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PERFORMANCE LIMITATIONS IN THE RIPPLE-FREE ROBUST SERVOMECHANISM PROBLEM FOR SAMPLED LTI SYSTEMS

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Abstract:
Fundamental limitations for error tracking/regulation are obtained for the ripple-free robust servomechanism problem (RFRSP) for a sampled system. In studying this problem, the robust servomechanism problem is considered for a multi-input/multi-output sampled LTI system, using a cheap control problem formulation. Explicit expressions are obtained for the limiting performance costs associated with error tracking/regulation in the RSP, and application of these results is then made to obtain explicit expressions for the limiting performance costs associated with error tracking/regulation in the RFRSP. These limitations can be characterized completely by the number and location of the non-minimum phase transmission zeros.

Keywords: Sampled systems; cheap control problem; optimal control; performance limitations; servomechanism problem; non-minimum phase.

1. INTRODUCTION

Various properties of the robust servomechanism problem have been extensively studied for continuous linear time-invariant (LTI) systems (Davison and Scherzinger, 1987; Qiu and Davison, 1993; Seron, and Middleton, 1998; Shaked, 1980; Qiu, and Toker, 1997; Qiu and Chen, 1998), and for sampled LTI systems (Ben Jemaa and Davison, 1996, 1999, 2000) in recent years. In this case, on ignoring non-linear effects, for minimum phase square continuous LTI systems, perfect asymptotic tracking/disturbance rejection and arbitrarily good transient response, for a specified class of reference and disturbance signals, can be achieved (Davison and Scherzinger, 1987). The robust servomechanism problem for right-invertible non-minimum phase continuous systems and non-right invertible systems, however, has fundamental performance limitations for error tracking/regulation. Explicit expressions for the limiting optimal cost for right-invertible non-minimum phase systems were obtained in (Qiu and Davison, 1993; Shaked, 1980), and it was shown in (Qiu and Davison, 1993; Qiu and Chen, 1998) that the robust servomechanism tracking cost, for the case when the performance cost is the integral of the deviation of the system outputs, can be expressed as a function of the right-hand plane transmission zeros of the system. Reference (Seron, and Middleton, 1998) considered non-right-invertible systems, and expressed the limiting tracking cost for constant set-points as a function of the non-minimum phase zeros, and of the variation with frequency of the plant direction.

The robust servomechanism problem (RSP) was considered for multi-input/multi-output sampled LTI systems using a cheap control problem formulation, in (Qiu and Chen, 1998; Ben Jemaa and Davison, 1996, 1999, 2000), and in this case it was shown (Ben Jemaa and Davison, 1996, 1999, 2000) that the fundamental performance limitations of the behavior of the RSP for minimum phase sampled systems is independent of the order of the plant and of the infinite transmission zero structure of the continuous plant model. In (Qiu and Chen, 1998; Ben Jemaa and Davison, 1999, 2000), it was shown that the fundamental performance limitations on the behavior of the RSP for non-minimum phase sampled systems is characterized by the number and location of the transmission zeros of the sampled open loop system which lie outside of the unit circle.

The ripple-free robust servomechanism problem (RFRSP) consists of finding a robust servomechanism controller which achieves exact asymptotic tracking/disturbance rejection for a linear sampled-data system at and between the sampling instants in the presence of a specified class of tracking/disturbance signals (Doraiswami, 1981, 1982, 1997; Emami-Naeni, 1984). A hybrid control strategy is proposed in (Doraiswami, 1982) to solve the RFRSP; it consists of a digital controller which acts as a stabilizer and a digital servocompensator controller to ensure that asymptotic tracking and disturbance rejection occurs at the sampling instances, along with an analog servocompensator type controller to ensure zero asymptotic error occurs between sampling instances. In this paper, the performance limitations on the behavior of such a RFRSP for minimum and non-minimum phase sampled systems is considered.

2. THE ROBUST SERVOMECHANISM PROBLEM (RSP) FOR SAMPLED LTI SYSTEMS

The robust servomechanism problem (RSP) was studied in (Ben Jemaa and Davison, 1996, 1999, 2000; Qiu and Chen, 1998) for multi-input/multi-output sampled LTI systems, using a cheap control problem formulation. The following gives an overview of some of the results obtained.

