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RECEDING HORIZON LINEAR QUADRATIC CONTROL WITH FINITE INPUT CONSTRAINT SETS

Daniel E. Quevedo * **José A. De Doná** *
Graham C. Goodwin *

* *Department of Electrical & Computer Engineering, The University of Newcastle, Callaghan, NSW 2308, Australia*

Abstract: By exploring the geometry of the underlying constrained optimization, a finitely parameterized solution to the discrete time receding horizon linear quadratic control problem with a finite input constraint set is obtained. The resulting controller gives rise to a closed loop system which is piece-wise affine in the plant state. The switching regions are polytopes and are related to those obtained when dealing with ℓ_∞ (saturation-like) constraint sets.

Keywords: predictive control, quadratic control, constraints, binary control, finite fields, multilevel controllers, quantized signals, switching values.

1. INTRODUCTION

On-off and relay feedback control systems are widespread and have been studied extensively, see e.g. (Bockman, 1991; Gonçalves *et al.*, 2001). An interesting generalization corresponds to the finite control set case analyzed by Chitour and Piccoli (2001). Here, the constraint set is allowed to contain any finite number of elements. This situation arises in many practical contexts, e.g. when a finite number of control levels are available. The same problem arises when considering digital control systems affected by quantization and saturation (Sznaier and Sideris, 1994; Feng and Loparo, 1997). In addition, finite alphabet control laws form a precursor to hybrid systems as studied e.g. in (Branicky, 1998; Bemporad and Morari, 1999). They also have close links to the issue of control with communication constraints (Liu and Wong, 1997) and predictive analog to digital data converters such as the $\Sigma\Delta$–Modulator described in (Norsworthy *et al.*, 1997).

In this contribution, the discrete time receding horizon quadratic control problem with a finite alphabet is studied. The approach is in the spirit of recent results on Model Predictive Control, which are aimed at providing insight into the nature of the control law via an elucidation of the inherent structure of the mapping between plant states and optimal controls. In particular, for the case of ℓ_∞ constraints (e.g. saturation) one obtains a characterization in terms of a partition of the state space into polytopal regions in which the control input is piece-wise affine in the state (Seron *et al.*, 2000; Bemporad *et al.*, 2002; Johansen *et al.*, 2000). The present work is also related to the work of Beck and Teboulle (2000), where relations between optimizers of quadratic programmes with ℓ_∞ and with binary constraints are explored.

The key-result obtained here is to show that a similar structure holds for the case of control with a finite constraint set. Moreover, there exists a close connection between the partitions induced by the finite alphabet–, and the ℓ_∞–constrained solution. Advantages accruing from the explicit description of the solution, as detailed here, include:

- giving a complete characterization of the control policy, which enhances verifiability

of the complete range of behavior without relying on on-line optimization,
- providing a structure which opens the door to alternative tools for study the dynamic behavior of the closed loop system, including issues such as stability.

The outline of the paper is as follows. The general problem is formulated in the next section. The unconstrained solution is reviewed in Section 3. In Sections 4 and 5 the finite set constrained solution is characterized. Relations to the ℓ_∞–constrained case are examined in Section 6. Section 7 contains an example and Section 8 draws some conclusions.

2. GENERAL PROBLEM FORMULATION

Consider a system with scalar input $u(k)$ and state vector $x(k) \in \mathbb{R}^n$ described by:

$$x(k+1) = Ax(k) + Bu(k), \quad u(k) \in \mathbb{U}, \tag{1}$$

where $\mathbb{U}$ is a set representing input constraints.

The finite horizon (open loop) quadratic regulator problem at time $t = k$ consists of obtaining the optimizing sequence of present and future control inputs

$$\mathbf{u}^\star(x(k)) = \arg \min_{\mathbf{u}(k)\in\mathbb{U}^N} \{V_N(x(k), \mathbf{u}(k))\}, \tag{2}$$

where:

$$\mathbf{u}(k) = \begin{bmatrix} u(k) \\ u(k+1) \\ \vdots \\ u(k+N-1) \end{bmatrix}, \ \mathbb{U}^N \triangleq \mathbb{U} \times \cdots \times \mathbb{U} \tag{3}$$

and V_N is the finite horizon quadratic cost functional

$$\begin{aligned} V_N(x(k), \mathbf{u}(k)) &= x^T(k+N)Px(k+N) \\ &+ \sum_{t=k}^{k+N-1} \{x^T(t)Qx(t) + u^T(t)Ru(t)\}, \end{aligned} \tag{4}$$

with $Q = Q^T \geq 0$, $P = P^T > 0$ and $R > 0$.

