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OPTIMAL MODEL REFERENCE ADAPTIVE CONTROLLERS FOR SYSTEMS WITH INPUT NONLINEARITY

Chih-Chiang Cheng [1], **Chiu-Ju Teng**

Department of Electrical Engineering, National Sun Yat-Sen University, No. 70, Lien-Hai Road, Kaohsiung, 804, Taiwan, Republic of China

Abstract: Based on the Lyapunov stability theorem, an optimal model reference adaptive control (OMRAC) scheme designed for a class of multi-input (MI) systems with input non-linearity for solving robust tracking problems is presented in this paper. The proposed control scheme contains an optimal controller designed for nominal systems, and an adaptive mechanism used to automatically adapt the unknown upper bound of perturbation. The asymptotical stability of tracking error is guaranteed for the controlled system . A numerical example is given for demonstrating the feasibility of the proposed control scheme.

Keywords: model reference, adaptive control, tracking control, Lyapunov theorem

1. INTRODUCTION

Reference model has been widely used in the design of control systems. Various adaptive control algorithms using reference model have been developed for counteracting instability and improving robustness with respect to unmodeled dynamics and external disturbances. Among these algorithms are projection (Naik *et al.*, 1992), persistent excitation (Ioannou *et al.*, 1989), sign-following for the non-minimum phase systems (Feng *et al.*, 1996), as well as variable structure model reference adaptive control (VS-MRAC) (Hsu *et al.*, 1989). All the aforementional works only guarantee that the tracking error is bounded and small on average in the presence of unmodeled dynamics or bounded external disturbances. They do not guarantee that the tracking error either can approach to zero asymptotically or is confined within a sufficiently small region whose size can be tuned by control parameters.

It is also observed in practice that there do exist nonlinearities in the control input due to physical limitation, e.g., saturation, quantization, backlash, deadzone, etc., and time-delay term due to finite speed of information processing and mechanism of the plant. The effects of input non-linearity and time-delay frequently become a source of instability that cannot be ignored during the design of a control system. As a result, the control problem for systems with input non-linearity and time-delay argument has received considerable attentions by many authors in recent years. Hsu (1997; 1998a; 1998b) has proposed VSC schemes for uncertain dynamic systems with series nonlinearities for solving regulation prob-

[1] Corresponding author. Tel: +886-7-5252000 ext. 4133; fax: +886-7-5254199; E-Mail: chengcc@ee.nsysu.edu.tw

lems. These control schemes can possess insensitivity to matching uncertainties and disturbances. However, the information of upper bound of uncertainties is required. Sun *et al.* (1995; 1997) presented a composite feedback control for a class of uncertain nonlinear systems so that the property of global exponential stability can always be achieved. However, the information of the upper bound of uncertainties is still required.

In this paper the idea of Hsu(1998a; 1998b), Sue *et al.*(1997) is extended to design an OMRAC law for a class of perturbed linear systems with input non-linearity and time-delay argument so that the tracking errors of state variables can have the property of asymptotical stability and the adaptive gains can have pre-specified convergence rate. The proposed control scheme contains two types of controllers. One is a linear feedback optimal controller, the other is an adaptive state feedback controller, which will automatically adapt the gain needed to overcome the perturbations, so that the information of upper bound of perturbations is not required.

2. SYSTEM DESCRIPTION AND PROBLEM FORMULATION

Consider a class of perturbed multi-input linear dynamic systems with input nonlinearity. The dynamic equation of the plant is described by

$$\begin{aligned}\dot{\mathbf{x}}(t) &= [\mathbf{A} + \Delta\mathbf{A}(t,\mathbf{x})]\mathbf{x}(t) + \Delta\mathbf{A}_h\mathbf{x}(t-h(t)) \\ &\quad +[\mathbf{B} + \Delta\mathbf{B}(t,\mathbf{x})]\boldsymbol{\phi}(\mathbf{u}) \\ &\quad +\Delta\mathbf{f}(t,\mathbf{x}(t),\mathbf{x}(t-h(t))), \qquad (1) \\ \mathbf{x}(t) &= \boldsymbol{\theta}(t), \quad -H \le t \le 0,\end{aligned}$$

where $\mathbf{x}(t) \in R^n$ is the state of the plant, $\mathbf{u}(t) \in R^m$ is the control input. The constant matrices $\mathbf{A} \in R^{n\times n}$, $\mathbf{B} \in R^{n\times m}$ are known, whereas $\Delta\mathbf{A}(\cdot)$, $\Delta\mathbf{A}_h(\cdot)$, $\Delta\mathbf{B}(\cdot)$ are unknown real-valued matrix functions with appropriate dimensions representing time-varying model uncertainties. $\boldsymbol{\phi}(\mathbf{u})$, the unknown input non-linearity, is a continuous function, and $\Delta\mathbf{f}(\cdot)$ is the uncertain extraneous disturbance or nonlinearity of the plant. The pair $(\mathbf{A},\mathbf{B})$ is completely controllable, and $h(t)$ is the unknown delay with $0 \le h(t) \le H$, where H is also an unknown constant. The vector $\boldsymbol{\theta}(t)$ is given for specifying the initial condition. In addition, all the states of the plant are assumed to be available for measurement.

