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OPEN-LOOP WORST-CASE IDENTIFICATION OF NON SCHUR PLANTS [1]

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Abstract: This paper presents an LMI based algorithm for deterministic worst–case identification of non Schur plants in an open-loop setting. Contrary to other approaches dealing with this problem, the proposed technique does not require prior knowledge of a stabilizing controller. The main result of the paper shows that, as the information is completed, i.e. in the limit when the number of samples tends to infinity and the noise level tends to zero, the identified model converges, in the ℓ_2–induced topology, to the actual plant. Additional results include an algorithm outlining how to use the results of finite–lenght experiments to synthesize stabilizing controllers. *Copyright © 2002 IFAC*

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1. INTRODUCTION

The problem of worst–case identification of non–Schur (i.e marginally stable or unstable) plants has been addressed several times in the literature (see the survey by Mäkilä *et al.* (1995)). All of these papers pursue a closed-loop approach, where the unknown plant is prestabilized prior to performing the identification step.

Under the assumption that a stabilizing controller is known, one possible approach to solve this problem uses coprime factorizations of both the unknown system and the given controller. By considering the unknown plant as a member of the set of all plants stabilized by the assumed controller, the problem can be reduced to the identification of a stable system, namely the Youla-Kucera parameter. This approach was first proposed by Hansen *et al.* (1989) and later by Schrama (1991) for MIMO plants in the context of identification for control. Here the objective was estimating a model of the plant –based on the given *a priori* compensator– to design a *new* controller, in the former case to minimize some function of the closed-loop error (since the controller was actually designed for a model of the system), and as a part of an iterative identification-design procedure in the latter. The idea was further extended by Dasgupta and Anderson (1996) to Nonlinear Time Varying plants.

Based on this approach and for a specific choice of the input to the closed-loop system, Partington and Mäkilä (1994) showed the existence of robustly convergent algorithms in the ℓ_1 sense, and obtained different error bounds between the unstable plant and the identified model in several metrics, such as the subspace gap, the projection gap and the graph metric (see for example (Mäkilä and Partington, 1993) for definitions and some properties of these metrics.)

In a $\mathcal{H}_\infty$ setting, Mäkilä and Partington (1992) dealt with the problem of getting a model of an unstable system from measurements of the closed-loop stable transfer function $H \doteq PS$, where P is the open-loop unstable plant and S is the closed-loop sensitivity.

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Under the additional assumption that the unknown plant is strongly stabilizable, it was shown that it is possible to guarantee that an approximation $\hat{H}$ to the transfer function H leads to an approximation $\hat{P}$ to the unstable plant, which converges in the graph, gap and chordal metrics as $\hat{H}$ approaches the actual closed-loop transfer function in the $\mathcal{H}_\infty$ norm.

A common feature of all the approaches mentioned above, is that they require prior knowledge of a controller that stabilizes the unknown plant. However, this requirement can be too restrictive, specially in cases where the plant is not strongly stabilizable (and thus the controller itself has to be open loop unstable). To avoid this difficulty, this paper takes a different approach, directly identifying the plant from some *a priori* assumptions and time-domain measurements of its output over a finite horizon $[0,N]$. Note that many practical situations involve plants that are either marginally stable or mildly unstable and thus it is possible to carry time–domain experiments over reasonably long horizons, without exceeding physical limits on the plant. Formally, the proposed approach is similar to the one used by Chen and Nett (1995) and by Parrilo *et al.* (1998) for worst–case identification of stable plants. The main result of this paper shows that *even when used for open–loop unstable plants*, the identification algorithm converges in the ℓ_2–induced topology as the information is completed, i.e. as the noise level tends to zero and the number of data points to infinity. In addition, it provides worst–case identification error bounds and illustrates how to use these bounds to synthesize controllers with guaranteed finite–horizon worst–case performance. Finally, this work outlines an algorithm for synthesizing stabilizing controllers for the unknown plant, based on following the identification step with a model (in)validation one.

The paper is organized as follows. Section 2 presents the notation and some required results. Section 3 states the problem and presents the main results. Section 4 illustrates these results with two simple examples. Finally, Section 5 presents some conclusions and points out to some issues open for further research.

2. PRELIMINARIES

2.1 Notation

$\ell_2[0,N]$ denotes the space of square summable real, one-sided finite sequences $x = \{\mathbf{x}_i\}_{i=0}^N$ equipped with the norm $\|x\|_{\ell_2}^2 \doteq \sum_{i=0}^N \mathbf{x}_i^T \mathbf{x}_i < \infty$. Similarly, $\ell_\infty[0,N]$ denotes the space of bounded sequences equipped with the norm $\|x\|_{\ell_\infty} \doteq \sup_{i \geq 0} |\mathbf{x}_i| < \infty$, and $\mathcal{B}\ell_\infty(N,\varepsilon)$ denotes the origin centered ε radius ball in this space.