The optimization provides the optimal sequence, which is a function of the current state $x(k)$ and is denoted as $\mathbf{u}^\star(x(k))$. The solution therefore corresponds to an open loop control law. A closed loop law is usually obtained by implementing $\mathbf{u}^\star(x(k))$ in a *receding horizon* manner, i.e. by applying only the first control action

$$u^\star(x(k)) = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \mathbf{u}^\star(x(k))$$

and repeating the optimization at the next time instant with a new *initial* state and the finite horizon shifted by one, see e.g. the review by Mayne *et al.* (2000).

In this paper, the main interest resides in the case where $\mathbb{U}$ is a finite set so that the optimization problem (2) is non-convex due to the fact that the constraint set is non-convex. In order to derive the solution to this problem, it is useful to first briefly review the unconstrained solution.

3. UNCONSTRAINED SOLUTION

The optimizer to the Linear Quadratic Regulator problem without constraints, i.e. where $\mathbb{U} = \mathbb{R}$, is well known and can be obtained by using dynamic programming. It involves the solution of a Riccati Equation, see e.g. (Kwakernaak and Sivan, 1972). This problem can also be recast as a static optimization problem by rewriting the system equations and the cost functional as follows.

Given the current state $x(k)$, and equation (1), the (predicted) future states up to time $t = k+N$ satisfy:

$$\mathbf{x}(k) = \begin{bmatrix} x(k+1) \\ x(k+2) \\ \vdots \\ x(k+N) \end{bmatrix} = \Phi\mathbf{u}(k) + \Lambda x(k),$$

where:

$$\Phi = \begin{bmatrix} B & 0 & \dots & 0 & 0 \\ AB & B & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A^{N-1}B & A^{N-2}B & \dots & AB & B \end{bmatrix}, \ \Lambda = \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix}.$$

Hence, the cost function (4) can be written in vector form according to:

$$\begin{aligned} V_N(x(k), \mathbf{u}(k)) = \overline{V}_N(x(k)) + \mathbf{u}^T(k)W\mathbf{u}(k) \\ + 2\mathbf{u}^T(k)Fx(k), \end{aligned} \tag{5}$$

where:

$$W = \Phi^T\mathbf{Q}\Phi + \mathbf{R} \in \mathbb{R}^{N\times N}, \ F = \Phi^T\mathbf{Q}\Lambda \in \mathbb{R}^{N\times n},$$
$$\mathbf{Q} = \mathrm{diag}(Q, \dots, Q, P) \in \mathbb{R}^{Nn\times Nn}$$
$$\mathbf{R} = \mathrm{diag}(R, \dots, R) \in \mathbb{R}^{N\times N}$$

and $\overline{V}_N(x(k))$ does not depend on $\mathbf{u}(k)$.

Since $W > 0$ and the optimization (5) depends on the vector of *parameters* $\mathbf{x}(k)$, this problem is called a *multi-parametric* quadratic programme (Bemporad *et al.*, 2002). By direct calculation it follows that the solution to this unconstrained problem is:

$$\mathbf{u}^\star_{uc}(x(k)) = -W^{-1}Fx(k) \tag{6}$$

and yields the receding horizon control law:

$$u^\star_{uc}(x(k)) = -\begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} W^{-1}Fx(k).$$

In the next section the geometry of the constrained optimization problem is exploited in order to derive the *constrained solution* based upon the *unconstrained solution* (6).

4. CONSTRAINED SOLUTION

Given the current state $x(k)$, the level curves of $V_N(x(k), \mathbf{u}(k))$ in (5) are ellipsoids centered at $\mathbf{u}^\star_{\mathrm{uc}}(x(k))$ in the input sequence space $\mathbb{R}^N$.

