ellipsoids. *Copyright © 2002 IFAC*

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1. INTRODUCTION

In the literature, most parameter or state estimation problems are solved via a stochastic approach, with the noise assumed to be random. Kalman filtering is the typical technique for such an approach. Often, however, the underlying probabilistic assumptions are not realistic (the main perturbation may for instance be deterministic). It is then more natural to assume that these perturbations are unknown-but-bounded and to characterize the set of all values of the parameter or state vector that are consistent with this hypothesis. This corresponds to *guaranteed estimation*, first considered in the early seventies (Schweppe, 1973; Bertsekas and Rhodes, 1971). Russian researchers A.B. Kurzhanskii and F.L. Chernousko further developed ellipsoidal techniques for guaranteed parameter and state estimation (Kurzhanskii, 1977; Chernousko, 1994; Kurzhanskii and Valyi, 1996). Important contributions have been presented in (Fogel and Huang, 1982), in the context of parameter estimation. At present, the theory of guaranteed estimation is a well developed and mature area of control theory, see, e.g. the survey monograph (Milanese *et al.*, 1996), special issues of journals (Norton, 1994, 1995; Walter, 1990), chapters in the book (Walter and Pronzato, 1997) and the references therein. The most recent results can be found in (Durieu *et al.*, 2001).

However, most of the above mentioned works deal with problems where the plant model (its structure, in the case of parameter estimation) is assumed to be precisely known and where all the uncertainty relates to external perturbations and measurement noise. This assumption seems unrealistic for most real-life problems. The lack of precise information is the fundamental paradigm of modern control theory, where the concept of robustness plays a key role. The goal of this paper is to develop a robust approach to guaranteed estimation, i.e. to find methods to take into account unavoidable but specified model uncertainty with ellipsoidal techniques.

Consider a linear discrete-time dynamic system with the state equation

$$x_{k+1} = A_k x_k + w_k \tag{1}$$

and the measurement equation

$$y_k = C_k x_k + v_k. \tag{2}$$

A sequence of known inputs u_k could easily be incorporated in the state equation (1). It is not introduced for the sake of notational simplicity. The classical unknown-but-bounded approach (Schweppe, 1973;

Bertsekas and Rhodes, 1971; Kurzhanskii, 1977; Chernousko, 1994; Kurzhanskii and Valyi, 1996; Fogel and Huang, 1982; Milanese *et al.*, 1996; Norton, 1994, 1995; Walter, 1990; Walter and Pronzato, 1997; Durieu *et al.*, 2001) is based on the assumption that the matrices A_k, C_k are known while the external perturbation vector w_k and measurement noise vector v_k are subjected to the constraints

$$||w_k|| \le \alpha_k, \quad ||v_k|| \le \beta_k, \tag{3}$$

with α_k, β_k known. The problem is then to find a guaranteed estimate for x_k provided that measurements $y_1, \dots, y_k$ and some prior information relating to x_0 are available. Particular cases are: estimation of the attainability set (without measurements: $C_k \equiv 0$, $v_k \equiv 0$), parameter estimation (without dynamics: $A_k \equiv I$, $w_k \equiv 0, x_k \equiv \theta$) and parameter tracking ($A_k \equiv I, x_k = \theta_k$). The traditional technique is recursively to compute ellipsoids guaranteed to contain x_k.

The present research deals with a more general problem where in addition to state perturbation and measurement uncertainty (3) the model matrices are also uncertain:

$$A_k \in \mathscr{A}_k, \quad C_k \in \mathscr{C}_k, \tag{4}$$

where $\mathscr{A}_k, \mathscr{C}_k$ are some classes of matrices. Particular cases of the general problem have been considered in the literature (Clement and Gentil, 1990; Cerone, 1993; Kurzhanskii and Valyi, 1996; Chernousko, 1996; Rokityanskii, 1997; Norton, 1999; Chernousko and Rokityanskii, 2000). Serious difficulties have been recognized. For instance, consider a class of uncertainty described by interval matrices. An interval matrix A with entries a_{ij} is given by a nominal matrix A^0 with entries a^0_{ij} and by associated ranges α_{ij}:

$$\mathscr{A}_{int} = \{A:\ |a_{ij} - a^0_{ij}| \le \alpha_{ij} \quad \forall i,j\}. \tag{5}$$

Then the simplest attainability set for one step with x_0 fixed:

$$X_1 = \{x_1 = Ax_0 :\ A \in \mathscr{A}_{int}\} \tag{6}$$

is already not convex and its detailed description meets combinatorial difficulties for large dimensions.