This paper considers single input-single output (SISO), discrete-time, finite dimensional, causal, linear time-invariant (LTI) systems, represented in the frequency domain by the $\mathcal{Z}$ transform $H(z) = \sum_{k=0}^\infty h_k z^{-k}$, where h_k denotes the impulse response at k; and in the time domain by the convolution kernel $y_k = (h*u)_k$ or alternatively by the infinite lower triangular Toeplitz matrix mapping input to output sequences:

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} h_0 & 0 & 0 & \cdots \\ h_1 & h_0 & 0 & \\ h_2 & h_1 & h_0 & \\ \vdots & & & \ddots \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \end{bmatrix}. \tag{1}$$

$\mathbf{T}_h^N$ denotes the upper left finite submatrix obtained from the matrix in (1) of size $(N+1)\times(N+1)$. This matrix will be useful while dealing with finite input-output sequences on the horizon $[0,N]$.

$\mathcal{H}_{\infty,\rho}$ denotes the space of complex–valued functions essentially bounded on $|z| = \rho$ and with bounded analytic continuation in $|z| > \rho$, equipped with the usual norm $\|.\|_{\infty,\rho} \doteq \sup_{|z|>\rho} |H(z)|$. $\mathcal{H}_{\infty,\rho}(K)$ denotes the open K–ball in $\mathcal{H}_{\infty,\rho}$, i.e.

$$\mathcal{H}_{\infty,\rho}(K) \doteq \{f \in \mathcal{H}_{\infty,\rho} : \|f\|_{\infty,\rho} < K\} \tag{2}$$

In the sequel, $\mathcal{H}_\infty$ and $\|.\|_\infty$ will be used when $\rho = 1$.

Let $\mathcal{L}(N)$ denote the space of causal, LTI bounded operators in $\ell_2[0,N]$. $\|\cdot\|_{\ell_2[0,N]\to\ell_2[0,N]}$ denotes the $\ell_2[0,N]$ induced norm in this space, i.e:

$$\|h\|_{\ell_2[0,N]\to\ell_2[0,N]} \doteq \sup_{\substack{u\in\ell_2[0,N] \\ u\neq 0}} \frac{\|(h*u)\|_{\ell_2}}{\|u\|_{\ell_2}}. \tag{3}$$

The projection operator $\mathcal{P}_N : \mathcal{L}(\infty) \to \mathcal{L}(N)$ is defined by $\mathcal{P}_N[h] \doteq \{h_0, h_1, \ldots, h_{N-1}\}$.

Given a subset A of a metric space (X,m) its diameter is defined as $d(A) = \sup_{x,a\in A} m(x,a)$ and $\overline{A}$ denotes its closure. Finally given a matrix $\mathbf{M}$, $\mathbf{M}^T$ denotes its transpose, $(\mathbf{M})_i$ its i^{th} row, $\mathbf{M}^\dagger$ its Moore–Penrose pseudoinverse, and $\|\mathbf{M}\|_1 = \max_i \sum_j |(\mathbf{M})_{i,j}|$. As usual $\mathbf{M} > 0$ ($\mathbf{M} \geq 0$) indicates that $\mathbf{M}$ is positive definite (positive semi-definite), and $\mathbf{M} < 0$ that $\mathbf{M}$ is negative definite.

2.2 Some Required Results

Lemma 1. (Bounded Real). Consider a proper, finite dimensional, LTI, stable system h with minimal state space realization

$$\left(\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right).$$

The the following statements are equivalent:

a.- $\|h\|_\infty < \gamma$.
b.- The following LMI:

$$\begin{bmatrix} \mathbf{A}^T\mathbf{X}\mathbf{A} - \mathbf{X} + \mathbf{C}^T\mathbf{C} & \mathbf{A}^T\mathbf{X}\mathbf{B} + \mathbf{C}^T\mathbf{D} \\ \mathbf{B}^T\mathbf{X}\mathbf{A} + \mathbf{D}^T\mathbf{C} & \mathbf{B}^T\mathbf{X}\mathbf{B} - \gamma^2\mathbf{I} + \mathbf{D}^T\mathbf{D} \end{bmatrix} < 0$$

admits a symmetric positive solution $\mathbf{X} = \mathbf{X}^T > 0$.

