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LMI BASED MPC

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Abstract: In this work, we present a Model Predictive Controller (MPC) based on Linear Matrix Inequalities (LMI's). As in the standard MPC algorithms, at each (sampling) time, a convex optimization problem is solved to compute the control law. The optimization involves constraints written as LMI's, including those normally associated to MPC problems, such as input and output limits. Even though a state space representation is used, only the measurable output and some statistic properties of the not measurable states are used to determine the controller, hence it is an output feedback control design method. Stability of the closed loop system is demonstrated. The design technic is illustrated with a numerical example.

Keywords: Linear Matrix Inequalities, Predictive Control, Output Feedback Control, Convex Programming.

1. INTRODUCTION

MPC is the most popular industrial MIMO control strategy (Camacho and Bordons, 1995). Among the reasons for the popularity we may mention that all real systems are subjected to physical constraints, such as actuator's operation limits, and they may be explicitly considered, in the MPC formulation. It is mainly, a control technic for systems with slow dynamics, even though this drawback is rapidly changing (Maciejowsky, 2002).

In the MPC scheme, the control law is obtained from an optimization problem whose objective function weights the control efforts and the deviations from the set point. The optimization problem normally includes constraints on the input (normally hard constraints), output and state (usually soft constraints). The optimization is performed over a (prediction) horizon which is continuously moved forward in time, since only the first control law is applied (out of those calculated over the horizon) to the system (R. de Keyser and Dumortier, 1988; Clarke and Mohtadi, 1989).

The MPC was introduced in the 70's (J. Richalet and Papon, 1978), and ever since much research have been done in the area to assure, among other, stability and feasibility of the problem (Rawlings and Muske, 1993), (D.W. Clarke and Scattolini, 1991), (Clarke and Scattolini, 1991). MPC is a methodology that, working always in the time domain, lets the operator handle easily, physical performance requirements, such as upper and lower bounds on the process variables, and tuning of the closed loop. At the same time, very little knowledge of the theory involved is required. Being these, maybe the reason for its industrial use.

In this work, an extension to the output feedback case, of the methodology proposed in (M. Kothare and Morari, 1996) is featured. Here we con-

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sider the constrained case –input and output restrictions are included explicitly in the problem formulation–. To assure stability, and infinite horizon is considered in the objective –quadratic– function. The optimization problem is formulated in terms of LMI's. There exist already dedicated powerful algorithms that allow to obtain a solution in polynomial time, many times comparable to those necessary to get an analytical solution of a similar problem (S. Boyd and Balakrishnan, 1994). It is important to keep in mind that an "on line" and on time solution is fundamental to MPC.

2. PROBLEM STATEMENT

Consider the discrete linear time invariant system represented by:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) \end{aligned} \tag{1}$$

where $x(k) \in \mathrm{IR}^{\mathrm{n}}$, $u(k) \in \mathrm{IR}^{\mathrm{m}}$ and $y(k) \in \mathrm{IR}^{\mathrm{q}}$ are respectively, the state, input and output of the system. $A \in \mathrm{IR}^{\mathrm{n\times n}}$, $B \in \mathrm{IR}^{\mathrm{n\times m}}$ and $C \in \mathrm{IR}^{\mathrm{q\times n}}$ are constant matrices.

We want to find, at each sampling time k, a control law in the MPC framework, that stabilizes 1, with the following representation (dynamic controller):

$$\begin{aligned} x_c(k+1) &= A_c x_c(k) + B_c y(k) \\ u(k) &= C_c x_c(k) \end{aligned} \tag{2}$$

where $x_c(k) \in \mathrm{IR}^{\mathrm{n}}$ and A_c, B_c, C_c are matrices of appropriate dimensions. The problem reduces, for each sampling time, to determine the matrices A_c, B_c, C_c, so that the closed loop system is stable.