We employ another model of uncertainty, which simplifies the analysis. It is assumed that the matrix uncertainty is combined with the uncertainty due to state perturbations and measurement noise by ellipsoidal constraints:

$$\frac{||A_k - A^0_k||^2}{\varepsilon_A^2} + \frac{||w_k||^2}{\delta_w^2} \le 1, \tag{7}$$

$$\frac{||C_k - C^0_k||^2}{\varepsilon_C^2} + \frac{||v_k||^2}{\delta_v^2} \le 1, \tag{8}$$

where A^0_k, C^0_k are nominal matrices, while ε_A, δ_w and ε_C, δ_v are prespecified weights. In (7, 8) and below, the vector norm $||x||$, $x \in R^n$, is understood as Euclidean: $||x||^2 = \sum x_i^2$, while the operator norm is used for matrices: for $A \in R^{m\times n}$, $||A|| = \max_{||x||\le 1} ||Ax|| = \max(eig(A^TA))^{1/2}$. Similar models arise in other problems related to systems under uncertainty, such as total least squares (Golub and Van Loan, 1980; El Ghaoui and Lebret, 1997) and robust optimization (Ben-Tal *et al.*, 2000; El Ghaoui and Calafiore, 1999). It can be proved that the set of all states x_k consistent with a given numerical value of the data vector y_k is described by a quadratic constraint with indefinite matrix. A technique to treat such constraints is developed in (Polyak, 1998); thus the minimal ellipsoid containing the intersection of this set with an ellipsoid can be constructed effectively. Such an approach provides an opportunity to extend ellipsoidal outer-bounding techniques to uncertain models; this is the main contribution of the paper.

2. PRELIMINARIES

The notation $P > 0$ ($P \ge 0$) for a matrix $P = P^T$ means that P is positive definite (nonnegative definite). An ellipsoid is denoted by

$$E(c,P) = \{x:\ (x-c)^T P(x-c) \le 1\}, \tag{9}$$

where the vector $c \in R^n$ is the center of the ellipsoid and the matrix $P \ge 0$ characterizes its shape and size.

A matrix H and a vector w are said to be admissible (for given values of ε and δ) if

$$\frac{||H||^2}{\varepsilon^2} + \frac{||w||^2}{\delta^2} \le 1. \tag{10}$$

Each of the inequalities (7), (8) can obviously be expressed in the form of (10). The set of all admissible pairs will be denoted by S. In (10), $\varepsilon = 0$ is understood as $H = 0$ and $||w|| \le \delta$, while $\delta = 0$ means $w = 0$ and $||H|| \le \varepsilon$.

The following simple assertions will be used in the paper.

Lemma 1. For any given $x \in R^n$, the set of points $Hx + w$ for all admissible H and w is a ball:

$$\{z = Hx + w:\ H, w \in S\} = \tag{11}$$

$$\{z:\ ||z||^2 \le \varepsilon^2 ||x||^2 + \delta^2\}. \tag{12}$$

Lemma 2. Assume $B_i \in R^{n\times n}$, $B_i > 0$, $i = 0,1,2$ and there exist $\tau_1 > 0$, $\tau_2 > 0$, $\tau_1 + \tau_2 \le 1$ such that

$$\begin{pmatrix} B_0 - \tau_1 B_1 & B_0 \\ B_0 & B_0 - \tau_2 B_2 \end{pmatrix} \le 0. \tag{13}$$