For achieving the design of model reference control system, a reference model is given by

$$\begin{aligned}\dot{\mathbf{x}}_m(t) &= \mathbf{A}_m\mathbf{x}_m(t) + \mathbf{B}_m\mathbf{r}(t) \qquad (2) \\ \mathbf{x}_m(0) &= \mathbf{x}_{m0},\end{aligned}$$

where $\mathbf{x}_m(t) \in R^n$ is the state of the reference model, $\mathbf{r}(t) \in R^r$ is piecewise continuous and bounded reference input. The constant matrices $\mathbf{A}_m \in R^{n\times n}$, $\mathbf{B}_m \in R^{n\times r}$ are known, and $\mathbf{A}_m$ is stable matrix.

In this paper, the matrices $\mathbf{A}^T$, $\mathbf{A}^{-1}$, and $\lambda(\mathbf{A})$ represent the matrix transpose, matrix inverse, and the eigenvalue of $\mathbf{A}$, respectively. $\|\mathbf{x}(t)\|$ stands for the Euclidean norm of a vector, and induced two-norm is used to compute a matrix norm $\|\mathbf{A}\|$. In order to design a model reference control system successfully, the following assumptions are also made:

A1. (Hsu, 1998a; 1998b; Sun *et al.*, 1997) The uncertain matrix functions $\Delta\mathbf{A}$, $\Delta\mathbf{A}_h$, $\Delta\mathbf{B}$, and $\Delta\mathbf{f}(\cdot)$ are continuous in $\mathbf{x}$ and piecewise continuous in t, there also exist continuous matrix functions $\Delta\bar{\mathbf{A}}$, $\Delta\bar{\mathbf{A}}_h$, $\bar{\mathbf{B}}_m$, and $\bar{\mathbf{f}}(\cdot)$ of appropriate dimensions such that the following matching conditions are satisfied:

$$\begin{aligned}&\mathbf{A}_m - \mathbf{A} - \Delta\mathbf{A} = \mathbf{B}\Delta\bar{\mathbf{A}}, \quad \Delta\mathbf{A}_h = \mathbf{B}\Delta\bar{\mathbf{A}}_h, \\ &\mathbf{B}_m = \mathbf{B}\bar{\mathbf{B}}_m, \quad \Delta\mathbf{B} = \mathbf{B}\Delta\bar{\mathbf{B}}, \quad \Delta\mathbf{f} = \mathbf{B}\Delta\bar{\mathbf{f}}\end{aligned}$$

Remark: (Corless, 1993) The assumption of matching condition will indeed restrict the application of the proposed control scheme, however, these matched uncertainties do appear frequently in many manipulator systems.

A2. (Hsu, 1998a; 1998b; Sun *et al.*, 1997) The unknown input non-linearity $\boldsymbol{\phi}(\cdot) : R^m \to R^m$ is any continuous operator satisfying

$$\gamma_1\mathbf{u}^T\mathbf{u} \le \mathbf{u}^T\boldsymbol{\phi}(\mathbf{u}), \quad \forall\ \mathbf{u} \in R^m,$$

where γ_1 is a positive number.

The tracking error $\mathbf{e}(t)$ is defined as

$$\mathbf{e}(t) \equiv \mathbf{x}_m(t) - \mathbf{x}(t). \qquad (3)$$

From (1)-(3) and assumption A1, the dynamic equation of tracking error can be derived as

$$\begin{aligned}\dot{\mathbf{e}}(t) &= \mathbf{A}_m\mathbf{e}(t) - \mathbf{B}\boldsymbol{\phi}(\mathbf{u}) \\ &\quad +\mathbf{B}[\Delta\boldsymbol{\psi}(t,\mathbf{x}(t),\mathbf{x}(t-h(t)))], \qquad (4)\end{aligned}$$

where

$$\begin{aligned}\Delta\boldsymbol{\psi}(\cdot) &\equiv \Delta\bar{\mathbf{A}}\mathbf{x}(t) + \bar{\mathbf{B}}_m\mathbf{r}(t) - \Delta\bar{\mathbf{A}}_h\mathbf{x}(t-h(t)) \\ &\quad -\Delta\bar{\mathbf{B}}\boldsymbol{\phi}(\mathbf{u}) - \Delta\bar{\mathbf{f}}(t,\mathbf{x}(t),\mathbf{x}(t-h(t))) \qquad (5)\end{aligned}$$

is the lumped perturbation of the system. Based on the knowledge of the bound on the lumped perturbation described in (5), the following assumption is introduced:

