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AMPLITUDE DEPENDENT ANALYSIS AND STABILIZATION FOR NONLINEAR SAMPLED-DATA CONTROL SYSTEMS

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Abstract: The robust stability condition for sampled-data control systems with a sector nonlinearity was presented in our previous paper. Although it is applicable only to the sampled-data control system of a certain class, a usual discrete-time control system can belong to this type of class. This paper analyzes the amplitude dependent behavior of nonlinear sampled-data (i.e., discrete-time) control systems in a frequency domain. First, the robust stability condition which was derived in our previous papers is applied to a sampled-data system containing a single time-invariant nonlinear element. Then, an instability condition for that type of nonlinear feedback system is derived. By considering restricted areas (two sectors) in the nonlinear characterisitic, the existence of a sustained oscillation is estimated (whether it is periodic or not), and the relationship between the stable (unstable) conditions and the result which is derived from the classic describing function is compared. Based on these considerations, the stabilization of nonlinear discrete-time control systems is examined in the frequency domain. *Copyright ©2002 IFAC*

Keywords: Nonlinear sampled-data systems; input-output stability; Popov criterion; instability; describing function

1. INTRODUCTION

This paper analyzes the amplitude dependent behavior of nonlinear sampled-data control systems in a frequency domain. In actuality, a sustained oscillation (whether it is periodic or not) cannot be avoided in the response of nonlinear dynamical systems. Nonetheless, the practical analysis and design method is only a graphical and approximated version for a periodic oscillation in respect to continuous-time systems, that is, *describing function*, in other words, the harmonic balance method(Atherton, 1975). However, as for discrete-time system, there is no method in particular to analyze and design such a control system.

In this paper, first, the robust stability condition which was derived in our previous papers is applied to a sampled-data system containing a single time-invariant nonlinear element. Then, an instability condition for that type of nonlinear feedback system is derived as an inverse problem. By considering restricted areas (two sectors) in the nonlinear characterisitic, the existence of a sustained oscillation is estimated, and the relationship between the stable (unstable) conditions and the approximated result which is derived from the classic describing function is compared. Based on these considerations, the stabilization of nonlinear discrete-time control systems is examined in the frequency domain.

2. EQUIVALENT CLOSED LOOP SYSTEM

In our previous paper(Okuyama *et al.*, 1996; Okuyama *et al.*, 1999), the robust stability for nonlinear sampled-data control systems was analyzed in the frequency domain as a natural expansion of Popov's criterion for continuous-time systems. The control system to be considered here is a sampled-data control system with time-invariant nonlinear characteristic $N(\cdot)$ as shown in Fig. 1. Here, $\mathcal{H}$ is the zero-order-hold which is usually performed in A/D(D/A) converter and $G(s)$ is the transfer function of the system to be controlled, which is expressed by continuous-time.

In addition, suppose that nonlinear characteristic

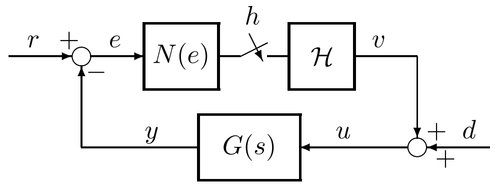

Fig. 1. Nonlinear sampled-data control system.

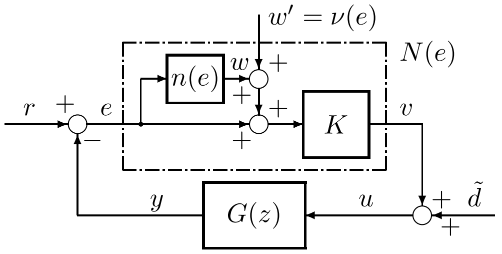

Fig. 2. Equivalent nonlinear discrete-time system.