In the case of a finite constraint set $\mathbb{U}$ containing $n_{\mathbb{U}}$ elements, the sequence $\mathbf{u}(k)$ is constrained to belong to the finite set $\mathbb{U}^N$ defined in (3).

The constrained optimization problem can be geometrically interpreted as follows: Find the point $\mathbf{u} \in \mathbb{U}^N$, which belongs to the smallest ellipsoid defined by (5) (i.e. the point which provides the smallest cost while satisfying the constraints).

This geometrical problem can be simplified if, as in (Seron *et al.*, 2000), the following transformation is used:

$$\tilde{\mathbf{u}} = W^{1/2}\mathbf{u}, \tag{7}$$

so that the cost function becomes:

$$\begin{aligned} V_N(x(k), \tilde{\mathbf{u}}(k)) = \overline{V}_N(x(k)) + \tilde{\mathbf{u}}^T(k)\tilde{\mathbf{u}}(k) \\ + 2\tilde{\mathbf{u}}^T(k)W^{-1/2}Fx(k). \end{aligned} \tag{8}$$

The level curves of this function are spheres in the transformed input sequence space $\mathbb{R}^N$, centered at

$$\tilde{\mathbf{u}}^\star_{\mathrm{uc}}(x(k)) = -W^{-1/2}Fx(k). \tag{9}$$

As a consequence, an explicit characterization of the constrained solution can be obtained. For this purpose it is useful to introduce a nearest neighbor vector quantizer as follows.

Definition 1. (Nearest Neighbor Vector Quantizer). Given a countable (not necessarily finite) set of non-equal vectors $\mathcal{B} = \{b_1, b_2, \dots\} \subset \mathbb{R}^{n_B}$, the nearest neighbor quantizer is defined as a mapping $q_{\mathcal{B}} \colon \mathbb{R}^{n_B} \to \mathcal{B}$ which assigns to each vector $a \in \mathbb{R}^{n_B}$ the closest element of $\mathcal{B}$ (as measured by the Euclidean norm), i.e., $q_{\mathcal{B}}(a) = b_i \in \mathcal{B}$ if and only if a belongs to the set:

$$\begin{aligned} &\{c \in \mathbb{R}^{n_B} : \|c - b_i\| \le \|c - b_j\|, \forall b_j \ne b_i, b_j \in \mathcal{B}\} \\ &\setminus \{c \in \mathbb{R}^{n_B} : \exists j < i \text{ such that } \|c - b_i\| = \|c - b_j\|\}. \end{aligned} \tag{10}$$

Note that the zero measure sets of points which satisfy (10) with equality have been arbitrarily assigned to the element having the smallest index. This is done in order to avoid ambiguity in case of *frontier points*, that is, points which are equidistant to two, or more, elements of $\mathcal{B}$.

This definition allows one to state the following result.

Theorem 2. Suppose $\mathbb{U}^N = \{v_1, v_2, \dots, v_r\}$, where $r = n_{\mathbb{U}}^N$, then the optimizing sequence for the constrained problem (1)–(4) is given by:

$$\mathbf{u}^\star(x(k)) = W^{-1/2} q_{\tilde{\mathbb{U}}^N}(-W^{-1/2}Fx(k)), \tag{11}$$

where $q_{\tilde{\mathbb{U}}^N}(\cdot)$ is the nearest neighbor quantizer (10), mapping $\mathbb{R}^N$ to $\tilde{\mathbb{U}}^N$. The image of the mapping $q_{\tilde{\mathbb{U}}^N}(\cdot)$ is the set of *transformed* vertices, defined as:

$$\tilde{\mathbb{U}}^N \triangleq \{\tilde{v}_1, \tilde{v}_2, \dots, \tilde{v}_r\}, \; \tilde{v}_i = W^{1/2}v_i, \; v_i \in \mathbb{U}^N. \tag{12}$$