A3. (Wheeler *et al.*, 1998) There exists a function $\mathbf{w} \in R^{p+1}$ and an unknown constant vector $\mathbf{k} \in R^{p+1}$ such that for all $\mathbf{x} \in R^n$ and all $t \ge 0$

$$\|\Delta\boldsymbol{\psi}(t,\mathbf{x}(t),\mathbf{x}(t-h(t)))\| \le \mathbf{w}^T\mathbf{k}, \tag{6}$$

where

$$\mathbf{w} \equiv [1 \quad \|\mathbf{x}\| \quad \|\mathbf{x}\|^2 \quad \cdots \quad \|\mathbf{x}\|^p]^T,$$
$$\mathbf{k} \equiv [k_0 \quad k_1 \quad k_2 \quad \cdots \quad k_p]^T, \tag{7}$$

and $k_i, i = 0, 1, 2, \cdots, p$ are unknown positive constants.

The main objective of this paper is to design an optimal model reference adaptive control law based on Lyapunov stability theorem without the requirement of the information of upper bound of perturbation, the dynamic equation (4) will be stabilized in spite of the existence of perturbations.

3. OPTIMAL CONTROLLER DESIGN FOR THE NOMINAL SYSTEMS

The case when there is no perturbation and $\boldsymbol{\phi}(\mathbf{u}) = \gamma_1\mathbf{u}$ for deriving an optimal control effort for the dynamic equation (4) is considered first in this section. The nominal dynamic equation of (4) is represented as

$$\dot{\mathbf{e}}(t) = \mathbf{A}_m\mathbf{e}(t) - \gamma_1\mathbf{B}\mathbf{u}. \tag{8}$$

In order to derive an optimal control law for (8), a performance index is considered as

$$J(\mathbf{e}(t_0), \mathbf{u}(\cdot), t_0) = \int_{t_0}^{\infty} \exp(2\alpha t)(\mathbf{u}^T\mathbf{u} + \mathbf{e}^T\mathbf{Q}\mathbf{e})dt, \tag{9}$$

where $\mathbf{Q}$ is a positive definite symmetric constant matrix, and α is a nonnegative constant. Note that the pair $[(\mathbf{A}_m + \alpha\mathbf{I}), -\gamma_1\mathbf{B}]$ is completely stabilizable. The problem now is to find an optimal control law which can minimize the performance index function (9). Define

$$\hat{\mathbf{e}}(t) \equiv \exp(\alpha t)\mathbf{e}(t), \quad \hat{\mathbf{u}}(t) \equiv \exp(\alpha t)\mathbf{u}(t). \tag{10}$$

The performance index function (9) then can be rewritten as

$$\hat{J}(\hat{\mathbf{e}}(t_0), \hat{\mathbf{u}}(\cdot), t_0) = \int_{t_0}^{\infty} (\hat{\mathbf{u}}^T\hat{\mathbf{u}} + \hat{\mathbf{e}}^T\mathbf{Q}\hat{\mathbf{e}})dt. \tag{11}$$

Suppose that $\mathbf{u}^*(t)$ is the optimal control at time t which minimize (9). Then according to (10) the optimal control at time t which minimize (11) is $\hat{\mathbf{u}}^*(t) = \exp(\alpha t)\mathbf{u}^*(t)$, and the resultant value of tracking error at time t is given by $\hat{\mathbf{e}}(t) = \exp(\alpha t)\mathbf{e}(t)$. Now the following lemma is utilized to obtain the optimal control law for (11).

Lemma (Anderson *et al.*, 1989)**:** Suppose that there exists a control law $\mathbf{u}(t)$ that stabilizes the system

$$\dot{\mathbf{x}}(t) = \mathbf{F}\mathbf{x}(t) + \mathbf{G}\mathbf{u}, \quad \mathbf{x}(t_0) = \mathbf{x}_0$$

where the system is completely stabilizable. Then the optimal control law is uniquely given by

$$\mathbf{u}(t) = -\mathbf{R}^{-1}\mathbf{G}^T\mathbf{P}\mathbf{x}(t),$$

which minimizes the following performance index

$$J(\mathbf{x}(t_0), \mathbf{u}(\cdot), t_0) = \int_{t_0}^{\infty} (\mathbf{u}^T\mathbf{R}\mathbf{u} + \mathbf{x}^T\mathbf{Q}\mathbf{x})dt,$$

where $\mathbf{R}$ and $\mathbf{Q}$ are positive definite symmetric constant matrices, and $\mathbf{P} > 0$ is the unique solution of the Riccati equation given by

$$\mathbf{PF} + \mathbf{F}^T\mathbf{P} - \mathbf{PGR}^{-1}\mathbf{G}^T\mathbf{P} + \mathbf{Q} = \mathbf{0}. \qquad \square \tag{12}$$