$N(\cdot)$ is time-invariant and can be written as

$$N(e) = K(e + n(e) + \nu(e)), \quad 0 < K < \infty \quad (1)$$

$$|w| = |n(e)| \le \alpha |e|, \quad 0 < \alpha < 1 \quad (2)$$

$$|w'| = |\nu(e)| < \infty, \quad (3)$$

where $n(e)$ and $\nu(e)$ are nonlinear terms relative to nominal linearized gain K. By rearranging the nonlinear sampled-data control system, Fig. 2 can be obtained, where $G(z)$ is the z-transform of $G(s)$ together with zero-order-hold $\mathcal{H}$. In Fig. 2, r, e, w, $\cdots$ denote discrete-time variables $r(kh)$, $e(kh)$, $w(kh)$, $\cdots$. (Hereafter, these will be abbreviated as $r(k)$, $e(k)$, $w(k)$, $\cdots$).

As for the robust stability (in other words, absolute stability) analysis, it is sufficient to consider only nonlinear term $n(e)$, because nonlinear term $\nu(e)$ can be treated as a disturbance signal. (Althuogh the absolute value of nonlinear term $|K(e + n(e))|$ will be unbounded for $|e| \to \infty$, the absolute value of $|K\nu(e)|$ is bounded. Thus, the global stability of that type of nonlinear system is defined by the nonlinear characteristic $K(e + n(e))$).

Consider new sequences $e_m^*(k)$ and $w_m^*(k)$ ($k = 1, 2 \cdots, N$) which satisfy the following equation:

$$e_m^*(k) = e_m(k) + q \cdot \frac{\Delta e(k)}{h}, \quad (4)$$

$$w_m^*(k) = w_m(k) - \alpha q \cdot \frac{\Delta e(k)}{h}, \quad (5)$$

where q is a non-negative number, $e_m(k)$ and $w_m(k)$ are neutral points of sequences $e(k)$ and $w(k)$, respectively, i.e.,

$$e_m(k) = \frac{e(k) + e(k-1)}{2}, \qquad w_m(k) = \frac{w(k) + w(k-1)}{2},$$

and $\Delta e(k) = e(k) - e(k-1)$ is the backward difference of error. The relationship between these

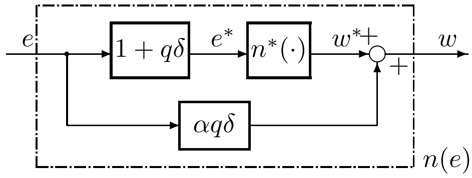

Fig. 3. Nonlinear subsystem.

equations is shown by the block diagram in Fig. 3. In this figure, δ is defined as

$$\delta(z) := \frac{2}{h} \cdot \frac{1 - z^{-1}}{1 + z^{-1}}. \quad (6)$$

Eq. (6) corresponds to the bilinear transformation approximation between z and δ when relating δ to Laplace transform variable s for a continuous-time system. Then, the loop transfer function from w^* to e^* can be given by $F(\alpha, q, z)$ as shown in Fig. 4. Here,

$$F(\alpha, q, z) = \frac{(1 + q\delta(z))KG(z)}{1 + (1 + \alpha q \delta(z))KG(z)}, \quad (7)$$

and r', d' are transformed exogenous inputs.

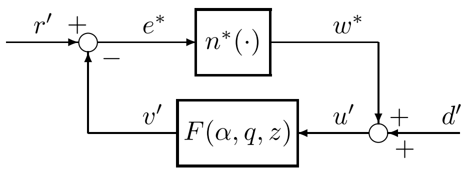

Fig. 4. Equivalent closed loop system.

3. PRELIMINARIES

Let us define a new nonlinear function such as $f(e) := n(e) + \alpha e$. This function belongs to the first and third quadrants. Considering the equivalent linear characteristic which varies with discrete-time $k = 1, 2, \cdots$, it can be written as

$$0 \le \gamma(k) := \frac{f(e(k))}{e(k)} \le 2\alpha. \quad (8)$$

When this type of $\gamma(k)$ is used, sector inequality (2) can be expressed as $n(e(k)) = (\gamma(k) - \alpha)e(k)$.

The following assumption will be provided in regard to the nonlinear characteristics to avoid the difficult problems that are peculiar to nonlinear sampled-data control systems (Kalman, 1957).