PROOF. See (Gahinet and Apkarian, 1994). □

Lemma 2. (Bounded Real, Difference version). Consider a proper, finite dimensional, LTI, stable system h with minimal state space realization

$$\left(\begin{array}{c|c}\mathbf{A} & \mathbf{B}\\ \hline \mathbf{C} & \mathbf{D}\end{array}\right).$$

Assume that the following functional LMI admits a positive semidefinite solution $\mathbf{X}_k = \mathbf{X}_k^T \geq 0$:

$$\begin{bmatrix}\mathbf{A}^T\mathbf{X}_{k+1}\mathbf{A}-\mathbf{X}_k+\mathbf{C}^T\mathbf{C} & \mathbf{A}^T\mathbf{X}_{k+1}\mathbf{B}+\mathbf{C}^T\mathbf{D}\\ \mathbf{B}^T\mathbf{X}_{k+1}\mathbf{A}+\mathbf{D}^T\mathbf{C} & \mathbf{B}^T\mathbf{X}_{k+1}\mathbf{B}-\gamma^2\mathbf{I}+\mathbf{D}^T\mathbf{D}\end{bmatrix} < 0. \tag{4}$$

Then

$$\|h\|_{\ell_2[0,N]\to\ell_2[0,N]} < \gamma. \tag{5}$$

PROOF. Let $u \in \ell_2[0,N]$ denote an arbitrary input sequence and x, z the corresponding state and output sequences. Pre and post-multiplying (4) by $\begin{bmatrix}\mathbf{x}_k^T & \mathbf{u}_k^T\end{bmatrix}$ and $\begin{bmatrix}\mathbf{x}_k^T & \mathbf{u}_k^T\end{bmatrix}^T$, and reordering terms gives:

$$\begin{aligned}0 >& \mathbf{x}_k^T\mathbf{A}^T\mathbf{X}_{k+1}\mathbf{A}\mathbf{x}_k + \mathbf{x}_k^T\mathbf{A}^T\mathbf{X}_{k+1}\mathbf{B}\mathbf{u}_k - \mathbf{x}_k^T\mathbf{X}_k\mathbf{x}_k\\ & \mathbf{u}_k^T\mathbf{B}^T\mathbf{X}_{k+1}\mathbf{A}\mathbf{x}_k + \mathbf{u}_k^T\mathbf{B}^T\mathbf{X}_{k+1}\mathbf{B}\mathbf{u}_k - \gamma^2\mathbf{u}_k^T\mathbf{u}_k\\ & \mathbf{x}_k^T\mathbf{C}^T\mathbf{C}\mathbf{x}_k + \mathbf{x}_k^T\mathbf{C}^T\mathbf{D}\mathbf{u}_k + \mathbf{u}_k^T\mathbf{D}^T\mathbf{C}\mathbf{x}_k + \mathbf{u}_k^T\mathbf{D}^T\mathbf{D}\mathbf{u}_k\\ =& \mathbf{x}_{k+1}^T\mathbf{X}_{k+1}\mathbf{x}_{k+1} + \mathbf{z}_k^T\mathbf{z}_k - \mathbf{x}_k^T\mathbf{X}_k\mathbf{x}_k - \gamma^2\mathbf{u}_k^T\mathbf{u}_k.\end{aligned}$$

Summing this last inequality from $k=0$ to $k=N$ and using the facts that $\mathbf{x}_0 = 0$ and $\mathbf{X}_k > 0, \forall k$ yields:

$$\begin{aligned}0 > \mathbf{x}_{N+1}^T\mathbf{X}_{N+1}\mathbf{x}_{N+1} + \sum_{k=0}^N \mathbf{z}_k^T\mathbf{z}_k - \gamma^2\sum_{k=0}^N \mathbf{u}_k^T\mathbf{u}_k >\\ \sum_{k=0}^N \mathbf{z}_k^T\mathbf{z}_k - \gamma^2\sum_{k=0}^N \mathbf{u}_k^T\mathbf{u}_k \Rightarrow \sum_{k=0}^N \mathbf{z}_k^T\mathbf{z}_k < \gamma^2\sum_{k=0}^N \mathbf{u}_k^T\mathbf{u}_k\end{aligned}$$

which is equivalent to (5). □

Lemma 3. (Carathéodory-Fejér). Given a sequence $\{h_k\}_{k=0}^N$, and numbers $K > 0$, $\rho > 0$, there exists a complex valued function $H(z) \in \mathcal{H}_{\infty,\rho}(K)$ such that

$$H(z) = h_0 + h_1 z^{-1} + h_2 z^{-2} + \ldots + h_N z^{-N} + \ldots$$

if and only if

$$\mathbf{R}_\rho^2 - \frac{1}{K^2}(\mathbf{T}_h^N)^T\mathbf{R}_\rho^2\mathbf{T}_h^N \geq 0, \tag{6}$$

with $\mathbf{R}_\rho \doteq \operatorname{diag}(1\ \rho\ \cdots\ \rho^N)$ and $\mathbf{T}_h^N$ defined as in Section (2.1).

PROOF. See (Parrilo *et al.*, 1998) and references therein. □

3. IDENTIFICATION OF NON–SCHUR PLANTS

In this section we present the proposed algorithm, provide some simple worst-case bounds on the identification error over the finite horizon and analyze its convergence properties. We begin by precisely defining the problem under consideration.