Now differentiating (10) and using (8) yields

$$\dot{\hat{\mathbf{e}}}(t) = (\mathbf{A}_m + \alpha\mathbf{I})\hat{\mathbf{e}}(t) - \gamma_1\mathbf{B}\hat{\mathbf{u}}. \tag{13}$$

According to the previous Lemma, the optimal control law of the system (13) is

$$\hat{\mathbf{u}}^*(t) = \gamma_1\mathbf{B}^T\mathbf{P}\hat{\mathbf{e}}(t).$$

From (10) it is easy to see that the optimal control $\mathbf{u}^*(t)$ of the system (8) is

$$\mathbf{u}^*(t) = \exp(-\alpha t)\hat{\mathbf{u}}^*(t) = \gamma_1\mathbf{B}^T\mathbf{P}\mathbf{e}(t), \tag{14}$$

where the matrix $\mathbf{P}$ satisfies the following Riccati equation

$$\mathbf{P}(\mathbf{A}_m + \alpha\mathbf{I}) + (\mathbf{A}_m^T + \alpha\mathbf{I})\mathbf{P} - \gamma_1^2\mathbf{PBB}^T\mathbf{P} + \mathbf{Q} = \mathbf{0}. \tag{15}$$

4. DESIGN OF ROBUST OMRAC SCHEME

Since the nominal system (8) is subject to the perturbation $\Delta\boldsymbol{\psi}(\cdot)$ as shown in (5), the results of the previous section is utilized to design a robust optimal adaptive controller to stabilize the perturbed dynamic system (4). The controller proposed for system (1) is designed as

$$\mathbf{u}(t) = \mathbf{u}_1(\mathbf{e}(t)) + \mathbf{u}_2(\mathbf{e}(t)), \tag{16}$$

where

$$\mathbf{u}_1(\mathbf{e}(t)) = \gamma_1\mathbf{B}^T\mathbf{P}\mathbf{e}(t), \tag{17}$$
$$\mathbf{u}_2(\mathbf{e}(t)) = \gamma_1\beta_1(t)\mathbf{B}^T\mathbf{P}\mathbf{e}(t), \tag{18}$$
$$\beta_1(t) = \frac{1}{\gamma_1^2}\frac{(\mathbf{w}^T\hat{\mathbf{k}}(t))^2}{(\mathbf{w}^T\hat{\mathbf{k}}(t))\|\mathbf{B}^T\mathbf{P}\mathbf{e}(t)\| + \varepsilon}, \tag{19}$$
$$\frac{d}{dt}\hat{\mathbf{k}}(t) = -2\rho\boldsymbol{\Gamma}\hat{\mathbf{k}}(t) + \boldsymbol{\Gamma}\mathbf{w}\|\mathbf{B}^T\mathbf{P}\mathbf{e}(t)\|, \tag{20}$$

and $\hat{\mathbf{k}}(t) \equiv [\hat{k}_0(t) \quad \hat{k}_1(t) \quad \cdots \quad \hat{k}_p(t)]^T \in R^{p+1}$ is the adaptive gain of the unknown vector $\mathbf{k}$ in (7), ε and ρ are any non-negative constants. Note that the constant matrix $\boldsymbol{\Gamma} \in R^{(p+1)\times(p+1)}$ is a symmetric and positive definite matrix specified by the designer.

It is obvious that the proposed controller (16) consists of two parts. The first part $\mathbf{u}_1(\cdot)$ is a linear feedback optimal controller obtained from (14). This part is used to stabilize the nominal system (8). The second part $\mathbf{u}_2(\cdot)$ is a bounded continuous (non-linear) adaptive state feedback controller which is used to overcome the perturbations in order to increase the robustness of stability of the controlled system. The following theorem shows that the proposed control scheme can indeed guarantee the stability of the perturbed systems (1).

Theorem: Consider the perturbed dynamic equation (1) and the reference model (2) with aforementioned Assumptions A1-A3. If the control effort $\mathbf{u}(t)$ is designed as in (16), then the dynamic equation of tracking error (4) is (a) asymptotically stable if $\rho = \varepsilon = 0$ (b) uniformly ultimately bounded if $\rho \neq 0$ and $\varepsilon \neq 0$.