PROOF. The proof follows directly from the foregoing discussion. Since the level curves of (8) are spheres centered at $\tilde{\mathbf{u}}^\star_{\mathrm{uc}}(x(k))$, the constrained optimizer, among the transformed sequences $\tilde{\mathbf{u}} \in \tilde{\mathbb{U}}^N$, is given by $q_{\tilde{\mathbb{U}}^N}(\tilde{\mathbf{u}}^\star_{\mathrm{uc}}(x(k)))$. By using (7) and (9), $\mathbf{u}^\star = \mathbf{u}^\star(x(k)) = W^{-1/2}\tilde{\mathbf{u}}^\star(x(k)) = W^{-1/2}q_{\tilde{\mathbb{U}}^N}(-W^{-1/2}Fx(k))$ is obtained. □

It should be noted that $q_{\tilde{\mathbb{U}}^N}(\cdot)$ is a memoryless nonlinearity, so that the control law (11) corresponds to a time-invariant nonlinear state feedback law.

The receding horizon law corresponding to (11) is:

$$u^\star(x(k)) = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} W^{-1/2} q_{\tilde{\mathbb{U}}^N}(-W^{-1/2}Fx(k)).$$

The above solution uses nearest neighbor quantizers. These have been studied and used extensively in the context of signal encoding, see e.g. the book by Gersho and Gray (1992). Due to their simplicity they are suitable for on-line implementation, which can be carried out directly by performing $r-1$ comparisons at each time step. Moreover, they introduce a polytopal partition (called *Voronoi* partition) of their domain into equivalence classes. This property will be used in the sequel to derive an explicit characterization of the resulting optimal control law, more suitable for analysis and design purposes.

5. EXPLICIT STATE-SPACE CHARACTERIZATION OF THE CONSTRAINED SOLUTION

The solution (11) obtained in the previous section is described according to the composition of transformations:

$$x \in \mathbb{R}^n \xrightarrow{-W^{-\frac{1}{2}}F} \tilde{\mathbf{u}}^\star_{\mathrm{uc}} \xrightarrow{q_{\tilde{\mathbb{U}}^N}(\cdot)} \tilde{\mathbf{u}}^\star \xrightarrow{W^{-\frac{1}{2}}} \mathbf{u}^\star \in \mathbb{R}^N, \tag{13}$$

where $\mathbf{u}^\star$ is the constrained optimum open-loop sequence, $\tilde{\mathbf{u}}^\star$ is the constrained optimum *transformed* open-loop control sequence and $\tilde{\mathbf{u}}^\star_{\mathrm{uc}}$ is the unconstrained optimum transformed open-loop control sequence.

The quantizer $q_{\tilde{\mathbb{U}}^N}(\cdot)$ induces a partition of its domain. Since the constrained optimizer in (11) (see also (13)) is defined in terms of $q_{\tilde{\mathbb{U}}^N}(\cdot)$, an equivalent partition of the state space can be obtained.

By combining (10) and (11), it follows that $\mathbf{u}^\star(x(k)) = \nu_i$, if and only if

$$x(k) \in \mathcal{R}_i.$$

The regions $\mathcal{R}_i$ are defined as:

$$\begin{aligned}\mathcal{R}_i \triangleq \Big\{ & z \in \mathbb{R}^n \colon \| - W^{-1/2}Fz - \tilde{\nu}_i\| \\ & \leq \| - W^{-1/2}Fz - \tilde{\nu}_j\|,\ \forall \tilde{\nu}_j \neq \tilde{\nu}_i,\ \tilde{\nu}_j \in \tilde{\mathbb{U}}^N \Big\} \\ \setminus \Big\{ & z \in \mathbb{R}^n \colon \exists j < i \text{ such that } \| - W^{-1/2}Fz - \tilde{\nu}_i\| \\ & = \| - W^{-1/2}Fz - \tilde{\nu}_j\| \Big\}.\end{aligned}$$

The relation (12) allows one to establish an explicit state-space characterization of the finite-set constrained solution:

$$\mathbf{u}^\star(x(k)) = \nu_i \Longleftrightarrow x(k) \in \mathcal{R}_i,$$

with:

$$\begin{aligned}\mathcal{R}_i = \Big\{ & z \in \mathbb{R}^n \colon -2(\nu_j - \nu_i)^T Fz \\ & \leq (\nu_j + \nu_i)^T W(\nu_j - \nu_i),\ \forall \nu_j \neq \nu_i,\ \nu_j \in \mathbb{U}^N \Big\} \\ & \setminus \Big\{ z \in \mathbb{R}^n \colon \exists j < i \text{ such that} \\ & -2(\nu_j - \nu_i)^T Fz = (\nu_j + \nu_i)^T W(\nu_j - \nu_i) \Big\}.\end{aligned} \tag{14}$$

It is worth noting that some of the inequalities in (14) may be redundant. In these cases the corresponding regions do not share a common border, i.e. are not adjacent. For example, see Figure 3, for the case $N = 2$, where the specification of $\mathcal{R}_1$ (and $\mathcal{R}_4$) require only 2 inequalities instead of 3.