3.1 *Problem Statement*

Consider the problem of identifying a non Schur plant h from a set of noisy measurements, over a finite horizon $[0,N]$, of the output y to a known but arbitrary input sequence $u \in \ell_2[0,N]$:

$$y_k = (h * u)_k + \eta_k, \quad k = 0, 1, \ldots, N \tag{7}$$

corrupted by additive bounded noise η in the set

$$\mathcal{N} \doteq \ell_\infty[0,N](\varepsilon). \tag{8}$$

Further, the plant is known to belong to a given set of candidate models $\mathcal{S}$. This identification problem can be precisely stated as follows.

Problem 1. Given an unknown non Schur plant, the *a priori* sets of candidate models and noise $(\mathcal{S}, \mathcal{N})$ and a finite set of samples of the input and output of the plant $(\mathbf{u}, \mathbf{y})$:

- Determine whether the consistency set
$$\mathcal{T}(\mathbf{y}) \doteq \{h \in \mathcal{S} : y_k - (h*u)_k \in \mathcal{N},\ k = 0, 1, \ldots, N\} \tag{9}$$
is nonempty.
- If $\mathcal{T}(\mathbf{y}) \neq \emptyset$, find a model $\hat{h} \in \mathcal{T}(\mathbf{y})$.

In the sequel, we consider the following two different characterizations of the *a priori* set of models $\mathcal{S}$:

$$\mathcal{S}_1 = \mathcal{H}_{\infty,\rho}(K) \text{ for some given } \rho \geq 1, K \tag{10}$$

and

$$\mathcal{S}_2 \doteq \left\{h \in \mathcal{H}_{\infty,\rho} : |h_k| \leq K\rho^k, \forall k\right\}. \tag{11}$$

The first case above leads to a computable necessary and sufficient condition for checking consistency, which as shown next is formally identical to the one arising in the case of stable models. However, in the case of unstable plants (as opposed to marginally stable), it may be difficult to check the validity of this assumption. On the other hand, condition (11) is easily testable, since it only involves checking that the growth of the impulse response [2] is bounded by $K\rho^k$. The trade-off here is that this second condition leads only to sufficient conditions: feasibility of the LMIs (13) guarantees consistency of the *a priori* sets $(\mathcal{S}, \mathcal{N})$ and the *a posteriori* experimental information $(\mathbf{u}, \mathbf{y})$, since $\mathcal{S}_1 \subset \mathcal{S}_2$.

Notice that the algorithm proposed by Chen and Nett (1995) and Parrilo *et al.* (1998) can still be applied to establish consistency of the data and obtain an unstable model, since stability of the unknown plant is used only to obtain worst-case error bounds and establish convergence. More precisely, there exists at least one $h \in \mathcal{S}_1$ which can reproduce the available experimental data within the assumed error bounds if and only if equation (6) together with

$$\|\mathbf{y} - \mathbf{T}_u^N\mathbf{h}\|_{\ell^\infty} \leq \varepsilon \tag{12}$$

[2] clearly any other fixed signal can be used here

hold (see Parrilo *et al.* (1998) for details). A potential problem here is that the condition number of the matrix in condition (6) grows as ρ^{4N} [3]. This difficulty can be solved by noticing that if the LMIs (6)–(12) hold for some $K, \rho, \mathbf{h}$, then they hold for $K, \tilde{\rho}, \mathbf{h}_{\tilde{\rho}}$, where $\mathbf{h}_{\tilde{\rho}} \doteq \left[h_0 \; h_1/\tilde{\rho} \; \cdots \; h_N/\tilde{\rho}^N\right]$.

Thus, Problem 1 can be solved using the following scaled LMIs:

$$\begin{gathered}\mathbf{R}^2_{\frac{\rho}{\tilde{\rho}}} - \frac{1}{K^2}(\mathbf{T}^N_{h_{\tilde{\rho}}})^T \mathbf{R}^2_{\frac{\rho}{\tilde{\rho}}} \mathbf{T}^N_{h_{\tilde{\rho}}} \geq 0 \\ (\mathbf{R}^{-1}_{\tilde{\rho}}\mathbf{y} - \mathbf{T}^N_{\tilde{u}}\mathbf{h}_{\tilde{\rho}}) \in \mathcal{N}_{\tilde{\rho}},\end{gathered} \tag{13}$$

where $\mathcal{N}_{\tilde{\rho}} \doteq \{\eta_{\tilde{\rho}} : |\eta_{\tilde{\rho}_k}| \leq \varepsilon/\tilde{\rho}^k\}$, $\mathbf{T}^N_{\tilde{u}} = \mathbf{R}^{-1}_{\tilde{\rho}}\mathbf{T}^N_u\mathbf{R}_{\tilde{\rho}}$ and $\tilde{\rho} \sim \rho$, combined with the mapping:

$$H_{id}(z) = H_{\tilde{\rho}}(\frac{z}{\tilde{\rho}}). \tag{14}$$

When $\tilde{\rho} > \rho$ the algorithm outlined above can be formally interpreted as solving the problem of obtaining a model of a *stable* plant $h_{\tilde{\rho}} \in \mathcal{S}_{\tilde{\rho}}$, where:

$$\mathcal{S}_{\tilde{\rho}} \doteq \{H(\tilde{\rho}z) \colon h \in \mathcal{S}\} = \mathcal{H}_{\infty,\frac{\rho}{\tilde{\rho}}}(K), \tag{15}$$

using the experimental data $\{u_k/\tilde{\rho}^k\}$, $\{y_k/\tilde{\rho}^k\}$, corrupted by noise in the set $\mathcal{N}_{\tilde{\rho}}$.