Proof: A Lyapunov function candidate is chosen as

$$V(\mathbf{e}, \tilde{\mathbf{k}}) = \frac{1}{2}\mathbf{e}^T\mathbf{P}\mathbf{e} + \frac{1}{2}\tilde{\mathbf{k}}(t)^T\boldsymbol{\Gamma}^{-1}\tilde{\mathbf{k}}(t), \tag{21}$$

where $\tilde{\mathbf{k}}(t) \equiv \hat{\mathbf{k}}(t) - \mathbf{k}$ is the adaptation error. Then, taking the derivative of $V(\cdot)$ along the trajectories of the tracking error's dynamic equation (4), and noting that

$$\frac{d}{dt}\tilde{\mathbf{k}}(t) = -2\rho\boldsymbol{\Gamma}\tilde{\mathbf{k}}(t) + \boldsymbol{\Gamma}\mathbf{w}\|\mathbf{B}^T\mathbf{P}\mathbf{e}(t)\| - 2\rho\boldsymbol{\Gamma}\mathbf{k},$$

one can obtain

$$\begin{aligned}\frac{d}{dt}V(\mathbf{e}, \tilde{\mathbf{k}}) &= \frac{1}{2}\mathbf{e}^T(\mathbf{P}\mathbf{A}_m + \mathbf{A}_m^T\mathbf{P})\mathbf{e}\\ &+\mathbf{e}^T\mathbf{P}\mathbf{B}\Delta\boldsymbol{\psi}(\cdot) - \mathbf{e}^T\mathbf{P}\mathbf{B}\boldsymbol{\phi}(\mathbf{u}) - 2\rho\|\tilde{\mathbf{k}}(t)\|^2\\ &+\tilde{\mathbf{k}}^T(t)\mathbf{w}\|\mathbf{B}^T\mathbf{P}\mathbf{e}(t)\| - 2\rho\tilde{\mathbf{k}}^T(t)\mathbf{k}.\end{aligned} \tag{22}$$

From (16) to (18), and Assumption A2, it is noted that

$$\begin{aligned}&-\mathbf{e}^T\mathbf{P}\mathbf{B}\boldsymbol{\phi}(\mathbf{u})\\ &= -(\gamma_1 + \gamma_1\beta_1)^{-1}(\gamma_1\mathbf{B}^T\mathbf{P}\mathbf{e} + \gamma_1\beta_1\mathbf{B}^T\mathbf{P}\mathbf{e})^T\boldsymbol{\phi}(\mathbf{u})\\ &= -(\gamma_1 + \gamma_1\beta_1)^{-1}\mathbf{u}^T\boldsymbol{\phi}(\mathbf{u})\\ &\leq -(\gamma_1 + \gamma_1\beta_1)^{-1}\gamma_1\mathbf{u}^T\mathbf{u}\\ &= -\gamma_1^2\mathbf{e}^T\mathbf{P}\mathbf{B}\mathbf{B}^T\mathbf{P}\mathbf{e} - \gamma_1^2\beta_1\|\mathbf{B}^T\mathbf{P}\mathbf{e}\|^2.\end{aligned} \tag{23}$$

According to (6) and (23), (22) can further be derived as

$$\begin{aligned}&\frac{d}{dt}V(\mathbf{e}, \tilde{\mathbf{k}})\\ &\leq \frac{1}{2}\mathbf{e}^T(\mathbf{P}\mathbf{A}_m + \mathbf{A}_m^T\mathbf{P})\mathbf{e} + \|\mathbf{B}^T\mathbf{P}\mathbf{e}\|\|\Delta\boldsymbol{\psi}(\cdot)\|\\ &\quad -\gamma_1^2\mathbf{e}^T\mathbf{P}\mathbf{B}\mathbf{B}^T\mathbf{P}\mathbf{e} - \gamma_1^2\beta_1\|\mathbf{B}^T\mathbf{P}\mathbf{e}\|^2 - 2\rho\|\tilde{\mathbf{k}}(t)\|^2\\ &\quad +\mathbf{w}^T\tilde{\mathbf{k}}(t)\|\mathbf{B}^T\mathbf{P}\mathbf{e}(t)\| + 2\rho\|\tilde{\mathbf{k}}^T(t)\|\|\mathbf{k}\|\\ &= \frac{1}{2}\mathbf{e}^T[\mathbf{P}(\mathbf{A}_m + \alpha\mathbf{I}) + (\mathbf{A}_m^T + \alpha\mathbf{I})\mathbf{P}\\ &\quad -\gamma_1^2\mathbf{P}\mathbf{B}\mathbf{B}^T\mathbf{P} + \mathbf{Q}]\mathbf{e} - \alpha\mathbf{e}^T\mathbf{P}\mathbf{e} - \frac{1}{2}\mathbf{e}^T\mathbf{Q}\mathbf{e}\\ &\quad -\frac{1}{2}\gamma_1^2\mathbf{e}^T\mathbf{P}\mathbf{B}\mathbf{B}^T\mathbf{P}\mathbf{e} + \|\mathbf{B}^T\mathbf{P}\mathbf{e}\|\|\Delta\boldsymbol{\psi}(\cdot)\|\\ &\quad -\gamma_1^2\beta_1\|\mathbf{B}^T\mathbf{P}\mathbf{e}\|^2 - 2\rho\|\tilde{\mathbf{k}}(t)\|^2\\ &\quad +\mathbf{w}^T\tilde{\mathbf{k}}(t)\|\mathbf{B}^T\mathbf{P}\mathbf{e}(t)\| + 2\rho\|\tilde{\mathbf{k}}^T(t)\|\|\mathbf{k}\|\\ &\leq -\alpha\mathbf{e}^T\mathbf{P}\mathbf{e} + \mathbf{w}^T\mathbf{k}\|\mathbf{B}^T\mathbf{P}\mathbf{e}\| - \gamma_1^2\beta_1\|\mathbf{B}^T\mathbf{P}\mathbf{e}\|^2\\ &\quad -2\rho\|\tilde{\mathbf{k}}(t)\|^2 + \mathbf{w}^T\tilde{\mathbf{k}}(t)\|\mathbf{B}^T\mathbf{P}\mathbf{e}(t)\|\\ &\quad +2\rho\|\tilde{\mathbf{k}}^T(t)\|\|\mathbf{k}\|\\ &= -\alpha\mathbf{e}^T\mathbf{P}\mathbf{e} + \mathbf{w}^T\hat{\mathbf{k}}(t)\|\mathbf{B}^T\mathbf{P}\mathbf{e}\|\\ &\quad -\gamma_1^2\beta_1\|\mathbf{B}^T\mathbf{P}\mathbf{e}\|^2 - 2\rho\|\tilde{\mathbf{k}}(t)\|^2\\ &\quad +2\rho\|\tilde{\mathbf{k}}^T(t)\|\|\mathbf{k}\|, \quad \forall\, t \geq 0.\end{aligned} \tag{24}$$