[Assumption-1] Error sequence $e(k)$ passes the origin. Specifically, the relationship $\gamma(k-1) = \gamma(k)$ is maintained whenever $e(k-1)e(k) < 0$. Therefore, the line between coordinates $(e(k-1), f(e(k-1)))$ and $(e(k), f(e(k)))$ by linear interpolation also passes the origin. □

This assumption is not too inaccessible. If the sampling period is shorter than the transient response of the system, variations of error $\Delta e(k)$ are also expected to be small when the sequence passes the origin. Hence, Assumption-1 will be satisfied. Even if the sampling period is relatively long, it will be satisfied when nonlinear characteristics are gentle around the origin. Therefore, the above covers a considerably wide range of problems.

Based on the above premise, the following properties can be shown.

[Lemma-1] For a positive integer N (the number of steps), the following inequality holds:

$$\|w_m(k)\|_N \leq \alpha \|e_m(k)\|_N. \tag{9}$$

(Proof) The proof is omitted. □

[Lemma-2] If the following inequality is satisfied in regard to the inner product of the neutral points of $f(e)$ and the backward difference of error:

$$\langle\ w_m(k) + \alpha e_m(k), \Delta e(k)\ \rangle_N \geq 0, \tag{10}$$

the following can be obtained

$$\|w_m^*(k)\|_N \leq \alpha \|e_m^*(k)\|_N \tag{11}$$

for any $q \geq 0$. In the above, $< \cdot, \cdot >_N$ and $\| \cdot \|_N$ denote the inner product and the ℓ_2 norm in the N dimensional space, respectively.
(Proof) The proof was written in our previous papers (Okuyama *et al.*, 1996; Okuyama *et al.*, 1999). □
The left side of Eq. (10) can be expressed in terms of nonlinear function $f(\cdot)$.
[Lemma-3] For any step N, the following equation is valid:

$$\langle\ w_m(k) + \alpha e_m(k), \Delta e(k)\ \rangle_N = \frac{1}{2}\sum_{k=1}^{N}(f(e(k)) + f(e(k-1)))\Delta e(k). \tag{12}$$

(Proof) The proof is omitted. □

If $\sigma(N)$ was used for the right side of Eq. (12), it can be shown that $\sigma(N)$ is the total area of the trapezoid formed by sampling point $(f(e(k-1)), f(e(k)))$ on nonlinear curve $f(e)$ and error step width $\Delta e(k)$. The total area of trapezoid $\sigma(N)$ can be rewritten by the following.
[Lemma-4] For any step N,

$$\sigma(N) = \frac{1}{2}(f(e(N))e(N) - f(e(0))e(0)) + \epsilon(N), \tag{13}$$

where $\epsilon(N) = \frac{1}{2}\sum_{k=1}^{N} f_0(k) \cdot \Delta e(k)$. Here, $f_0(k)$ is an intercept at which the straight line passing sample points p_k and p_{k-1} on the nonlinear function $f(e)$ intersects the vertical axis.
(Proof) The proof is omitted. □

4. CLASSES OF SAMPLED-DATA SYSTEMS

[Assumption-2] The total area of a trapezoid, allowing for signs of coordinate $(e(k), f(e(k)))$, $(k = 0, 1, 2, \cdots, N)$ which traces a nonlinear curve is always non-negative (i.e., $\sigma(N) \geq 0$ regardless of the transient response. □

Although this Assumption seems to be too inaccessible, some of the following sampled-data systems can satisfy it.

(1) Nonlinear sampled-data systems, of which point $(e(k), f(e(k)))$ traces the same points on the nonlinear curve belongs to Class $\mathbf{S}_c$.
(2) Nonlinear sampled-data systems (which satisfy $\epsilon(N) = 0$ at any step N, i.e., $f_0(k) = 0$ $(k = 1, 2, \cdots, N)$) are classified into Class $\mathbf{S}_l$.
(3) Nonlinear sampled-data systems which satisfy $\epsilon(N) \geq 0$, at any step N (i.e., $f_0(k) \cdot \Delta e(k) \geq 0$, $(k = 1, 2, \cdots, N)$) are classified into Class $\mathbf{S}_r$.

Note: The fulfillment of **(3)** is expected from systems where response $(e(k), f(e(k)))$ on a nonlinear curve turns in a clockwise direction. The systems in Class $\mathbf{S}_r$ naturally contain the above-mentioned systems of Class $\mathbf{S}_l$ which satisfies $\epsilon(N) = 0$.