Then

$$B_0^{-1} \ge \tau_1^{-1} B_1^{-1} + \tau_2^{-1} B_2^{-1}. \tag{14}$$

Lemma 3. (Schweppe, 1973). Consider two quadratic functions $f_i(x) = (x - c_i)^T P_i (x - c_i)$, $i = 0,1$, $P_0 > 0$ and their weighted sum $f_\tau(x) = (1-\tau) f_0(x) + \tau f_1(x)$, $0 \le \tau \le 1$. Then the set $E = \{x:\ f_\tau(x) \le 1\}$ is an ellipsoid $E(c,P)$ with

$$
\begin{aligned}
P &= (1-\nu)^{-1}P_\tau, \\
P_\tau &= (1-\tau)P_0 + \tau P_1, \\
c &= P_\tau^{-1}[(1-\tau)P_0c_0 + \tau P_1c_1], \\
\nu &= (1-\tau)c_0^TP_0c_0 + \tau c_1^TP_1c_1 - c^TP_\tau c,
\end{aligned} \tag{15}
$$

provided that $P_\tau > 0$.

Note that we do not assume that P_1 is positive definite. Note also that if $c_0 = c_1 = 0$, then $c = 0$, $\nu = 0$, $P = P_\tau$.

The so called S-procedure is a well-known tool in system and control applications (Boyd *et al.*, 1994); it has been introduced by Yakubovich at the end of the sixties. We need the following version of it. Given two quadratic forms $f_i(x) = x^TA_ix$, $i = 1,2$ in R^N and real numbers α_i, $i = 1,2$, the problem is then to characterize all quadratic forms $f_0(x) = x^TA_0x$ in R^N and real numbers α_0 such that

$$
\left.\begin{array}{l} f_1(x) \le \alpha_1 \\ f_2(x) \le \alpha_2 \end{array}\right\} \Rightarrow f_0(x) \le \alpha_0. \tag{16}
$$

To say it another way, the problem is to describe quadratic forms such that $x \in E_1 \cap E_2$ implies $x \in E_0$, where $E_i = \{x : f_i(x) \le \alpha_i\}$, $i = 0,1,2$. The matrices A_i are not required to be positive definite, thus the sets E_i are not necessarily ellipsoids. Taking the weighted sum of f_1 and f_2 (with weights $\tau_1 \ge 0$ and $\tau_2 \ge 0$), we obtain an obvious sufficient condition for (16) to be satisfied:

$$
A_0 \le \tau_1A_1 + \tau_2A_2 \tag{17}
$$

and

$$
\alpha_0 \ge \tau_1\alpha_1 + \tau_2\alpha_2. \tag{18}
$$

More interesting is that under some mild assumptions this sufficient condition is also necessary, as indicated by the following lemma.

Lemma 4. (Polyak, 1998). Suppose $N \ge 3$ and there exist $\mu_1, \mu_2 \in R$, $x^0 \in R^N$ such that

$$
\mu_1A_1 + \mu_2A_2 > 0, \tag{19}
$$

$$
f_1(x^0) < \alpha_1, \quad f_2(x^0) < \alpha_2. \tag{20}
$$

Then (16) holds if and only if there exist $\tau_1 \ge 0$, $\tau_2 \ge 0$ such that the inequalities (17) and (18) are satisfied.

3. ELLIPSOIDAL STATE ESTIMATION

This section provides the main parts of the technique for state estimation; the detailed algorithm for the recursive estimation of the state of the system (1, 2) subject to the constraints (7, 8) can be easily constructed from these blocks.

3.1 *Approximation of sum*

Suppose no observations take place, so only the state equation (1) of the dynamic system is available. Assume the initial state vector $x_0 \in R^n$, $n \ge 2$ belongs to an ellipsoid E_0. The problem is recursively to estimate the sequence of attainability sets by outer-bounding ellipsoids. We treat one step of such a procedure.