The regions defined in (14) are polytopes. Without taking into account constraint borders, they can be written in compact form as:

$$\mathcal{R}_i = \{x \in \mathbb{R}^n \colon D_i x \leq h_i\},$$

where the rows of D_i are equal to all terms $-2(\nu_j - \nu_i)F$ as required, while the vector h_i contains the scalars $(\nu_j + \nu_i)^T W(\nu_j - \nu_i)$.

On the other hand, some of the regions $\mathcal{R}_i$ may be empty. However, if the pair (A, B) is completely controllable and A is invertible, then the rank of F is equal to $\min(N, n)$. Hence, if $n \geq N$, then $-W^{-1/2}F$ is onto, so that $\forall \tilde{\nu}_j \in \tilde{\mathbb{U}}^N$ there exist, at least one, x such that $q_{\tilde{\mathbb{U}}^N}(-W^{-1/2}Fx) = \tilde{\nu}_j$. In this case, none of the regions $\mathcal{R}_i$ are empty.

If $N > n$, the rank of F is equal to n and the transformation does not span the whole space $\mathbb{R}^N$, so that some regions may be empty. This is illustrated in Figure 1 for the case $n = 1$, $n_{\mathbb{U}} = 2$ and $N = 2$. As can be seen, depending on the *unconstrained optimum locus* given by the (dashed) line $-W^{-1/2}Fx$, $x \in \mathbb{R}$, there are situations in which some sequences $\tilde{\nu}_j$ will never be optimal, yielding empty regions in the state space (see also Figure 3 below).

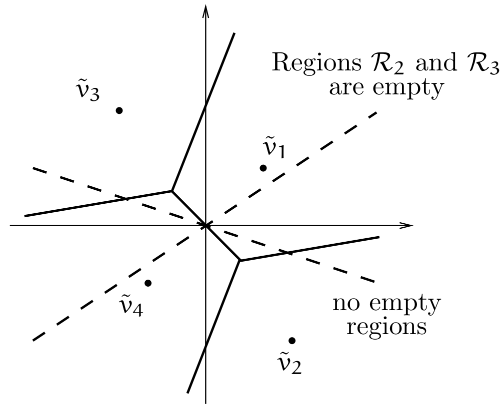

Fig. 1. Partition of the transformed input sequence space with $N = 2$ (solid lines) and two examples of $-W^{-1/2}Fx$, $x \in \mathbb{R}$ (dashed lines)

Next consider the receding horizon case. To implement this policy only $u^\star(k)$, the first element of $\mathbf{u}^\star(k)$, is used. As a consequence, only $n_{\mathbb{U}}$ instead of $n_{\mathbb{U}}^N$ regions are needed to characterize the receding horizon control law. Each of these regions is given by the union of all regions $\mathcal{R}_i$ corresponding to vectors ν_i, having the same first element. The following result is immediate, hence, it is stated without a proof.

Corollary 3. Let the constraint set $\mathbb{U} = \{u_1, \ldots, u_{n_{\mathbb{U}}}\}$, and consider its partition into equivalence classes:

$$\begin{aligned}\mathbb{U}_i^N &\triangleq \{\nu \in \mathbb{U}^N \colon \begin{bmatrix}1 & 0 & \cdots & 0\end{bmatrix} \nu = u_i\}, \\ \mathbb{U}^N &= \bigcup_{i=1,\ldots,n_{\mathbb{U}}} \mathbb{U}_i^N.\end{aligned}$$