3.2 Identification Error and Convergence Properties

The identification procedure proposed in Section 3.1 is interpolatory since it generates a model in the consistency set $\mathcal{T}(\mathbf{y})$. Recall that (see for instance (Sánchez Peña and Sznaier, 1998) and references therein) the worst–case identification error of an interpolatory algorithm $\mathcal{A}$ is bounded by:

$$e_{id}(\mathcal{A}) \doteq \sup_{\mathbf{y}\in\mathcal{Y}} \left\{ \sup_{g\in\mathcal{T}(\mathbf{y})} \|g - \mathcal{A}(\mathbf{y})\| \right\} \leq \mathcal{D}(I), \tag{16}$$

where $\mathcal{Y}$ is the set of all possible experimental data consistent with the *a priori* information $(\mathcal{S},\mathcal{N})$, $\mathcal{D}(I)$ denotes the diameter of information:

$$\mathcal{D}(I) \doteq d(\mathcal{T}(\mathbf{y})) = \sup_{\mathbf{y}} d\left(\mathcal{T}(\mathbf{y})\right) \tag{17}$$

and where the norm of interest in this framework is the induced $\ell_2[0,N]$ norm. Moreover, since the *a priori* sets $(\mathcal{S},\mathcal{N})$ are convex and symmetric, with points of symmetry $h_s = 0$ and $\eta_s = 0$ respectively, the worst-case diameter is attained when the available *a posteriori* information is the null experiment $\mathbf{y}_0$, i.e.

$$\mathcal{D}(I) = 2 \sup_{h\in\mathcal{T}(\mathbf{y}_0)} \|h\|_{\ell_2[0,N]\to\ell_2[0,N]}. \tag{18}$$

Next result provides an upper bound on the induced $\ell_2[0,N]$ norm of a system $h \in \mathcal{S}$, given an upper bound on the $\mathcal{H}_\infty$ norm of the stable system $h_{\tilde{\rho}} \in \mathcal{S}_{\tilde{\rho}}$. When combined with (16) and (18), it provides an upper bound on the identification error of the proposed method over the finite horizon.

Theorem 4. Consider a proper, finite dimensional, LTI, not necessarily stable system h with minimal state space realization

$$\left(\begin{array}{c|c}\mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D}\end{array}\right).$$

Let $\tilde{\rho} > 1$ be such that the system $H_{\tilde{\rho}}(z) \doteq H(\tilde{\rho}z)$ with state space realization

$$\left(\begin{array}{c|c}\frac{\mathbf{A}}{\tilde{\rho}} & \frac{\mathbf{B}}{\tilde{\rho}} \\ \hline \mathbf{C} & \mathbf{D}\end{array}\right)$$

is stable, with $\|h_{\tilde{\rho}}\|_\infty \leq \gamma$. Let $u \in \ell_2[0,N]$ be an arbitrary input sequence and z its output sequence. Then $\|h\|_{\ell_2[0,N]\to\ell_2[0,N]} < \gamma\,\tilde{\rho}^N$.

PROOF. Since $\|\tilde{\rho}^N h_{\tilde{\rho}}\|_\infty \leq \gamma\tilde{\rho}^N$, from the Bounded Real Lemma 1 there exists $\mathbf{X}_{\tilde{\rho}} = \mathbf{X}^T_{\tilde{\rho}} > 0$ such that:

$$\begin{bmatrix}\mathbf{A}^T_{\tilde{\rho}}\mathbf{X}_{\tilde{\rho}}\mathbf{A}_{\tilde{\rho}} - \mathbf{X}_{\tilde{\rho}} + \tilde{\rho}^{2N}\mathbf{C}^T\mathbf{C} & \mathbf{A}^T_{\tilde{\rho}}\mathbf{X}_{\tilde{\rho}}\mathbf{B}_{\tilde{\rho}} + \tilde{\rho}^{2N}\mathbf{C}^T\mathbf{D} \\ \mathbf{B}^T_{\tilde{\rho}}\mathbf{X}_{\tilde{\rho}}\mathbf{A}_{\tilde{\rho}} + \tilde{\rho}^{2N}\mathbf{D}^T\mathbf{C} & \mathbf{B}^T_{\tilde{\rho}}\mathbf{X}_{\tilde{\rho}}\mathbf{B}_{\tilde{\rho}} - \gamma^2\tilde{\rho}^{2N}\mathbf{I} + \tilde{\rho}^{2N}\mathbf{D}^T\mathbf{D}\end{bmatrix} < 0$$