The closed loop system (1 and 2) may be represented by:

$$\begin{aligned} \hat{x}(k+1) &= \hat{A}\hat{x}(k) + \hat{B}u(k) \\ u(k) &= K\hat{x}(k) \\ y(k) &= \hat{C}\hat{x}(k) \end{aligned} \tag{3}$$

where

$$\hat{A} = \begin{bmatrix} A & 0 \\ B_c C & A_c \end{bmatrix}, \; \hat{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \qquad K = \begin{bmatrix} 0 \; C_c \end{bmatrix}, \; \hat{C}\begin{bmatrix} C \; 0 \end{bmatrix}, \tag{4}$$

and

$$\hat{x}(k) = \begin{bmatrix} x(k) \\ x_c(k) \end{bmatrix} \in \mathrm{IR}^{2\mathrm{n}} \tag{5}$$

As mentioned above, we consider an infinite horizon objective function to assure stability. In terms of the variables of the closed loop system, the function is given by:

$$J_\infty = \sum_{i=0}^{\infty} \{\hat{x}(k+i|k)^T \hat{Q}\hat{x}(k+i|k) + u(k+i|k)^T Ru(k+i|k)\} \tag{6}$$

where

$$\hat{Q} = \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}, \tag{7}$$

$Q \geq 0$, $R > 0$ and $\hat{x}(k+i|k)$ represents the prediction of $\hat{x}$ at instant $k+i$, given $\hat{x}(k)$. Evidently, $\hat{x}(k|k) = \hat{x}(k)$.

Let us introduce the quadratic function:

$$V(\hat{x}(k|k)) = \hat{x}(k|k)^T P\hat{x}(k|k), \; P > 0. \tag{8}$$

If we have that $\hat{x}(\infty|k) = 0$ then $V(\hat{x}(\infty|k)) = 0$.

Let us suppose for the moment, that for any instant k and $i \geq 0$, the following condition is satisfied,

$$\begin{aligned} V(\hat{x}(k+i+1|k)) - V(\hat{x}(k+i|k)) < \\ -\{\hat{x}(k+i|k)^T \hat{Q}\hat{x}(k+i|k) + \\ u(k+i|k)^T Ru(k+i|k)\} \end{aligned} \tag{9}$$

If we sum inequality 9 from $i = 0$ up to $i = \infty$ with $\hat{x}(\infty|k) = 0$, we obtain:

$$-V(\hat{x}(k|k)) < -J_\infty(k), \tag{10}$$

that is, 8 is an upper bound for the objective function 6. The algorithm proposed, much as in (M. Kothare and Morari, 1996), is that of minimizing 8 subject to conditions that assure 9, at each sampling time.

Even if not all states are measurable, some are. For the rest, even if unknown, some statistics must be available in order to define its condition in probabilistic terms.

It is possible then to partition the state vector $x(k)$ as:

$$x(k) = \begin{bmatrix} x_r(k) \\ x_m(k) \end{bmatrix} \in \mathrm{IR}^{\mathrm{n}} \tag{11}$$

where $x_m(k) \in \mathrm{IR}^{\mathrm{p}}$ represents the measurable (known) states and $x_r(k) \in \mathrm{IR}^{\mathrm{n-p}}$ the non measurable states, $p \leq q$. The unknown states will be characterized by probabilistic parameters such as probability density, mean ($E < x_r >$) and correlation matrix ($E < x_r x_r^T >$).

As mentioned, all real processes have constraints in their variables, those restrictions may be included in the algorithm as "sufficient" conditions expressed as LMI's. Input ($u(k)$) constraints normally represent physical limits (such as valve saturation, etc.) They are usually considered "hard" constraints since they have to be satisfied. We will consider this constraint through the euclidean norm, given by:

$$\|u(k+i|k)\|_2 \leq u_{\max}, \quad i \geq 0.$$

Output constraints are less restrictive, since they normally represent performance requirements. We will consider them, similarly, through the euclidean norm, given in this case by:

$$\|y(k+i|k)\|_2 \leq y_{\max} \quad i \geq 1.$$

Vector $y(k+i|k)$, represents system's predicted output at time $k+i$, based on the output at time k, $y(k)$. The output constraints are imposed on future values ($i \geq 1$), since it does not make any sense to apply it to the actual value ($i = 0$).