We shall initially consider the square continuous plant modeled by

$$\begin{aligned} \dot{x} &= \mathcal{A}x + \mathcal{B}(u + \omega) \\ y &= \mathcal{C}x + \eta, \qquad e = y - y_{ref} \end{aligned} \tag{1}$$

where $u \in \mathbf{R}^r$, $y \in \mathbf{R}^r$, $x \in \mathbf{R}^n$, $\omega \in \mathbf{R}^r$, $\eta \in \mathbf{R}^r$, u is the input, ω and η are constant unmeasurable disturbances, y is the output, $y_{ref} \in \mathbf{R}^r$ is a constant set point, and e is the error in the system. The following existence results are now obtained:

Lemma 1. (Davison, 1976) There exists a solution to the RSP for (1) iff the following conditions are all satisfied:

(i) $(\mathcal{C}, \mathcal{A}, \mathcal{B})$ is stabilizable and detectable

(ii) $rank \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & 0 \end{bmatrix} = n + r$

(iii) y is measurable.

Assume now that the plant is sampled with a Zero Order Hold (ZOH), with sampling interval $h > 0$, so that the resultant sampled system is described by :

$$\begin{aligned} x_{k+1} &= Ax_k + B(u_k + \omega) \\ y_k &= Cx_k + \eta \\ e_k &= y_k - y_{ref} \end{aligned} \tag{2}$$

where $A := e^{h\mathcal{A}}$, $B := \int_0^h e^{\mathcal{A}(h-\tau)}\mathcal{B}d\tau$, $C := \mathcal{C}$, and ω, η are constant unmeasurable disturbances, and y_{ref} is a constant set-point tracking signal.

Lemma 2. (Goldenberg and Davison, 1974) There exists a solution to the RSP for (2) iff the following conditions are all satisfied:

(i) (C, A, B) is stabilizable and detectable

(ii) $rank \begin{bmatrix} A - I & B \\ C & 0 \end{bmatrix} = n + r$

(iii) y_k is measurable.

The following existence result is obtained re a solution to the RSP for the sampled system (2) in (Ben Jemaa and Davison, 1999):

Lemma 3. (Ben Jemaa and Davison, 1999) Given the continuous system (1), assume that there exists a solution to the RSP for (1), i.e. the conditions of lemma 1 are all satisfied; then for almost all $h > 0$, there exists a solution to the RSP for the sampled system (2), i.e. the conditions of lemma 2 are satisfied.

The following intermediate result is obtained in (Ben Jemaa and Davison, 1999):

Lemma 4. (Ben Jemaa and Davison, 1999) Given the continuous LTI non-degenerate (Davison and Wang, 1974) system (1), then for almost all (Davison and Wang, 1973) plant parameters $(\mathcal{C},\mathcal{A},\mathcal{B})$ and almost all $h > 0$, the resultant sampled system (2) has the property that (CB) is non-singular; in particular if $(\mathcal{C},\mathcal{A},\mathcal{B})$ has a non-singular interactor matrix, then (CB) is non-singular for almost all $h > 0$.

In order to find a "high performance " digital controller to solve the RSP for (2) in the presence of constant disturbances ω and η, and constant set-points y_{ref}, the following "cheap performance index " is now defined:

$$J=h\sum_{k=o}^{\infty}\left\{e_{k-1}^{'}e_{k-1}+\epsilon(u_k-u_{k-1})^{'}(u_k-u_{k-1})\right\} \tag{3}$$

where $e_k := y_k - y_{ref}$ is the error in the system, and $\epsilon > 0$, for the following system:

$$\begin{aligned}\begin{bmatrix}x_{k+1}-x_k\\ e_k\end{bmatrix}&=\begin{bmatrix}A&0\\ C&I\end{bmatrix}\begin{bmatrix}x_k-x_{k-1}\\ e_{k-1}\end{bmatrix}+\begin{bmatrix}B\\ 0\end{bmatrix}(u_k-u_{k-1})\\ e(k-1)&=\begin{bmatrix}0&I\end{bmatrix}\begin{bmatrix}x_k-x_{k-1}\\ e_{k-1}\end{bmatrix}\end{aligned} \tag{4}$$

2.1 *Robust Servomechanism Control Results for Strictly Proper Non-minimum Phase Systems*

Assume now that the sampled system (2) has $\rho \in [1, n-1]$ transmission zeros, and is non-minimum phase with $p \in [1,\rho]$ non-minimum phase transmission zeros given by $\lambda_i,\ i = 1, 2, ..., p$.