Then, the receding horizon law implementation of the optimizer of (1)–(4) is:

$$u^\star(x(k)) = u_i, \quad \text{if } x(k) \in \mathcal{X}_i, \quad i = 1, 2, \ldots n_{\mathbb{U}},$$

where:

$$\begin{aligned}\mathcal{X}_i &= \bigcup_{j\colon \nu_j \in \mathbb{U}_i^N} \mathcal{X}_{ij} \\ \mathcal{X}_{ij} &= \Big\{ z \in \mathbb{R}^n \colon -2(\nu_k - \nu_j)^T Fz \\ &\qquad \leq (\nu_k + \nu_j)^T W(\nu_k - \nu_j), \forall \nu_k \in \mathbb{U}^N \setminus \mathbb{U}_i^N \Big\} \\ &\qquad \setminus \Big\{ z \in \mathbb{R}^n \colon \exists \nu_k \in \mathbb{U}^N \setminus \mathbb{U}_i^N,\ k < j, \text{ such that} \\ &\qquad -2(\nu_k - \nu_j)^T Fz = (\nu_k + \nu_j)^T W(\nu_k - \nu_j) \Big\}.\end{aligned}$$

Note that this description requires less evaluations of inequalities than the direct calculation of the union of all $\mathcal{R}_j$ (as defined in (14)) with $\nu_j \in \mathbb{U}_i^N$, since inequalities corresponding to *internal borders* are not evaluated.

The closed loop obtained from the above control law is a piece-wise affine system with polytopal switching regions and hence belongs to the class of discrete time switched hybrid systems, see e.g. (Branicky, 1998; Bemporad and Morari, 1999). It also fits into the class of systems having *symbolic dynamics* as e.g. in (Wu and Chua, 1994; Ramadge, 1990).

Regarding the closed loop dynamics, in general the state trajectories of the closed loop will be dominated by its nonlinear and *non-smooth* dynamics that results form the nearest neighbor quantizer $q_{\tilde{\mathbb{U}}^N}(\cdot)$. Indeed, the results of Ramadge (1990) and Wu and Chua (1994) suggest that for stable plants one will obtain limit cycles. (Note that, given bounded inputs, the states of an asymptotically stable plant are always bounded.) When A is not a stability matrix, global stability with a finite input constraint set cannot be ensured. The trajectories, if bounded, will in general be non-periodic. Thus, at best, one can hope for *ultimate boundedness* (Blanchini, 1999).

6. RELATIONSHIP TO SATURATION CONSTRAINTS

Seron *et al.* (2000), and also Bemporad *et al.* (2002), using a different methodology, have studied the case of input saturation (the convex problem defined over ℓ_∞ norm constraints). These contributors have shown that the receding horizon implementation of the optimal control problem can be finitely parameterized in closed loop and calculated off-line. The state space is partitioned into polytopes in which the receding horizon controller is piece-wise affine in the state.

The transformed input sequence space partition established by Seron *et al.* (2000) using a geometric argument similar to the one used in Section 4 is sketched in Figure 2 for the case $N = 2$ and saturation interval $[-\Delta,\ \Delta]$. The ℓ_∞–constrained optimizer $\tilde{\mathbf{u}}^\star(x(k))$ is related to the unconstrained optimizer $\tilde{\mathbf{u}}^\star_{\mathrm{uc}}(x(k))$ by a minimum Euclidean distance projection as follows.

If $\tilde{\mathbf{u}}^\star_{\mathrm{uc}}(x(k))$ lies inside of the *allowed* region Θ_o (the polytope formed by the transformed vertices and containing the origin), then $\tilde{\mathbf{u}}^\star(x(k)) = \tilde{\mathbf{u}}^\star_{\mathrm{uc}}(x(k))$. On the other hand, if $\tilde{\mathbf{u}}^\star_{\mathrm{uc}}(x(k)) \notin \Theta_o$, then the constrained solution is obtained by its orthogonal projection to the border of Θ_o.