with $\mathbf{A}_{\tilde{\rho}} \doteq \mathbf{A}/\tilde{\rho}$ and $\mathbf{B}_{\tilde{\rho}} \doteq \mathbf{B}/\tilde{\rho}$. Define $\mathbf{X}_k \doteq \mathbf{X}_{\tilde{\rho}}\tilde{\rho}^{-2k}$. Multiplying last inequality by $\tilde{\rho}^{-2k}$, it follows that $\mathbf{X}_k$ satisfies the following inequality:

$$\begin{gathered}\begin{bmatrix}\mathbf{A}^T\mathbf{X}_{k+1}\mathbf{A} - \mathbf{X}_k + \mathbf{C}^T\mathbf{C} & \mathbf{A}^T\mathbf{X}_{k+1}\mathbf{B} + \mathbf{C}^T\mathbf{D} \\ \mathbf{B}^T\mathbf{X}_{k+1}\mathbf{A} + \mathbf{D}^T\mathbf{C} & \mathbf{B}^T\mathbf{X}_{k+1}\mathbf{B} - \gamma^2\tilde{\rho}^{2N}\mathbf{I} + \mathbf{D}^T\mathbf{D}\end{bmatrix} + \\ \gamma^2\tilde{\rho}^{2N}(1-\tilde{\rho}^{-2k})\begin{bmatrix}0 & 0 \\ 0 & \mathbf{I}\end{bmatrix} + (\tilde{\rho}^{2(N-k)} - 1)\begin{bmatrix}\mathbf{C}^T & \mathbf{D}^T\end{bmatrix}\begin{bmatrix}\mathbf{C} \\ \mathbf{D}\end{bmatrix} < 0.\end{gathered}$$

Since $\tilde{\rho} > 1$, for $k \leq N$ the following inequality holds:

$$\begin{bmatrix}\mathbf{A}^T\mathbf{X}_{k+1}\mathbf{A} - \mathbf{X}_k + \mathbf{C}^T\mathbf{C} & \mathbf{A}^T\mathbf{X}_{k+1}\mathbf{B} + \mathbf{C}^T\mathbf{D} \\ \mathbf{B}^T\mathbf{X}_{k+1}\mathbf{A} + \mathbf{D}^T\mathbf{C} & \mathbf{B}^T\mathbf{X}_{k+1}\mathbf{B} - \gamma^2\tilde{\rho}^{2N}\mathbf{I} + \mathbf{D}^T\mathbf{D}\end{bmatrix} < 0.$$

The proof follows now directly from Lemma 2. □

A simple bound on the identification error over the finite horizon $[0,N]$ can be obtained by combining Theorem 4 with the bound provided by (Parrilo *et al.*, 1998) for the worst case identification error in the $\mathcal{H}_\infty$ sense, of exponentially stable plants:

$$\mathcal{D}(I) \leq 2\tilde{\rho}^N \left[\sum_{i=0}^{N} \nu_i + \frac{K(\frac{\rho}{\tilde{\rho}})^{N+1}}{1 - \frac{\rho}{\tilde{\rho}}}\right] \tag{19}$$

where $\nu_i \doteq \min\{K(\rho/\tilde{\rho})^i, \|(\mathbf{R}^{-1}_\rho\mathbf{T}^{-1}_u\mathbf{R}_\rho)_{i+1}\|_1\varepsilon\}$ and $\mathbf{T}_u$ is the finite Toeplitz matrix associated with the input sequence u (see (Parrilo *et al.*, 1998) for details).

Next we establish convergence of the algorithm when $N \to \infty$ and $\varepsilon \to 0$.

Theorem 5. If $\tilde{\rho}$ is selected such that $\mathcal{S}_{\tilde{\rho}} \subset \mathcal{H}_\infty$ then the proposed algorithm is convergent, i.e.

$$\lim_{\substack{N\to\infty \\ \varepsilon\to 0}} e_{id}(\mathcal{A}) = 0. \tag{20}$$

PROOF. Consider sequences $N_i \uparrow \infty$, $\varepsilon_i \downarrow 0$, and for a given (N,ε) denote by $\mathcal{T}(\mathbf{y}_0,N,\varepsilon)$ the set of plants consistent with the *a priori* information and the null outcome. Clearly if $h \in \mathcal{T}(\mathbf{y}_0,N,\varepsilon)$

[3] This follows from the fact that $\bar{\sigma}(R^2_\rho) = \rho^{2N}$, $\underline{\sigma}(R^{-2}_\rho) = \rho^{-2N}$ and the interlacing property of the eigenvalues of symetric matrices.