Since it is well known that

$$0 \leq \frac{ab}{a+b} \leq b, \quad \forall\, a, b > 0$$

Using (19) and previous inequality, one can obtain

$$\begin{aligned}&\mathbf{w}^T\hat{\mathbf{k}}(t)\|\mathbf{B}^T\mathbf{P}\mathbf{e}\| - \gamma_1^2\beta_1\|\mathbf{B}^T\mathbf{P}\mathbf{e}\|^2\\ &= \frac{(\mathbf{w}^T\hat{\mathbf{k}}(t))\|\mathbf{B}^T\mathbf{P}\mathbf{e}\|\varepsilon}{(\mathbf{w}^T\hat{\mathbf{k}}(t))\|\mathbf{B}^T\mathbf{P}\mathbf{e}\| + \varepsilon} \leq \varepsilon.\end{aligned} \tag{25}$$

On the other hand,

$$\begin{aligned}&-2\rho\|\tilde{\mathbf{k}}(t)\|^2 + 2\rho\|\tilde{\mathbf{k}}(t)\|\|\mathbf{k}\|\\ &= -\rho\|\tilde{\mathbf{k}}(t)\|^2 - \rho(\|\tilde{\mathbf{k}}(t)\| - \|\mathbf{k}\|)^2 + \rho\|\mathbf{k}\|^2\\ &\leq -\rho\|\tilde{\mathbf{k}}(t)\|^2 + \rho\|\mathbf{k}\|^2.\end{aligned} \tag{26}$$

Then, substituting (25) and (26) into (24) yields

$$\begin{aligned}&\frac{d}{dt}V(\mathbf{e}, \tilde{\mathbf{k}}) \leq -\alpha\mathbf{e}^T\mathbf{P}\mathbf{e} + \varepsilon - \rho\|\tilde{\mathbf{k}}(t)\|^2 + \rho\|\mathbf{k}\|^2\\ &= -\alpha\mathbf{e}^T\mathbf{P}\mathbf{e} - \rho\|\tilde{\mathbf{k}}(t)\|^2 + \tilde{\varepsilon} \equiv -\phi(t), \quad \forall\, t \geq 0\end{aligned}$$

where $\tilde{\varepsilon} \equiv \varepsilon + \rho\|\mathbf{k}\|^2$. $\phi(t) = 0$ is in fact an ellipse (if two-dimensional plane is considered) on $\mathbf{e}-\|\tilde{\mathbf{k}}\|$ plane with center at origin. (a) If $\rho = \varepsilon = 0$, then V will decrease until $\mathbf{e}(t) = 0$. This implies that the dynamic equation (4) is asymptotically stable. (b) If $\rho \neq 0$, $\varepsilon \neq 0$, then $\dot{V}(\mathbf{e}, \tilde{\mathbf{k}}) < 0$ if the values of $\mathbf{e}$ and $\|\tilde{\mathbf{k}}\|$ are outside the region $\phi(t) = 0$. Since there exist functions ψ_1, $\psi_2 \in K_\infty$ and $\psi_3 \in K$, e.g., $\psi_1 = V/2$, $\psi_2 = 2V$, $\psi_3 = \phi(t)$, the signals $\mathbf{e}$ and $\tilde{\mathbf{k}}(t)$ (and hence $\hat{\mathbf{k}}(t)$) are uniformly ultimately bounded (Khalil, 1996). Noted that $V(\mathbf{e}, \tilde{\mathbf{k}})$ (and hence $\mathbf{e}$) has a convergence rate α outside the region $\phi(t) = 0$ in the presence of the uncertain $\Delta\boldsymbol{\psi}(\cdot)$. ■