3. MAIN RESULT

The following theorem gives conditions for the existence of a stabilizing controller:

Theorem 1. Given the matrix $\Phi \in \mathbb{R}^{2n}$ and the known limit vectors $y_{\max}$, and $u_{\max}$, system 1 is stabilized by a controller of the form 2 if there exist matrices symmetric positive definite $X, Y \in \mathbb{R}^{n}$, and matrices $F \in \mathbb{R}^{n\times m}$, $L \in \mathbb{R}^{q\times n}$ and $Z \in \mathbb{R}^{n\times n}$, solutions of the following optimization problem:

$$\max \gamma^{-1} \tag{12}$$

subject to

$$\begin{bmatrix} Y & I \\ I & X \end{bmatrix} - \gamma^{-1}\Phi > 0 \tag{13}$$

$$\begin{bmatrix} Y & I & \Gamma & Z & 0 & 0 \\ * & X & A & \Delta & 0 & 0 \\ * & * & \Pi & \Omega & 0 & 0 \\ * & * & * & X & L^T & XQ^{1/2} \\ * & * & * & * & R^{-1} & 0 \\ * & * & * & * & * & I \end{bmatrix} > 0 \tag{14}$$

$$\begin{bmatrix} \gamma^{-1}u_{max}^2 I & 0 & L \\ * & Y & I \\ * & * & X \end{bmatrix} > 0 \tag{15}$$

$$\begin{bmatrix} Y & I & A^T C^T \\ * & X & (CAX + CBL)^T \\ * & * & \gamma^{-1} y_{\max}^2 I \end{bmatrix} > 0, \tag{16}$$

where $\Gamma = YA + FC$, $\Delta = AX + BL$, $\Pi = Y - Q$ and $\Omega = I - QX$.

Remark 2. For simplicity and being all symmetric matrices, in must of the cases only the upper triangular terms are shown. The $*$ accounts for the other elements.

Proof: Since 8 is an upper bound for the objective function 6, we may try to minimize such bound by solving the following problem:

$$\min \gamma \tag{17}$$

subject to

$$\hat{x}(k|k)^T P \hat{x}(k|k) < \gamma, \tag{18}$$

using the Schur complement, inequality 18 maybe written as:

$$\begin{bmatrix} \gamma I & \hat{x}(k|k)^T \\ \hat{x}(k|k) & P^{-1} \end{bmatrix} > 0 \tag{19}$$

which, by using the Schur complement, is equivalent to:

$$\gamma^{-1}\hat{x}(k|k)\hat{x}(k|k)^T < P^{-1}. \tag{20}$$

Since not all states are measurable, we have to recourse to the knowledge we have from them, If it is of statistical nature, we take the expected value of 20, considering 5 and 11, to obtain:

$$P^{-1} - \gamma^{-1}\Phi > 0 \tag{21}$$

where

$$\Phi = \begin{bmatrix} E\langle x_r x_r^T \rangle & E\langle x_r \rangle x_m^T & E\langle x_r \rangle x_c^T \\ * & x_m x_m^T & x_m x_c^T \\ * & * & x_c x_c^T \end{bmatrix}. \tag{22}$$

Let us partition matrices P^{-1} and P in the form,

$$P^{-1} = \begin{bmatrix} X & U \\ U^T & \hat{X} \end{bmatrix}; \; P = \begin{bmatrix} Y & V \\ V^T & \hat{Y} \end{bmatrix}, \tag{23}$$

and define the matrix:

$$T = \begin{bmatrix} Y & I \\ V^T & 0 \end{bmatrix}. \tag{24}$$

Without loss of generality, we may suppose that matrix V is a non singular matrix (C. Scherer and Chilali, 1997), Hence T is also regular.

By Schur complement, 21 is equivalent to:

$$T^T P^{-1} T - \gamma^{-1}\Phi > 0 \tag{25}$$

from where we obtain 13.

To obtain 14, we replace 1 and 8 in 9 to obtain:

$$\begin{gathered}(\hat{A} + \hat{B}K)^T P (\hat{A} + \hat{B}K) - P + \\ \hat{Q} + K^T R K < 0.\end{gathered} \tag{26}$$

which readily shows that, if inequality 14 is satisfied, stability is assured.