5. ROBUST STABILITY CONDITION

As was described in our previous paper (when using the subsystem in Fig. 3 instead of nonlinear element $n(\cdot)$ in Fig. 2), the robust stability condition for the above system can be given by using a small gain theorem in regard to the closed loop system as shown in Fig. 4.
[Theorem-1] If there exists a $q \geq 0$ in which the sector parameter α in regard to nonlinear term $n(\cdot)$ satisfies the following inequality, then the nonlinear sampled-data control system in Fig. 1 (equivalent to Fig. 2) is robust (or absolute) stable in the ℓ_2 sense:

$$\xi(q,\omega) = \frac{U^2 + V^2}{-q\Omega V + \sqrt{q^2\Omega^2V^2 + (U^2+V^2)\{(1+U)^2 + V^2\}}} < \frac{1}{\alpha}, \qquad \forall\omega \in [0, \omega_c]. \tag{14}$$

Here, $\Omega(\omega)$ is the distorted frequency of ω, and is given as

$$\delta(\mathrm{e}^{j\omega h}) = j\Omega(\omega) = j\frac{2}{h}\tan\left(\frac{\omega h}{2}\right)$$

from Eq. (6), and ω_c is a cut-off frequency which is the range satisfying *Shannon's sampling theorem*. Moreover, U and V are the real and the imaginary parts of $KG(\mathrm{e}^{j\omega h})$, respectively.
(Proof) The proof is obtained from

$$|F(\alpha, q, \mathrm{e}^{j\omega h})| = \left|\frac{(1 + jq\Omega(\omega))KG(\mathrm{e}^{j\omega h})}{1 + (1 + j\alpha q\Omega(\omega))KG(\mathrm{e}^{j\omega h})}\right| < \frac{1}{\alpha}. \tag{15}$$

based on Eq. (7). □ (Okuyama *et al.*, 1999; Okuyama *et al.*, 1998a)

Theorem-1 corresponds to a discrete-time version of Popov's criterion (Harris *et al.*, 1983). Since inequality (14) in Theorem-1 is for all ω considered and a certain q, the condition results in the following min-max problem:

$$\xi(q_0,\omega_0) = \min_q \max_\omega \xi(q,\omega) < \frac{1}{\alpha}. \quad (16)$$

That is, if inequality (16) is satisfied, the nonlinear sampled-data system as shown in Fig. 1 is stable when the nominal linear sampled-data system with gain K is stable.

6. INSTABILITY CONDITION

On the contrary, in this section, instability problem of the nonlinear discrete-time system is examined when the nominal system with gain K is unstable(Desoer and Vidyasagar, 1975). Consider the frequency transfer function $F(\alpha, q, \mathrm{e}^{j\omega h})$ to be a linear causal operator $\mathcal{F}$ in an ℓ_2 space, i.e., $\mathcal{F}: \ \ell_2 \to \ell_2$. In addition, $\mathcal{F}$ is assumed to be unstable in the sense that

$$\mathcal{U} = \{u'_m \in \ell_2 \mid v'_m = \mathcal{F}u'_m \in \ell_2\} \quad (17)$$

is not all of ℓ_2. Obviously, $\mathcal{U}$ is a set of stabilizable inputs u'_m (which is a subspace of ℓ_2). Here, u'_m and v'_m are neutral points of sequences $u'(k)$ and $v'(k)$, respectively.

Since $\mathcal{U}$ is not all of ℓ_2, the orthogonal subspace of it, $\mathcal{U}^\perp$, is nontrivial in the ℓ_2 space. If exogenous input d'_m exists in the orthogonal subspace (i.e., $d'_m \in \mathcal{U}^\perp$), $\langle u'_m, d'_m\rangle_N = 0$ must hold. In such a case, from the relation $w^* = u' - d'$,

$$\begin{aligned}\|w^*_m\|^2_N &= \|u'_m\|^2_N - 2\langle u'_m, d'_m\rangle_N + \|d'_m\|^2_N,\\ &= \|u'_m\|^2_N + \|d'_m\|^2_N.\end{aligned}$$