Note also that if $\rho = 0$, from (20) it is found that the adaptive gain $\hat{\mathbf{k}}(t)$ in general will slowly increase boundedlessly if $\mathbf{e}(t)$ is not exactly equal to zero in finite time. It is also found that if $\varepsilon = 0$, the chattering phenomenon is enhanced, this can be see from (18) and (19). Therefore, there is a tradeoff among tracking accuracy, chattering phenomenon, and adaptation performance.

5. SIMULATIONS

Consider a perturbed dynamic system with input nonlinearity and time-delay argument as described by (1) with

$$\mathbf{A} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -2 & 3 \\ 0 & -4 & -2 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and the unknown model uncertainties, extraneous disturbance, input non-linearity, and time delay argument are assumed to be

$$\Delta\mathbf{A}(t, \mathbf{x}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0.2 & 0.4\cos(t) \\ 0 & 0.1\sin(t) & 0.3 \end{pmatrix}$$

$$\Delta\mathbf{B}(t, \mathbf{x}) = \begin{pmatrix} 0 & 0 \\ -0.1\cos(2t) & 0 \\ 0 & -(0.2 + 0.1\sin(t)) \end{pmatrix}$$

$$\Delta\mathbf{A}_h(t, \mathbf{x}) = \begin{pmatrix} 0 & 0 & 0 \\ 2 & (1 - 0.2\cos(t)) & 0.1 \\ 2 & 0.2 & -0.1\sin(0.5t) \end{pmatrix}$$

$$\Delta\mathbf{f} = \begin{pmatrix} 0 \\ \cos(t)x_3(t - h(t)) \\ \sin(t)x_2(t - h(t)) \end{pmatrix}$$

$$\phi_1(\mathbf{u}) = \left[0.2\cos(u_2) + e^{(|\cos(u_1+u_2)|)}\right] u_1$$

$$\phi_2(\mathbf{u}) = \left[1 + 0.3\sin(u_1 + u_2) + 0.2e^{(1+\sin(u_2))}\right] u_2$$

$$h(t) = 0.2 + 0.1\cos(t).$$

The reference model is given by

$$\begin{pmatrix} \dot{x}_{m1} \\ \dot{x}_{m2} \\ \dot{x}_{m3} \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -2 & 3 \\ 0 & -4 & -2 \end{pmatrix} \begin{pmatrix} x_{m1} \\ x_{m2} \\ x_{m3} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r_1(t) \\ r_2(t) \end{pmatrix}, \quad \forall\, t \in R$$

and the reference signal is

$$\begin{pmatrix} r_1(t) \\ r_2(t) \end{pmatrix} = \begin{pmatrix} 0.2\cos(2t) \\ 0.4\sin(t) \end{pmatrix}.$$

The objective of control is to use the proposed control technique to design an optimal model reference adaptive controller such that the states $x_1(t)$, $x_2(t)$ and $x_3(t)$ can track the desired reference signals $x_{m1}(t)$, $x_{m2}(t)$, and $x_{m3}(t)$ respectively. The following is the design procedure of the proposed control scheme.

1. Solve the Riccati Equation

The Riccati equation (15) is solved first with $\gamma_1 = 0.5$, $\alpha = 3$, $\mathbf{Q} = \mathbf{I}$ in order to obtain the matrix $\mathbf{P}$. By using the software MATLAB, one obtain

$$\mathbf{P} = \begin{pmatrix} 364.0746 & 29.8318 & 40.4481 \\ 29.8318 & 10.1308 & 2.1756 \\ 40.4481 & 2.1756 & 8.9009 \end{pmatrix}.$$

2. Design of controller

According to (16) to (18), the control efforts of the proposed control scheme are

$$u_1(t) = (0.5 + 0.5\beta_1(t))[14.916e_1(t) + 5.0654e_2(t) + 1.0878e_3(t)]$$

$$u_2(t) = (0.5 + 0.5\beta_1(t))[35.1399e_1(t) + 6.1532e_2(t) + 5.5382e_3(t)]$$

where $\beta_1(t)$ and $\hat{\mathbf{k}}(t)$ is given by (19) and (20) ($\hat{\mathbf{k}}(0) = \mathbf{0}$), respectively. The designed parameters are set to $(\gamma_1, \rho, \varepsilon, p) = (0.5, 0.01, 0.04, 2)$ and

$$\boldsymbol{\Gamma} = \begin{pmatrix} 0.4 & 0 & 0 \\ 0 & 0.4 & 0 \\ 0 & 0 & 0.2 \end{pmatrix}.$$