By Schur's complement, 26 is equivalent to:

$$\begin{bmatrix} P^{-1} & (\hat{A}+\hat{B}K)P^{-1} & 0 \\ * & P^{-1}-P^{-1}\hat{Q}P^{-1} & P^{-1}K^T \\ * & * & R^{-1} \end{bmatrix} > 0. \quad (27)$$

Premultiplying 27 by $\hat{T}_2^T$ and postmultiplying by $\hat{T}_2$, where:

$$\hat{T}_2 = \begin{bmatrix} T & 0 & 0 \\ 0 & T & 0 \\ 0 & 0 & I \end{bmatrix}, \quad (28)$$

we obtain:

$$\begin{bmatrix} T^T P^{-1} T & \Psi & 0 \\ * & \Upsilon & T^T P^{-1} K^T \\ * & * & R^{-1} \end{bmatrix} > 0. \quad (29)$$

where $\Psi = T^T(\hat{A}+\hat{B}K)P^{-1}T$ and $\Upsilon = T^T P^{-1} T - T^T P^{-1}\hat{Q}P^{-1}T$.

Replacing 4, 23 and 24 in 29 and by defining:

$$\begin{aligned} F &= VB_c \\ L &= C_c U^T \\ Z &= YAX + FCX + YBL + VA_cU^T \end{aligned} \quad (30)$$

we obtain:

$$\begin{bmatrix} Y & I & \Gamma & Z & 0 \\ * & X & A & \Delta & 0 \\ * & * & \Pi & \Omega & 0 \\ * & * & * & X - XQX & L^T \\ * & * & * & * & R^{-1} \end{bmatrix} > 0 \quad (31)$$

from which 14 is obtained by using Schur's complement on the term (4,4) –$X - XQX$–. Now, we recall that $\min\gamma$ is equivalent to $\max\gamma^{-1}$.

Regarding the input and output constraints, let us now define matrices T_1 and T_3 that will be used later.

$$T_1 = \begin{bmatrix} I & 0 \\ 0 & T \end{bmatrix}; \quad T_3 = \begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix}. \quad (32)$$

We remark that if conditions 9 and 13 are satisfied then:

$$\hat{x}(k+i|k)^T P \hat{x}(k+i|k) < \gamma, \quad i \geq 1. \quad (33)$$

that is, the ellipsoid:

$$\mathcal{E} = \{z | z^T P z \leq \gamma\}$$

is an invariant ellipsoid for the predicted values of the states.

Then we may, at instant k, impose the following euclidean constraint to all future control (even if we will only use the next one):

$$\|u(k+i|k)\|_2 \leq u_{\max}, \quad i \geq 0, \quad (34)$$

then it follows,

$$\begin{aligned} \max_{i\geq 0} \|u(k+i|k)\|_2^2 &= \max_{i\geq 0} \|K\hat{x}(k+i|k)\|_2^2 \\ &\leq \max_{z\in\mathcal{E}} \|Kz\|_2^2 \\ &= \lambda_{\max}(W^{1/2}K^TKW^{1/2}). \end{aligned} \quad (35)$$

where $W^{-1} = \gamma^{-1}P$. From 35, by using Schur's complement, we can write:

$$\begin{bmatrix} u_{\max}^2 I & KW \\ WK^T & W \end{bmatrix} > 0, \quad (36)$$

or equivalently:

$$\begin{bmatrix} \gamma^{-1}u_{\max}^2 I & KP^{-1} \\ P^{-1}K^T & P^{-1} \end{bmatrix} \quad (37)$$

Replacing 4, 23 and 24 in 37 and by using definitions 30 we obtain 15.

Regarding the output, and again using the euclidean norm, we want to assure $\|y(k+i|k)\|_2 \leq y_{\max}$, $i \geq 1$. We have that:

$$\begin{aligned} &\max_{i\geq 0} \|y(k+i|k)\|_2^2 = \\ &\max_{i\geq 0} \|\hat{C}(\hat{A}+\hat{B}K)\hat{x}(k+i|k)\|_2^2, \quad i \geq 0 \\ &\leq \max_{z\in\mathcal{E}} \|\hat{C}(\hat{A}+\hat{B}K)z\|_2^2, = \\ &\lambda_{\max}(W^{1/2}(\hat{A}+\hat{B}K)^T\hat{C}^T\hat{C}(\hat{A}+\hat{B}K)W^{1/2}) \end{aligned} \quad (38)$$

then $\|y(k+i|k)\|_2^2 \leq y_{\max}^2$, $i \geq 1$ if:

$$\begin{bmatrix} P^{-1} & P^{-1}(\hat{A}+\hat{B}K)^T\hat{C}^T \\ \hat{C}(\hat{A}+\hat{B}K)P^{-1} & \gamma^{-1}y_{\max}^2 I \end{bmatrix} > 0, \quad (39)$$

premultiplying 39 by T_3^T (defined in 32) and postmultiplying by T_3 and by using 4, 23, 24, 30 in the resulting inequality, we obtain 16.