As a consequence, in the case of a binary constraint set, $\mathbb{U} = \{-\Delta, \Delta\}$, and plant state $x(k)$ such that $-W^{-1/2}Fx(k) \notin \Theta_o$, the borders of the regions in the transformed input sequence space (included as dotted lines in Figure 2) are parallel and equidistant to the borders of those regions of the ℓ_∞–constrained case, which are adjacent to an $N-1$–dimensional hyper-face of Θ_o. These regions are denoted as Θ_{si} in Figure 2. Due to linearity of the transformation $W^{-1/2}F$, the binary constraint

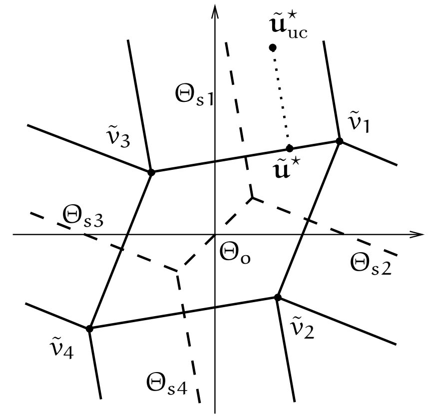

Fig. 2. Partitions of the transformed input sequence space with ℓ_∞–, and binary–constraint sets (solid and dashed lines, respectively)

regions $\mathcal{R}_i$ defined in (14) and the corresponding state space regions given ℓ_∞ constraints are similarly related. This is illustrated in the example of the next section which can be compared with examples in (Seron *et al.*, 2000).

7. EXAMPLE

A simple case of the above problem corresponds to the symmetric binary constraint set $\mathbb{U} = \{1, -1\}$. In this case $n_{\mathbb{U}} = 2$ and $r = 2^N$. Since $u^2 = 1, \forall u \in \mathbb{U}$, the value of R does not affect the optimization of (4), so that $R = 0$ can be chosen. (For the same reason, the solution of optimizing (5) does not depend on the diagonal entries of W.)

The elements of $\mathbb{U}^N$ can be ordered according to

$$v_i^T = \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix} - 2\begin{bmatrix} \alpha_{N-1} & \alpha_{N-2} & \dots & \alpha_0 \end{bmatrix},$$

with α_j, $j = 0, \dots, N-1$ defined implicitly by:

$$\sum_{j=0}^{N-1} \alpha_j 2^j = i - 1, \quad \alpha_j \in \{0, 1\}.$$

With this choice, in a receding horizon implementation, all the vertices $v_1, v_2, \dots, v_{r/2}$ provide the same control action and, due to symmetry, $\tilde{v}_i = -\tilde{v}_{(r+1-i)}$, $i = 1, \dots, r$ so that $\|\tilde{v}_i\|_2 = \|\tilde{v}_{(r+1-i)}\|_2$.

Moreover, redundancies in the inequalities (14) defining the regions $\mathcal{R}_i$, can be found by using the fact that the symmetrically opposed regions $\mathcal{R}_i$ and $\mathcal{R}_{r+1-i}$, do not have a common border, if

$$\|\tilde{v}_i\|_2 > \min_{k=1,\dots,r/2} \|\tilde{v}_k\|_2.$$

As an example, consider the set-up described in (1)–(4) for the stable plant:

$$A = \begin{bmatrix} 0.2 & 1 \\ 0 & 0.9 \end{bmatrix},\ B = \frac{1}{15}\begin{bmatrix} 2 \\ 1 \end{bmatrix},\ P = Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

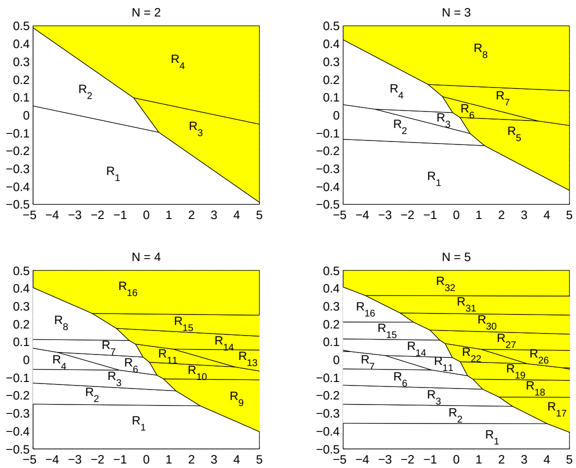

Fig. 3. State space partitions

(In principle there is no impediment to include higher order examples. Of course, at the expense of additional computation time.)