The results of simulation (with initial condition $\mathbf{x}(0) = \begin{pmatrix} 0.5 & -1 & -0.5 \end{pmatrix}^T$ are shown form Fig. 1 to Fig. 5. Fig. 1 shows all the tracking errors e_1, e_2, and e_3 are driven into a small bounded region respectively. Fig. 2 are the two control input functions u_1 and u_2, there is no chattering at all. The adaptation gain $\hat{\mathbf{k}}$ is shown in Fig. 3, which are all bounded as expected. If $(\rho, \varepsilon) = (0, 0)$, then there will be no tracking error theoretically since $\dot{V} < 0, \forall t$. However, from Fig. 4, one can see clearly that the chattering phenomenon is enhanced, and the adaptive gains $\hat{k}_i, i = 1, 2, 3$ all slowly increase boundlessly until the tracking error is exactly equal to zero, as shown in Fig 5.

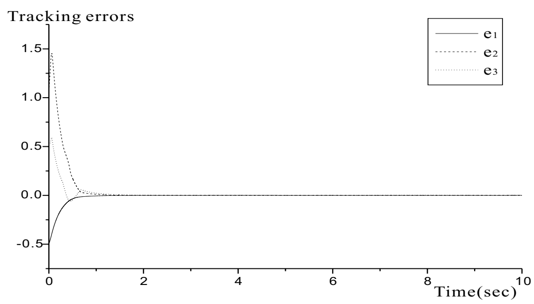

Fig. 1. Tracking error $\mathbf{e}$.

6. CONCLUSIONS

In this paper an optimal model reference adaptive controller for a class of MI linear systems with time-varying delay and input non-linearity is successfully proposed for solving robust tracking problems. The proposed control scheme can achieve asymptotical stability of tracking error,

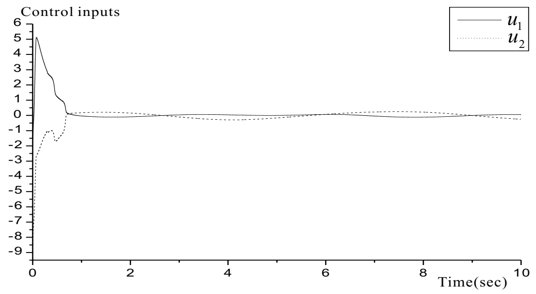

Fig. 2. Control input $\mathbf{u}$.

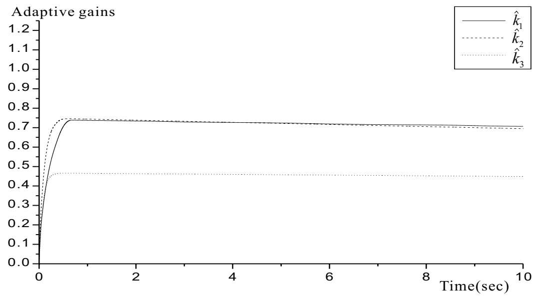

Fig. 3. Adaptive gain $\hat{\mathbf{k}}$.

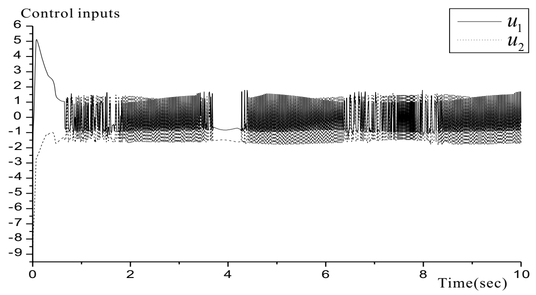

Fig. 4. Control input $\mathbf{u}$ with $(\rho, \varepsilon) = (0, 0)$.

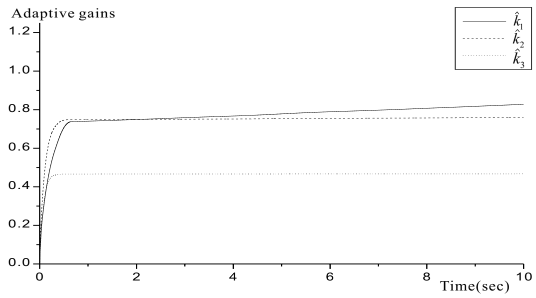

Fig. 5. Adaptive gain $\hat{\mathbf{k}}$ with $(\rho, \varepsilon) = (0, 0)$.

however, chattering phenomenon will arise and the adaptive gains will slowly increase boundlessly until all the tracking errors reach to zero. Therefore, the designer has a tradeoff among tracking accuracy, chattering phenomenon, and adaptation performance.

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