then $h_k \leq \min\{\|(\mathbf{T_u}^{-1})_k\|_1 \varepsilon, K\rho^k\}$. It follows that if $K\rho^k > \|(\mathbf{T_u}^{-1})_k\|_1 \varepsilon_i$ then $\mathcal{T}(\mathbf{y}_0, N_j, \varepsilon_j) \subset \mathcal{T}(\mathbf{y}_0, N_i, \varepsilon_i)$ for $j > i$ and thus using the result in (Aubin and Frankowska, 1990), page 18, the sequence of sets has a limit $\mathcal{T}^* = \cap_n \overline{\mathcal{T}(\mathbf{y}_0, N_n, \varepsilon_n)}$. If $\mathcal{T}^* \neq \{0\}$, then there exists some $g^* \in \mathcal{T}(\mathbf{y}_0, N_j, \varepsilon_j)$, $\forall j$ and such that, for some M and ξ,

$$\|g^*\|_{\ell^2[0,M] \to \ell^2[0,M]} > \xi > 0. \tag{21}$$

Let $\mathcal{T}_{\tilde{\rho}}(\mathbf{y}_0, N_j, \varepsilon) = \{g(\tilde{\rho}z) \colon g(z) \in \mathcal{T}(\mathbf{y}_0, N_j, \varepsilon)\}$. Since $g^*(\tilde{\rho}z) \in \overline{\mathcal{T}_{\tilde{\rho}}(\mathbf{y}_0, N_j, \varepsilon)}, \forall j$, using the error bound derived in Parrilo *et al.* (1998), it follows that there exists some N, ε such that $\|g^*(\tilde{\rho}z)\|_\infty \leq \frac{\xi}{\tilde{\rho}^M}$. This, combined with Theorem 4 implies that $\|g^*\|_{\ell^2[0,M] \to \ell^2[0,M]} \leq \xi$, which contradicts (21).

We finish this section by illustrating the use of the results above to synthesize stabilizing controllers for the unknown unstable plant. Recall that if a controller is designed to achieve robust performance for all plants in the transformed consistency set $S_{\tilde{\rho}}$, i.e. $\sup_{H \in S_{\tilde{\rho}}} \|T_{yw}\|_\infty \leq \gamma$, (where w and y denote some input/output pair of signal of interest, such as the tracking error to a class of inputs), then it follows that, *over a finite horizon M*, this controller guarantees

$$\|T_{yw}\|_{\ell_2[0,M] \to \ell_2[0,M]} \leq \tilde{\rho}^M \gamma \tag{22}$$

for the actual plant. Thus, the proposed algorithm can be used directly to identify plants or design controllers for cases where only the finite horizon behavior is of interest. An example of this situation is the identification, over a finite horizon, of a collection of coupled harmonic oscillators and the driving of such system from an initial to a final condition in a given time. This is a simplified, classical equivalence of a problem arising in the context of quantum computing, where it is desired to change the state of some q-bits and the additional dynamics accounts for the decoherence resulting from the interaction with the outside world (Sznaier *et al.*, 2001).

On the other hand, for a finite experiment length N, both bounds (19) and (22) tend to infinity as $M \to \infty$. Thus they are not very useful for controller synthesis, when the goal is to stabilize the (unknown) plant or guarantee performance over an infinite horizon. This difficulty can be solved by modelling the actual plant as the interconnection of the identified plant and stable dynamic uncertainty (for instance additive) and performing an additional model (in)validation step to test the validity of the assumption and to quantify the size of this uncertainty. Since the proposed algorithm is convergent, one will expect that this invalidation will succeed, by taking N large enough and tightening the bounds in the experimental noise, if necessary. If that is not the case, the *a priori* information on the set of candidate models should be improved.

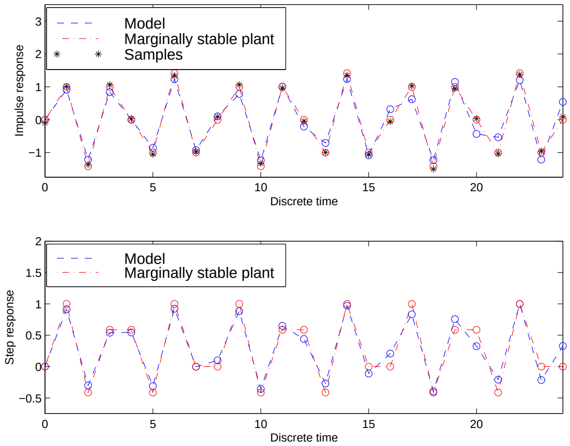

Fig. 1. Model vs. plant. top: impulse response, bottom: step response

4. EXAMPLES

This section illustrates the results with two simple examples. The first one involves a marginally stable plant, the second one a plant which is not strongly stabilizable.