Theorem 1. (Ben Jemaa and Davison, 1999) Given a sampled system (2) which has the property that there exists a solution to the RSP and is non-minimum phase, consider the optimal controller which minimizes the performance index (3); then

I) If $x(0) = 0$, $y_{ref} \neq 0$, $\eta \neq 0$, and $\omega = 0$, the optimal cost J as $\epsilon \to 0$ is given by:

$$J_{opt} = h(y_{ref}-\eta)^{'}M(y_{ref}-\eta) \tag{5}$$

where M is a constant matrix with

$$trace(M) = n + r - \rho + \sum_{i=1}^{p}\frac{(\lambda_i+1)}{(\lambda_i-1)}.$$

II) Assume in (2) that (CB) is invertible. Then

(a) if $x(0) = 0$, $y_{ref} \neq 0$, $\eta \neq 0$, and $\omega = 0$, the optimal cost J as $\epsilon \to 0$ is given by:

$$J_{opt} = h(y_{ref}-\eta)^{'}M(y_{ref}-\eta) \tag{6}$$

where M is a constant matrix with

$$trace(M) = 2r + \sum_{i=1}^{p}\frac{(\lambda_i+1)}{(\lambda_i-1)} \tag{7}$$

(b) if $x(0) = 0$, $y_{ref} = 0$, $\eta = 0$, and $\omega \neq 0$, the optimal cost J as $\epsilon \to 0$ is given by:

$$J_{opt} = h\omega^{'}M\omega. \tag{8}$$

where M is a constant matrix given by $M = \Theta^{'}\Theta$, where

$$\Theta = \prod_{i=p}^{1}(I_r + \frac{(\lambda_i\bar{\lambda}_i - 1)}{(\bar{\lambda}_i+1)}\bar{v}_i v_i^{'})CB \tag{9}$$

where the vector v_i is given by

$$[u_i^{'}\ v_i^{'}]\begin{bmatrix}\lambda_i I - A & -B\\ -C_m^{i-1} & 0\end{bmatrix} = 0 \tag{10}$$

and v_i is normalized,i.e. $\bar{v}_i^{'}v_i = 1$, and where

$$\begin{aligned}C_m^0 &= C\\ C_m^i &= C_m^{i-1} - \left(\frac{\lambda_i\bar{\lambda}_i - 1}{\bar{\lambda}_i+1}\right)\bar{v}_i u_i^{'}(A+I).\end{aligned}$$

Remark 1. An extension of theorem 5 for the case when CB is not invertible is also given in (Ben Jemaa and Davison, 1999).

3. RIPPLE-FREE ROBUST SERVOMECHANISM PROBLEM

It is desired now to study the fundamental limitations associated with solving the ripple-free robust servomechanism problem for sampled systems. Consider the square continuous plant (1). Then a hybrid control strategy proposed in (Doraiswami, 1982) to solve the RFRSP consists of a digital controller, which serves as a stabilizer and a servocompensator controller to ensure that asymptotic tracking and disturbance rejection occurs at the sampling instances, along with an analog servocompensator type of controller to ensure that zero asymptotic error occurs between the sampling instances. In the case of constant tracking/constant disturbances, the analog controller is just a simple servocompensator modeled by

$$\dot{\zeta} = v$$
$$u = \zeta \tag{11}$$

where u is the input to the plant (1), and the augmented continuous system (1),(11) is then modeled by

$$\begin{bmatrix} \dot{x} \\ \dot{\zeta} \end{bmatrix} = \hat{\mathcal{A}} \begin{bmatrix} x \\ \zeta \end{bmatrix} + \hat{\mathcal{B}} v + \hat{\mathcal{E}} \omega$$
$$y = \hat{\mathcal{C}} \begin{bmatrix} x \\ \zeta \end{bmatrix} + \eta \tag{12}$$

where v is a new control input, and $\hat{\mathcal{A}} := \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ 0 & 0 \end{bmatrix}$, $\hat{\mathcal{B}} := \begin{bmatrix} 0 \\ I \end{bmatrix}$, $\hat{\mathcal{C}} = \begin{bmatrix} \mathcal{C} & 0 \end{bmatrix}$ and $\hat{\mathcal{E}} := \begin{bmatrix} \mathcal{B} \\ 0 \end{bmatrix}$. Assume now that (12) is sampled with a (ZOH) with sampling interval $h > 0$, and let the sampled system be described by:

$$\begin{bmatrix} x_{k+1} \\ \zeta_{k+1} \end{bmatrix} = \hat{A} \begin{bmatrix} x_k \\ \zeta_k \end{bmatrix} + \hat{B} v_k + \hat{E} \omega_k$$
$$y_k = \begin{bmatrix} \mathcal{C} & 0 \end{bmatrix} \begin{bmatrix} x_k \\ \zeta_k \end{bmatrix} + \eta_k \tag{13}$$

where $\hat{A} := e^{h\hat{\mathcal{A}}}$, $\hat{B} := \int_0^h e^{(h-\tau)\hat{\mathcal{A}}} \hat{\mathcal{B}} d\tau$, and $\hat{E} := \int_0^h e^{(h-\tau)\hat{\mathcal{A}}} \hat{\mathcal{E}} d\tau$.