3.1 *Controller construction*

Once a solution to (12 - 16 is obtained, a controller might be readily built by assuming any regular matrix U (on which no restraint has been imposed, other than regularity (C. Scherer and Chilali, 1997)) and the controller is given by:

- $V = (I - YX)(U^T)^{-1}$
- $C_c = L(U^T)^{-1}$
- $B_c = V^{-1}F$
- $A_c = V^{-1}Z(U^T)^{-1}$

3.2 *Peak bound constraints*

In theorem 1 we have restrictions on the input and output variables through the euclidean norm. Other time domain specifications may be included. For instance, peak bounds on each component of the control variable, may be accounted for by:

$$|u_j(k+i|k)| \leq u_{j,\max} \quad i \geq 0,\ j = 1, 2, \ldots, \mathrm{m}.$$

Now,

$$\begin{aligned}\max_{i\geq 0} |u_j(k+i|k)|^2 &= \max_{i\geq 0} |(K\hat{x}(k+i|k))_j|^2 \\ &\leq \max_{z\in\mathcal{E}} |(Kz)_j|^2 \\ &\leq \|(KW^{1/2})_j\|_2^2, \\ &= (KWK^T)_{jj}\end{aligned}$$

Thus, the existence of a matrix $\mathcal{S}$ such that:

$$\begin{bmatrix} \mathcal{S} & K \\ K^T & W^{-1} \end{bmatrix} > 0, \qquad (40)$$

with $\mathcal{S}_{jj} \leq u_{j,\max}^2$, $i \geq 0$, $j = 1, 2, \ldots, \mathrm{m}$, guarantees that $|u_j(k+i|k)| \leq u_{j,\max}$, for all predicted values and all entries (i, j).

Premultiplication of 40 by T_1^T and postmultiplication by T_1, will lead to a LMI in the same variables as in theorem 1.

3.3 *Alternatives to handle unmeasured states*

Condition 13 arises from some statistical knowledge of the unmeasurable states. Some times, other information is also available, such as maxima and minima values of such states. In such cases condition 13 may be simply replace by:

$$\begin{bmatrix} \gamma I & \hat{x}_i^T \\ \hat{x}_i & P^{-1} \end{bmatrix} > 0; \qquad (41)$$

where $i = 1, 2, \ldots, 2^{n-p}$, i.e., all the extreme (possible) values of the state vector or of the unmeasurable part of it. In fact the range of extreme values may be reduced as long as the system approaches steady states values.

4. NUMERICAL EXAMPLE

Next, we present a numerical example to feature the algorithm. A model taken from (M. Kothare and Morari, 1996), slightly modified (a measurable output is included in the model, i.e., not all states are available for feedback), is used. The model is:

$$\begin{aligned} x(k+1) &= \begin{bmatrix} 1 & 0.1 \\ 0 & 0.99 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 0.0787 \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) \end{aligned} \qquad (42)$$

The sampling time is 0.1 s, and the state initial condition is $x(0) = x_c(0) = [0.05 \quad 0]^T$. Also $Q = I$ and $R = 0.00002$. For the unmeasured state we will suppose a Gaussian probability density function with mean zero and correlation matrix of 0.5., that is:

$$\begin{aligned} E < x_2 > &= 0 \\ E < x_2 x_2^T > &= [0.5] \end{aligned}$$

First we have impose no restrictions in the input/output variables obtaining the following profiles (the problem was solved, in each iteration, using Matlab –including its LMI Toolbox–):