Consider

$$
z = (A+H)x + w, \tag{21}
$$

where $x \in R^n$ and $A \in R^{n\times n}$ is a known matrix. We are interested in the set F of all such points z, when x lies in a non-degenerate ellipsoid $E(c,P)$, $P > 0$, while H, w is an admissible pair. We assume $E(c,P)$ to be centered at the origin, i.e. $c = 0$. This assumption is natural, if we deal with reachability problem with no measurements. Indeed, if $x_0 \in E(0,P_0)$, then $x_k \in E(0,P_k)$ due to the symmetry of the reachability set at each step. Thus:

$$
\begin{aligned}
F = \{(A+H)x + w : \\
x \in E(0,P),\ \frac{\|H\|^2}{\varepsilon^2} + \frac{\|w\|^2}{\delta^2} \le 1\}.
\end{aligned} \tag{22}
$$

This set is not an ellipsoid; in most cases it is not even convex (see example below). For the ellipsoidal technique to apply, it should be embedded in some ellipsoid $E(0,Q)$:

$$
F \subset E(0,Q). \tag{23}
$$

Moreover, we seek the ellipsoid with minimal size. The most natural objective functions are

$$
f_1(Q) = \mathrm{tr}\, Q^{-1}, \tag{24}
$$

$$
f_2(Q) = -\ln\det Q. \tag{25}
$$

Function $f_1(Q)$ is the sum of the squares of the ellipsoidal semi-axis (trace criterion) and $f_2(Q)$ relates to its volume (determinant criterion). Thus the problem is to minimize (24) or (25) subject to (23). The result below reduces the problem to one-dimensional optimization.

Theorem 1. Each ellipsoid in the family $E(0,Q(\tau))$ with

$$
\begin{aligned}
Q(\tau) = [A((1-\tau\delta^2)P - \tau\varepsilon^2I)^{-1}A^T + \\
\tau^{-1}I]^{-1}
\end{aligned} \tag{26}
$$

contains F for all τ such that

$$
0 < \tau < \tau^* = \frac{\lambda_{\min}}{\lambda_{\min}\delta^2 + \varepsilon^2}, \tag{27}
$$

where $\lambda_{\min}$ is the minimal eigenvalue of P.

Moreover, the minimization of the one-dimensional smooth and convex function $f_1(\tau) = \mathrm{tr}\, Q(\tau)^{-1}$ or $f_2(\tau) = -\ln\det Q(\tau)$ subject to (27) provides the minimal-trace or minimal-volume ellipsoid containing F.

Thus, finding the best ellipsoid containing F is reduced to a simple one-parameter optimization problem; this is the complete analog of the situation with no model uncertainty, where one-parametric optimization is required to construct the best ellipsoid for the prediction step, compare (Chernousko, 1994; Kurzhanskii and Valyi, 1996; Fogel and Huang, 1982; Durieu *et al.*, 2001; Polyak, 1998).

Some simpler approximations could also be considered. Chernousko and Rokityanskii (2000) proposed to obtain the ellipsoid $E(0,Q)$ as an approximation of the sum of $E(0,P)$ and a ball of radius r given by $E(0,r^{-2}I)$, $r=\varepsilon^2(\max_{x\in E(0,P)}||x||^2)+\delta^2$. Indeed, the sum of these two ellipsoids is already convex and contains F, and it is easy to construct an ellipsoid $\widetilde{Q}$ that contains this sum. The simplicity of such an approach is obvious. Moreover, it is valid if the initial ellipsoid is centered at any point in R^n. But, as illustrated by the next example, ellipsoidal estimation will then not be optimal among all ellipsoids (23).

Example 1. Let $P=\mathrm{diag}\{1/9,\ 1\}$, $A=I$ and $\varepsilon=1$, $\delta=0$. Then F is non-convex (see Fig. 1) and the ellipsoids $E(0,Q(\tau))$ with

$$Q(\tau)=\mathrm{diag}\{\tau(1-9\tau),\ \tau(1-\tau)\},\qquad (28)$$

$0<\tau<1/9$, contain F. In order to find the minimal ellipsoid according to (24) or (25), we need to solve a simple equation which has a unique root τ_{min} in the interval $(0,1/9)$. For example, for the trace criterion the optimal matrix is

$$Q(\tau_{\min})\simeq\mathrm{diag}\{2.69,\ 6.13\}\,10^{-2}.\qquad (29)$$

If, on the other hand, we seek an approximation of the sum of the ellipsoids $E(0,P)$ and $E(0,r^{-2}I)$, where $r=3$ in this case, then

$$\tilde{Q}(\gamma)=\mathrm{diag}\{\gamma(1-\gamma)/9,\ \gamma(1-\gamma)/(1+8\gamma)\},\qquad (30)$$