Figure 3 illustrates the state space partition obtained for optimization horizons $N = 2, 3, 4, 5$. Note that for $N \geq 4$, some regions are empty. The receding horizon control law is:

$$u(k) = \begin{cases} 1 & \text{if } x(k) \in \bigcup_{i=1,\ldots,r/2} \mathcal{R}_i, \\ -1 & \text{if } x(k) \in \bigcup_{i=r/2+1,\ldots,r} \mathcal{R}_i \end{cases}$$

and is also depicted in Figure 3.

8. CONCLUSIONS

This paper has studied the geometric structure of the discrete time receding horizon quadratic optimal control problem with finite input set constraints. The closed loop system obtained is piecewise affine, having polytopal switching regions which are related to those arising when considering ℓ_∞–constraint sets.

The results presented here are believed to open the door to future work on related issues including stability and a characterization of solutions to more general hybrid control problems in the same general framework.

9. REFERENCES

Beck, A. and M. Teboulle (2000). Global optimality conditions for quadratic optimization problems with binary constraints. *SIAM Journal on Optimization* **11**(1), 179–188.

Bemporad, A. and M. Morari (1999). Control of systems integrating logic, dynamics, and constraints. *Automatica* **35**, 407–427.

Bemporad, A., M. Morari, V. Dua and E. N. Pistikopoulos (2002). The explicit linear quadratic regulator for constrained systems. *Automatica* **38**, 3–20.

Blanchini, F. (1999). Set invariance in control. *Automatica* **35**, 1747–1767.

Bockman, S. F. (1991). On-Off, discrete time control of a stable first-order plant. In: *Proceedings of the ACC, Boston, MA*. pp. 1258–1262.

Branicky, M. S. (1998). Multiple Lyapunov functions and other analysis tools for switched and hybrid systems. *IEEE Transactions on Automatic Control* **43**(4), 475–482.

Chitour, Y. and B. Piccoli (2001). Controllability for discrete systems with a finite control set. *Math. Control Signals Systems* **14**, 173–193.

Feng, X. and K. A. Loparo (1997). Active probing for information in control systems with quantized state measurements: A minimum entropy approach. *IEEE Transactions on Automatic Control* **42**, 216–238.

Gersho, A. and R. M. Gray (1992). *Vector Quantization and Signal Compression*. Kluwer.

Gonçalves, J. M., A. Megretski and M. A. Dahleh (2001). Global stability of relay feedback systems. *IEEE Transactions on Automatic Control* **46**(4), 550–562.

Johansen, T. A., I. Petersen and O. Slupphaug (2000). On explicit suboptimal LQR with state and input constraints. In: *Proceedings of the 39th CDC, Sydney, Australia*.

Kwakernaak, H. and R. Sivan (1972). *Linear Optimal Control Systems*. Wiley–Interscience.

Liu, X. and W. S. Wong (1997). Controllability of linear feedback control systems with communication constraints. In: *Proceedings of the 36th CDC, San Diego, CA*. pp. 60–65.

Mayne, D. Q., J. B. Rawlings, C. V. Rao and P. O. M. Scokaert (2000). Constrained model predictive control: Optimality and stability. *Automatica* **36**(6), 789–814.

Norsworthy, S. R., R. Schreier and G. C. Temes (editors) (1997). *Delta–Sigma Data Converters: Theory, Design and Simulation*. IEEE Press, Piscataway, N.J.

Ramadge, P. J. (1990). On the periodicity of symbolic observations of piecewise smooth discrete-time systems. *IEEE Transactions on Automatic Control* **35**(7), 807–813.

Seron, M. M., J. A. De Doná and G. C. Goodwin (2000). Global analytical model predictive control with input constraints. In: *Proceedings of the 39th CDC, Sydney, Australia*.

Sznaier, M. and A. Sideris (1994). Feedback control of quantized systems with applications to neuromorphic controllers design. *IEEE Transactions on Automatic Control* **39**(7), 1497–1502.

Wu, C. W. and L. O. Chua (1994). Symbolic dynamics of piecewise-linear maps. *IEEE Transactions on Circuits and Systems II* **41**(6), 420–424.