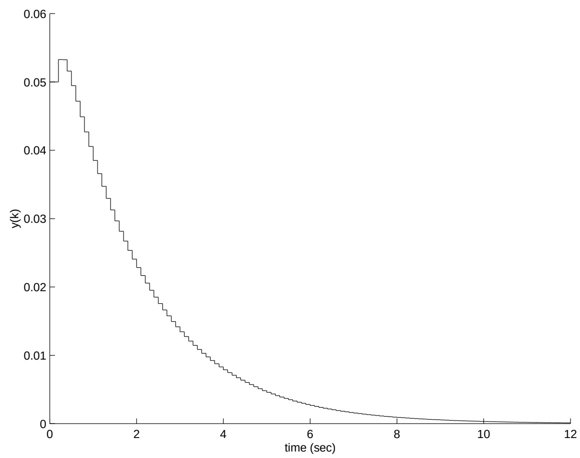

Fig. 1. Performance profile of the LMI based MPC. The output.

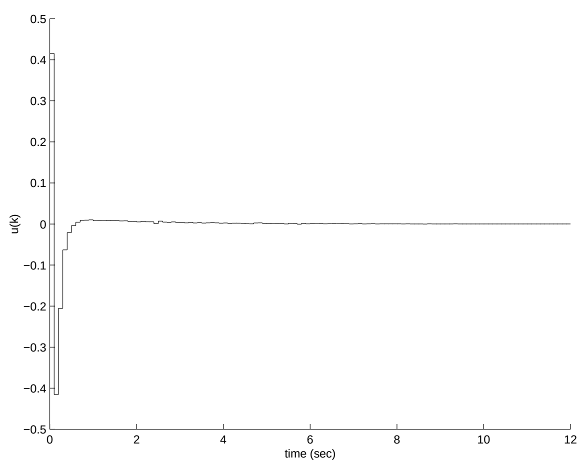

Fig. 2. Performance profile of the LMI based MPC. The control.

Then, we have impose $u_{\max} = 1$ and $y_{\max} = 3$, obtaining the following profile for the output y and control u of the system:

5. CONCLUSIONS

In this work an output feedback control design method is presented for Model Predictive Controllers based on Linear Matrix Inequality constraints. The controller assures stability of the closed loop system. Input and output constraints were included (as sufficient conditions) as additional LMIs. Other performance requirements

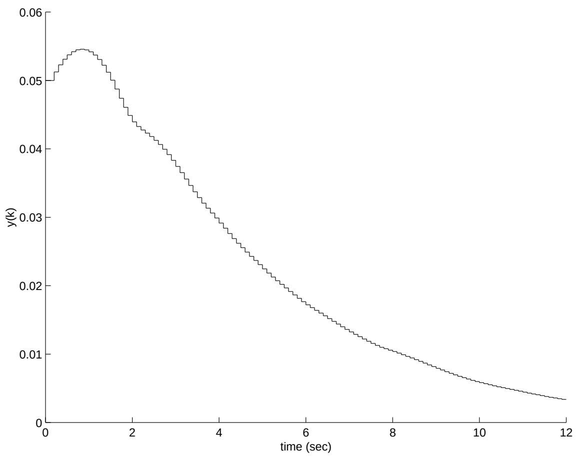

Fig. 3. Performance profile of the LMI based constrained MPC. The output.

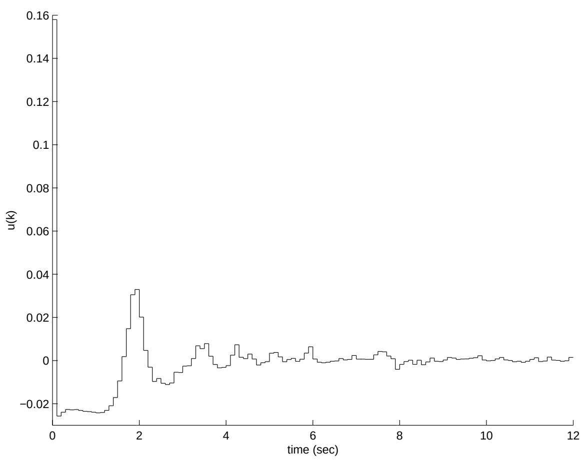

Fig. 4. Performance profile of the LMI based constrained MPC. The control.

normally formulated in the frequency domain such a H_∞ and H_2 may equally be incorporated. Since some of the states are not measurable, they (the unknown states) have been characterized by some probability properties. The unmeasured states can also be considered as Polytopic uncertainty, such an approach is also featured.

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