The following section describes various properties which result from augmenting (1) with the analog servocompensator (11), in order to solve the RFRSP.

3.1 *Properties of the Augmented Sampled System*

The number and location of any new transmission zeros introduced, resulting from augmenting (1) with the analog servocompensator (11) are now characterized in this section. Initially, continuous time systems (1), having a full rank $\mathcal{CB}$ matrix are considered.

Lemma 5.
Given a continuous time system (1), assume that the generic condition $rank(\mathcal{CB}) = r$ holds; then

1) the sampled system (2) has $n - r$ transmission zeros, and as $h \to 0$, all of the transmission zeros approach 1.
2) the sampled augmented system (13) has n transmission zeros, and as $h \to 0$,
   a) $n - r$ transmission zeros approach 1, and
   b) r transmission zeros approach -1.

The following lemma shows that the transmission zeros corresponding to the non generic case all lie on the negative real axis for SISO systems. In particular, given a SISO continuous time system, the following result provides a limiting relationship between the zeros of a continuous-time and sampled system.

Lemma 6. (Hagander and Sternby, 1984)
Given a SISO continuous time system (1) with ρ transmission zeros s_i, $i = 1, 2, \ldots, \rho$, consider the sampled system (2); then, as the sampling period $h \to 0$:

1) ρ zeros of (2) approach unity asymptotically as $\lim_{h \to 0} e^{hs_i}$, $i = 1, 2, \ldots, \rho$;
2) the remaining $n - 1 - \rho$ zeros of (2) approach the zeros of the Euler polynomial $\mathcal{E}_{n-\rho}(z)$,

where the Euler polynomials are defined by:

$$\mathcal{E}_1(z) = 1$$
$$\mathcal{E}_{k+1}(z) = (1 + kz)\mathcal{E}_k(z) + z(1 - z)\frac{d\mathcal{E}_k(z)}{dz}$$
$$k = 1, 2, \ldots \tag{14}$$

and have the property that the roots of $\mathcal{E}_k(z)$, $k = 2, 3, 4, \ldots$ are all real and negative.

The following corollary which follows directly from lemma 8, characterizes the transmission zeros of the SISO sampled augmented system (13).

Corollary 1. Consider the SISO continuous time system (1) with ρ transmission zeros s_i, $i = 1, 2, \ldots, \rho$, and the corresponding sampled augmented system (13); then, as the sampling period $h \to 0$:

1) ρ zeros of (13) approach 1 ;
2) the remaining $n - \rho$ zeros of (13) approach the zeros of the Euler polynomial $\mathcal{E}_{n-\rho+1}(z)$.

Given the sampled augmented system (13), the following section now describes any performance limitations associated with the performance index (3) as $\epsilon \to 0$.

3.2 *Ripple-free Robust Servomechanism Control Results*

The following existence result is now obtained:

Lemma 7.
Given the continuous system (1), assume that there exists a solution to the RSP for (1), i.e. conditions (i) to (iii) of lemma 1 all hold; then

a) there exists a solution to the RSP for the continuous time system (12) and,
b) there exists h_s so that for all h contained in $[0,\ h_s]$, there exists a solution to the RFRSP for the sampled system (13).

The following theorem shows that the new transmission zeros introduced by the analogue compensator, in the sampled augmented system (13) are innocent, i.e. they do not alter the limiting optimal performance cost obtained in (5) for the original sampled system (2).

Theorem 2. Consider the continuous system (1), and assume that it has a non-singular interactor matrix and that there exists a solution to the RSP for (1), i.e. the conditions of lemma 1 are all satisfied; then as $h \to 0$, the optimal performance costs (5), (6) associated with the sampled system (13), coincide respectively with the optimal performance costs (5), (6) associated with the sampled system (2).

Remark 2. Note that the assumptions imposed on the continuous system (1) are all generically satisfied.