Hence,

$$\|w^*_m(k)\|_N \geq \|u'_m(k)\|_N. \quad (18)$$

Furthermore, when considering e^* as a stabilizable input, the following set is given:

$$\mathcal{E} = \{e^*_m \in \ell_2 \mid v'_m = \mathcal{F}(w^*_m + d'_m) \in \ell_2\} \quad (19)$$

Since $\mathcal{E}$ is similarly not all of ℓ_2, the orthogonal subspace of it, $\mathcal{E}^\perp$, is nontrivial in the ℓ_2 space. If exogenous input r'_m exists in the orthogonal subspace (i.e., $r'_m \in \mathcal{E}^\perp$), $\langle r'_m, e^*_m\rangle_N = 0$ must hold. From the relation $v' = r' - e^*$,

$$\begin{aligned}\|v'_m\|^2_N &= \|r'_m\|^2_N - 2\langle r'_m, e^*_m\rangle_T + \|e^*_m\|^2_T,\\ &= \|r'_m\|^2_N + \|e^*_m\|^2_N.\end{aligned}$$

Hence,

$$\|v'_m(k)\|_N \geq \|e^*_m(k)\|_N. \quad (20)$$

By using inequalities (11), (18) and (20), the following relation is obtained:

$$\|u'_m(k)\|_N \leq \alpha\|v'_m(k)\|_N. \quad (21)$$

Then, inequality (21) can be rewritten as follows:

$$\|u'_m(k)\|_N \leq \alpha \sup_\omega |F(q,\alpha,\mathrm{e}^{j\omega h})| \cdot \|u'_m(k)\|_N. \quad (22)$$

However, if a small gain theorem (Desoer and Vidyasagar, 1975), i.e.,

$$\sup_\omega |F(q,\alpha,\mathrm{e}^{j\omega h})| < 1/\alpha \quad (23)$$

is satisfied for any $q \geq 0$, the above inequality (22) is contradicted for $N \to \infty$. Thus, the following should be written:

$$u'_m \notin \ell_2 \ \text{ and } \ v'_m \notin \ell_2. \quad (24)$$

It is obvious that the nonlinear discrete-time feedback system is unstable. Thus, the following theorem can be given.

[**Theorem-2**] If a small gain theorem (23) in regard to the closed loop system as shown in Fig. 4 is satisfied (i.e., inequalities (14), (16) are satisfied), the nonlinear discrete-time feedback system is unstable when the nominal linear discrete-time system with gain K is unstable.

(Proof) The proof would be obvious from the above derivation process. □

7. DESCRIBING FUNCTION AND STABILIZATION

A method of the amplitude dependent stability analysis for actual higher order nonlinear systems is harmonic balance, i.e., describing function. Although the analysis is based on an approximation in the Fourier series expansion, it is still a useful method for designing a nonlinear feedback system. In complex numbers, the describing function is defined as $N(A) = \frac{U_1}{A} \cdot \mathrm{e}^{j\phi_1}$, where A is the amplitude of input signal to the nonlinear function, $U_1 = \sqrt{a_1^2 + b_1^2}$ and $\phi_1 = -\tan^{-1}\frac{b_1}{a_1}$. When considering the above in a discrete-time domain, the following expression can be given:

$$a_1 = \frac{\triangle\theta}{2\pi}\sum_{\theta=-\pi}^{\pi}(u(\theta)\cos\theta + u(\theta+\triangle\theta)\cos(\theta+\triangle\theta)),$$

$$b_1 = \frac{\triangle\theta}{2\pi}\sum_{\theta=-\pi}^{\pi}(u(\theta)\sin\theta + u(\theta+\triangle\theta)\sin(\theta+\triangle\theta)).$$

Here, $\theta = k\omega h$ and $\triangle\theta = \omega h$. By using these equations, describing function, e.g., a_1 can be calculated numerically.