$0<\gamma<1$. The parameter γ can also be determined by direct calculation for the trace or determinant criterion. For the trace criterion, the optimal matrix is

$$\tilde{Q}(\gamma_{\min})\simeq\mathrm{diag}\{2.72,\ 5.52\}\,10^{-2}.\qquad (31)$$

The corresponding ellipsoid is only a suboptimal solution of the state estimation problem. Both ellipsoids are represented in Figure 1 (solid and pointed line). Note that the difference between them is small. But it becomes more significant when the algorithm is used recursively. □

3.2 *Approximation of intersection*

Consider now a linear dynamic system with measurements (2). An appropriate operation for the classical ellipsoidal state estimation at the correction step is the intersection of ellipsoids. In general, ellipsoids are considered as possibly degenerate, i.e. their matrices are only positive semi-definite. A scalar observation, for example, defines a set of possible locations of the state vector x which is a strip in R^n. The basic tool for the classical approximation of ellipsoid intersection is Lemma 3.

However, we assume here that the matrix in (2) is uncertain:

$$y=(C+H)x+w,\qquad (32)$$

where $x\in R^n$, $y\in R^m$, $C\in R^{m\times n}$ and H,w is an admissible pair. For a given vector of measurements y and the nominal matrix C, we need to estimate the set of all state vectors x that are consistent with the above data. According to Lemma 1

$$||y-Cx||^2\le\varepsilon^2||x||^2+\delta^2,\qquad (33)$$

or equivalently

$$x^T(C^TC-\varepsilon^2I)x-2x^TC^Ty+y^Ty-\delta^2 \le 0.\qquad (34)$$

Assume that $\mathrm{eig}\,C^TC\neq\varepsilon^2$, i.e. $C^TC-\varepsilon^2I$ is invertible. Then we can rewrite the last inequality in terms of a quadratic form:

$$(x-d)^TM(x-d)\le 1,\qquad (35)$$

where

$$M=\frac{R}{y^TCR^{-1}C^Ty-y^Ty+\delta^2},\quad d=R^{-1}C^Ty,\quad R=C^TC-\varepsilon^2I.\qquad (36)$$

The matrices R and M may not be positive or non-negative definite. Therefore, the set of all x that satisfy (35) is not necessarily an ellipsoid or a strip. It depends on the values of C and ε. Nevertheless, an ellipsoidal technique can be used to deal with the intersection of this set with some non-degenerate ellipsoid.

The main result of this section can now be stated.

Theorem 2. If x belongs to $E(c,P)$, $P>0$ and satisfies $y=(C+H)x+w$, where H,w is an admissible pair, then x also belongs to the ellipsoid $E(g,Q(\tau))$, with

$$\begin{aligned} Q(\tau)&=(1-\nu)^{-1}Q_\tau,\\ Q_\tau&=((1-\tau)P+\tau M),\\ g&=Q_\tau^{-1}((1-\tau)Pc+\tau Md),\\ \nu&=(1-\tau)c^TPc+\tau d^TMd-g^TQ_\tau g \end{aligned}\qquad (37)$$

for all τ such that $0\le\tau<\tau^*=\min\{1,1/(1-\lambda_{\min})\}$. Here $\lambda_{\min}$ is the minimal generalized eigenvalue of the matrix pair (M,P) (the generalized eigenvalues λ_i and eigenvectors v_i of the matrix pair (M,P) are defined as $Mv_i=\lambda_iPv_i$).