4. EXAMPLE

Consider the following continuous non-minimum phase system given by:

$$\mathcal{A}=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -0.2 & -0.2 & 0.1 & 0.01 & 0.1 & 0.01 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0.1 & -1 & -0.1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0.1 & 0 & 0 & -1 & -0.1 \end{bmatrix},\ \mathcal{B}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \tag{15}$$

and $$\mathcal{C}=\begin{bmatrix} -4 & -30 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The transmission zeros of this system are given by: {-0.0692, 4.1264}. If we sample this system with $h = 0.01$, the resulting sampled data system is non-minimum phase with transmission zeros given by:

{1.0421, 0.9993, -0.9997, -0.9856}

and has the property that (CB) in (2) is invertible. If $x(0) = 0$, $\omega = [0\ 0]'$, then it follows from (6), that the limiting optimal cost of the performance index (3), as $\epsilon \to 0$, is given by

$$J_{opt} = 0.01(y_{ref} - \eta)' M (y_{ref} - \eta) \tag{16}$$

where

$$M = \begin{bmatrix} 26.277 & 24.245 \\ 24.245 & 26.287 \end{bmatrix}$$
$$trace(M) = 52.56 \tag{17}$$

Now if we sample the augmented system (12) with $h = 0.01$, the resultant sampled system is non-minimum phase with transmission zeros given by:

{-3.7311, -3.6914, -0.2679, -0.2651, 1.0421, 0.9931}

and in this case, the new additional sampled transmission zeros, {-3.7311, -3.6914, -0.2679, -0.2651}, are close to the roots of the Euler polynomial $\mathcal{E}_3(z)$. The sampled augmented system also has the property that $(\hat{C}\hat{B})$ in (12) is invertible. Now if $x(0) = 0$, $\omega = [0\ 0]'$, it follows from (6), that the limiting optimal cost of the performance index (3), as $\epsilon \to 0$, is given by:

$$J_{opt} = 0.01(y_{ref} - \eta)' M (y_{ref} - \eta) \tag{18}$$

where

$$M = \begin{bmatrix} 26.813 & 24.236 \\ 24.236 & 26.813 \end{bmatrix}$$
$$trace(M) = 53.623 \tag{19}$$

It is observed that the limiting optimal cost (19) is approximately the same as (17) as predicted by theorem 11.

The closed loop system obtained using the robust servomechanism controller RSC is simulated with $x(0) = 0$, $y_{ref} = [1\ 1]'$, $\eta = [0\ 0]'$, and $\omega = 0$ in Figure 1 (a.1) and (a.2), which implies from (16) that $J_{opt} = 1.01$.

The closed loop system obtained by using the ripple-free robust servomechanism controller, is simulated for the case of zero initial conditions, $y_{ref} = [1\ 1]'$, $\eta = [0\ 0]'$, and $\omega = 0$ in Figure 1 (b.1) and (b.2), which implies from (18) that $J_{opt} = 1.02$.

It is seen from figure 1 that the optimal costs obtained do confirm the results obtained in (16) and (18).

5. CONCLUSIONS

An explicit expression is obtained for the limiting performance cost associated with the error tracking/regulation in the RFRSP. This cost is compared to the limiting performance cost associated with the error tracking/regulation in the RSP and it is shown that the two costs coincide.

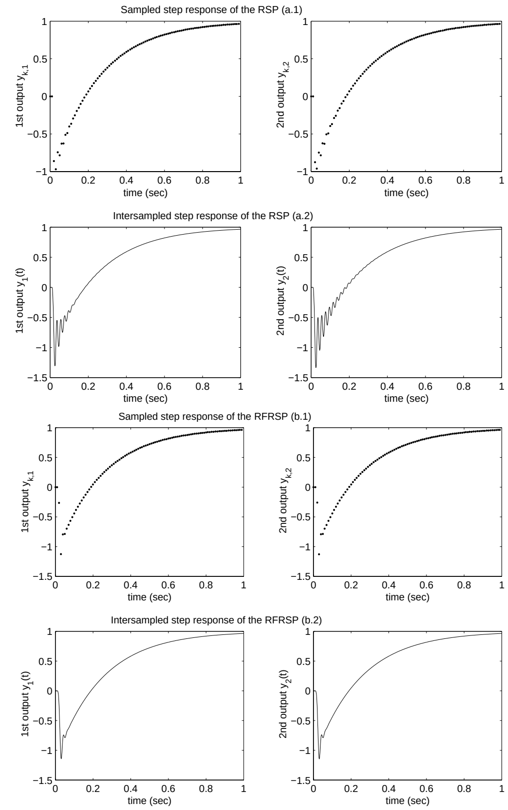

Fig. 1. Closed loop response with RSC and RFRSC for example 1: $y_{ref} = [1\ 1]'$, $\omega = [0\ 0]'$.

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