The results of Theorem-1 and Theorem-2 can be applied to the design of nonlinear discrete-time control systems. That is, an appropriate nonlinear gain and a stabilizing compensator $\mathcal{C}$ as shown in Fig. 4 are determined based on the above concept. In this paper, the following phase lead

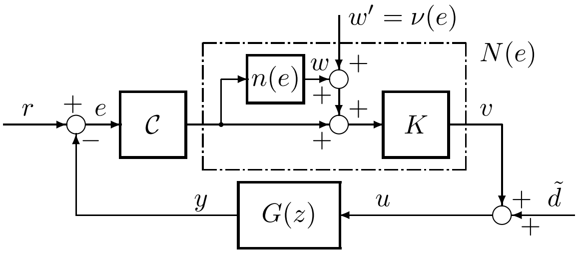

Fig. 5. Nonlinear discrete-time system with stabilizing compensator $\mathcal{C}$.

compensator is used(Okuyama *et al.*, 1998a):

$$\mathcal{C} = \frac{\delta + a}{\delta + b}, \quad 0 < a < b. \tag{25}$$

8. NUMERICAL EXAMPLES

[Example-1] Consider the following controlled system:

$$G(s) = \frac{(s+6)}{s(s+1)(s+2)}. \tag{26}$$

The stability region for linear gain K can be given by $0 < K < 1.2$ when the sampling period is chosen as $h = 0.2$. When choosing the nominal gain $K = 0.7$, $\xi(q_0, \omega_0) = 1.39$ and $\alpha < 0.72$ are obtained from Eqs. (14) and (16). The stability region (stable sector) becomes $[0.19, 1.20]$. (In this case, Aizerman's conjecture is valid(Okuyama *et al.*, 1998b)). On the other hand, when choosing the nominal gain $K = 1.7$, the instability region (unstable sector) can be determined as $[1.49, 1.91]$. Figure 6 shows the stable/unstable sectors and a nonlinear characteristic. Here, *sigmoid type function* $N(e) = \frac{4}{\pi} \cdot \tan^{-1}(4e^3(t))$ is considered as a nonlinear curve.

As shown in the figure, there is an area between stable and unstable sectors, which cannot be defined. Figure 7 shows time responses of the nonlinear discrete-time control system. In this example, stable/unstable (pseudo)periodic behaviors will be seen in the responses, which correspond to unstable/stable limit cycles for a continuous-time system in a state space. The amplitude of sustained oscillation can approximately be estimated from stable/unstable sectors shown in Fig. 6 and from describing function shown in Fig. 8. When using a phase lead compesator given in Eq. (25) such as $\mathcal{C} = \frac{\delta + 2.5}{\delta + 3.0}$, the stability region can be determined as, e.g., $0.12 < K < 1.68$. The sampled-data control system is stabilized as shown in Fig. 9.

[Example-2] Consider the following controlled

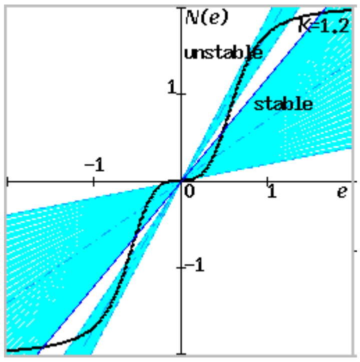

Fig. 6. Nonlinear characteristic and stable/unstable sectors for Example-1.

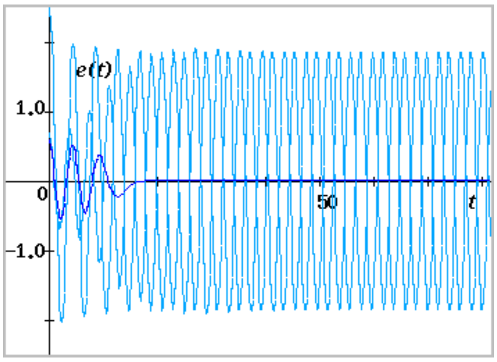

Fig. 7. Time responses for Example-1 ($r = e(0) = 0.6, 0.7, 2.5$).

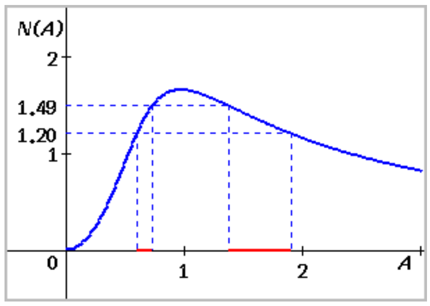

Fig. 8. Describing function for Example-1.