One-dimensional functions $f_1(\tau)=\mathrm{tr}\,Q(\tau)^{-1}$ and $f_2(\tau)=-\ln\det Q(\tau)$ are smooth and convex on the interval $0\le\tau<\tau^*$. Optimizing $E(g,Q(\tau))$ with respect to τ with (24) or (25) gives the optimal ellipsoid

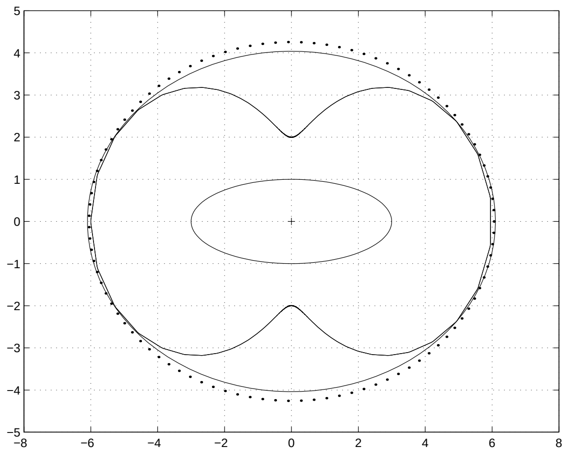

Fig. 1. Attainability set

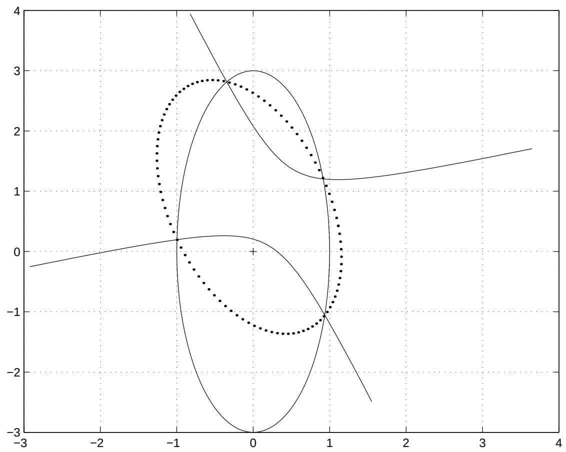

Fig. 2. Intersection

in this parametrized family containing all admissible points. However, in contrast with the solution for the prediction step (Theorem 1) it can be a suboptimal estimate in the class of all ellipsoids. It will be optimal only if the centers c and d of the sets become equal; an assumption that is not as natural as for the prediction step.

To illustrate specific features of measurements with an uncertain observation matrix, a scalar output, i.e. $m = 1$, is considered. In this case an admissible vector x (for fixed y) lies inside an hyperboloid in R^n.

Example 2. Take $m = 1$, $n = 2$, $y = 1$, $C = (1, 2)$, $\varepsilon = 1.5$ and $\delta = 0.5$. Assume a non-degenerate prior ellipsoid $E(c,P)$, with $c = 0$, $P = \mathrm{diag}\{1, 1/9\}$. The two eigenvalues of R as defined by (36) have different signs, so neither R nor M is positive or non-negative definite. We calculate $\lambda_{\min} = \min \mathrm{eig}(M,P) = -2.953$ and $\tau^* = \min\{1,\ 1/(1-\lambda_{\min})\} = 0.253 < 1$. From Theorem 2, the one-parametric family of ellipsoids with matrices $Q(\tau)$ and centers $g(\tau) \in R^n$ contains the intersection for all τ such that $0 \le \tau < \tau^*$. The minimal ellipsoid of this family could be calculated in terms of the trace or determinant criteria. Figure 2 shows the resulting approximation (for the trace criterion). □

4. CONCLUSIONS

In the present paper, an outer-bounding ellipsoidal technique for the estimation of the state of a linear dynamic system under model uncertainty (7, 8) has been proposed. As usual, two operations have been considered, which are at the core of construction of this state estimation. The first one is some generalized sum of ellipsoids and the second is an intersection. The combined quadratic constraints for uncertain model matrix and disturbances lead to one-step optimal estimates for systems with no measurements, while the correction step based on similar principles is suboptimal. We have addressed just one prediction step and one correction step. However, using the above techniques, it is trivial to develop recursive version of state estimation for linear discrete-time dynamic systems (1) with measurements (2) under uncertainties of the form (7, 8).

It is of interest to extend the above results for more general models of uncertainty, for instance, for w_k replaced with Bw_k in (1), and more realistic constraints than (7, 8).

5. ACKNOWLEDGEMENTS

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