Example 1. Consider the problem of identifying the marginally stable $(\rho = 1)$ plant:

$$S_1(z) = \frac{z}{z^2 + \sqrt{2}z + 1}, \tag{23}$$

from a finite set of samples of its impulse response $\mathbf{y} = \begin{bmatrix} y_0 & y_1 & \cdots & y_N \end{bmatrix}$ with $N = 24$, corrupted by additive noise bounded in amplitude by $\varepsilon = 0.15$:

$$y_k = s_k + \eta_k \quad \text{and} \quad |\eta_k| < \varepsilon. \tag{24}$$

By applying mapping (14) with $\tilde{\rho} = 1.2$, the problem is reduced to the identification of a stable plant analytic in $|z| > (1.2)^{-1}$. Both the consistency and identification problems were solved using MATLAB's LMI Toolbox, with the additional objective of finding the minimum value of the worst case gain K so that the set of LMIs (13) was feasible. Clearly, optimizing over K yields a smaller $\mathcal{H}_\infty$ norm of the identified model and therefore a smaller identification error.

The identification step led a model $\hat{S}_{1\tilde{\rho}}(z)$ with a stability margin $\rho = 0.8333$ and a minimum gain $K = 10.1978$, of order 25 (as many as the number of samples considered in the interpolation problem). Before applying the inverse mapping $\hat{S}_1(z) = \hat{S}_{1\tilde{\rho}}(z/\tilde{\rho})$, the stable model was balanced and reduced to avoid numerical problems and a bad behaviour –large peaks in its impulse response– beyond the finite horizon N. The resulting model $\hat{S}_1(z)$ is unstable and has two complex poles at $p_{1,2} = -0.7246 + / - j0.6941$:

$$\hat{S}_1(z) = \frac{0.0901z^2 + 1.0448z + 0.2002}{z^2 + 1.4492z + 1.0068}. \tag{25}$$

The top plot in Figure 1 shows the impulse responses of the identified unstable model and the marginally stable plant, and the original noisy samples. In order to test the quality of the model, different experiments over the same finite horizon were performed, comparing the outputs of the actual and identified plants to inputs not used in the identification step. Due to space limitations only the results to a step input are shown in the botton plot of Figure 1.

Example 2. Consider now the problem of getting a model of the following unstable –not strongly stabilizable– plant:

$$S_2(z) = 0.25\frac{z+1}{z^2+0.5z-1.5} \tag{26}$$

analytic in $|z| > \rho = 1.5$, from noisy measurements of its impulse response in the interval $k = 0,1,\ldots,24$. Applying the mapping (14) with $\tilde{\rho} = 1.65$, the problem is equivalent to the one of identifying a stable plant, analytic in $|z| > 1.5/1.65 = 0.9091$. Proceeding as in previous example, the consistency and identification steps led to a stable model $\hat{S}_{2\tilde{\rho}}(z)$ with a stability margin of $\rho = 0.9091$ and a minimum worst case gain of $K = 0.4606$. Before applying the inverse mapping, the stable model was balanced and reduced. The resulting unstable model $\hat{S}_2(z)$ is analytic in $|z| > 1.4895$ and of order 24. A further model order reduction would have led to a poor model, unable to follow the actual plant output for different experiments. The top plot of Figure 2 shows the impulse responses of the model and the unstable plant, together with the experimental samples. Finally, the bottom plot of Figure 2 compares the step responses of the obtained model and the actual plant on the finite horizon $[0,24]$.

5. CONCLUSIONS

This paper addresses the problem of identifying non Schur plants in a worst-case sense. Contrary to past work on this problem, the proposed method is intended to be applied in an open loop setting. Thus it avoids the need for assumptions, such as the knowledge of a stabilizing controller for the unknown plant, that can prove to be too restrictive in many practical situations. The algorithm is interpolatory, and in the limit as the number of measurements tends to infinity and the noise level to zero, convergent in the ℓ_2 induced topology. Efforts are currently under way to extend these ideas to the problem of (in)validation of possible unstable models.

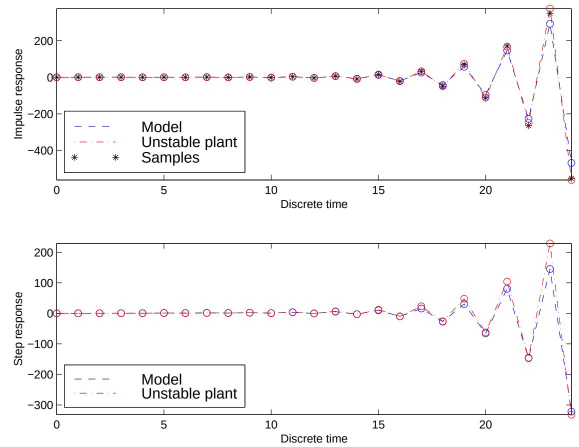

Fig. 2. Model vs. plant. top: impulse response, bottom: step response.

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