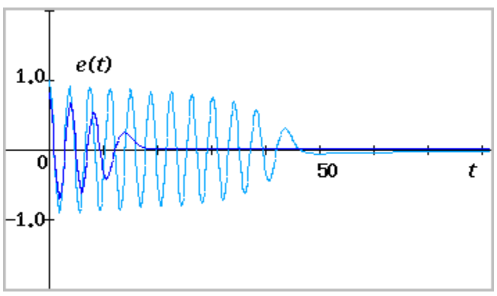

Fig. 9. Stabilized results for Example-1 using a phase lead compensator ($r = e(0) = 0.8, 1.0$).

system:

$$G(s) = \frac{2.5(s+0.5)}{s(s+2)(s-1)}. \tag{27}$$

In this example, the stability region can be given by $K > 1.91$ when the sampling period is chosen as $h = 0.05$. When choosing the nominal gain $K = 2.5$, $\xi(q_0, \omega_0) = 4.22$ and $\alpha < 0.24$ are obtained from Eqs. (14) and (16). The stable sector becomes $[1.91, 3.09]$. (The Aizerman conjecture is valid also in this case for the lower bound.) On the other hand, when choosing the nominal gain $K = 1.0$, $\xi(q_0, \omega_0) = 4.65$ and $\alpha < 0.21$ are obtained. Thus, the unstable sector becomes $[0.78, 1.21]$. Here, sigmoid type function $N(e) = 0.5e(t) + \tan^{-1}(10e^3(t))$ is considered as a nonlinear curve.

Figure 10 shows the nonlinear characteristic and the stable/unstable sectors. As is obvious from the figure, there is a considerable size of undefined area between the stable and the unstable sectors. However, we can also estimate a sustained oscillation which corresponds to a stable limit cycle for a continuous-time system in a state space. When using the same compesator as shown in Example-1 and adjusting slightly the nonlinear characteristic as $N(e) = 1.5e(t) + \tan^{-1}(10e^3(t))$, the sampled-data control system is stabilized as shown in Fig. 12.

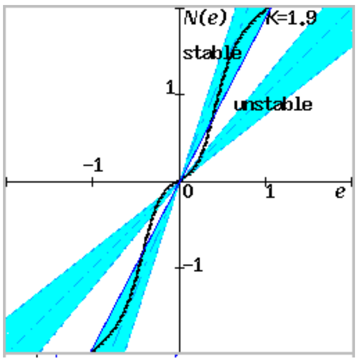

Fig. 10. Nonlinear characteristic and stable/unstable sectors for Example-2.

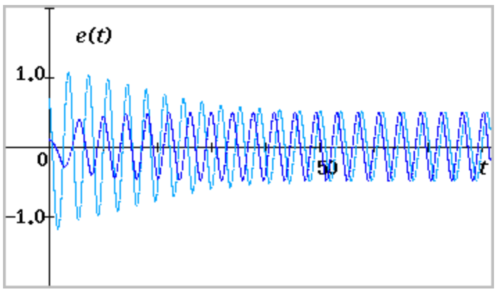

Fig. 11. Time responses for Example-2 ($r = e(0) = 0.1, 0.7$).

9. CONCLUSIONS

This paper analyzed the amplitude dependent behavior of nonlinear sampled-data control systems in a frequency domain. First, the robust stability condition which was derived in our previous papers was applied to a sampled-data control system containing a single time-invariant nonlinear element in the forward path. Then, an instability condition for that type of nonlinear feedback

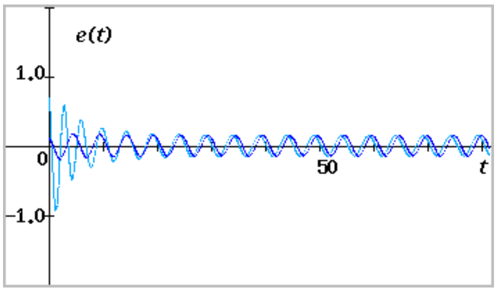

Fig. 12. Stabilized results for Example-2 ($r = e(0) = 0.1, 0.7$).

system was derived. By considering two sectors (stable and unstable) in the nonlinear characteristic, the existence of a sustained oscillation could be estimated. This conept will be extended to the multi-loop nonlinear discrete-time feedback systems(Okuyama *et al.*, 2001).

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