15th Triennial World Congress, Barcelona, Spain

CRAMÉR-RAO BOUND FOR STOCHASTIC VOLATILITY MODEL

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Abstract: Estimation problem for the stochastic volatility (SV) model, which is significant in financial econometrics, is discussed. Recursive relations for computation of the Cramér-Rao (CR) bound are derived for state and parameter estimation of this model. An attention is paid to regularity conditions for CR bound calculation. As the CR bound represents a lower bound of the mean-square error of an estimate, it can serve as a gauge of quality of nonlinear estimators for the SV model.

Keywords: nonlinear models, Cramér-Rao bound, nonlinear filters, financial systems, stochastic systems

1. INTRODUCTION

The class of the discrete time Stochastic Volatility (SV) models (Taylor, 1986) has its roots both in mathematical finance and financial econometrics. The SV models play the central role in asset return prediction (Danielsson, 1994; Andersen and Sørensen, 1996; Ruiz, 1994) and have become an attractive class of models and an alternative to other classes such as Autoregressive Conditional Heteroscedasticity (ARCH) models (Engle, 1982). In contrast to ARCH models the asset return variance representing the volatility is expressed in SV model separately from the demeaned return as an unobserved component and its logarithm follows an AR(1) process. The evaluating of the likelihood function of ARCH models is a straightforward task. However it is impossible to obtain explicit expressions for the likelihood function in the case of the SV models. Hence the maximum likelihood (ML) methods are not easy to implement.

The basic approach to a simple SV model estimation is the quasi-ML (QML) method developed by Ruiz (1994) and based on approximate linear filtering methods. Another approach relies on the method of moments (MM) (Andersen and Sørensen, 1996) to avoid the integration problems associated with the direct evaluation of the likelihood. The Monte Carlo evidence of (Jacquier *et al.*, 1994) suggests that these standard procedures suffer from poor finite sample performance because they do not depend on the exact likelihood and provide only optimal linear estimates. Alternatives based on the exact likelihood are simulation-based ML (Danielsson, 1994) and nonlinear filtering ML (Watanabe, 1999) methods. They are computationally intensive, but according to the Monte Carlo results of (Jacquier *et al.*, 1994) outperform QML and MM approaches.

Šimandl and Soukup (2001) successfully used the Gibbs sampler as a non-ML method for state and parameter estimation for discrete-time SV models. The Gibbs sampler belongs to the class of Markov chain Monte Carlo methods (Tanner, 1996) which represent pdf's by random samples drawn from these pdf's.

The performance of the above mentioned methods can be compared mutually but there is no idea how close are the estimates to the exact solution. Hence it would be very valuable to find an objective limit for performance quality and to gauge the estimators by their proximity to this limit. For this purpose, the Cramér-Rao (CR) bound can be used as a lower limit for the mean-square error of an estimate.

The CR bound, defined as the inverse of the Fisher information matrix under regularity conditions, is a common tool for estimate quality evaluation in constant parameter estimation. Van Trees (1968) extended the CR bound methodology for random parameters estimation and later the idea of the CR bound was successfully applied in state estimation for nonlinear stochastic dynamic systems by Bobrovsky and Zakai (1975) and Galdos (1980). These works are based on a certain kind of "equivalence" between pdf's of the original nonlinear system and an auxiliary linear Gaussian system. A survey of methods based on this approach was presented by Kerr (1989).

An alternative approach to derivation of the CR bound (called also posterior CR bound) for the filtering problem in discrete-time nonlinear systems was proposed by Tichavský *et al.* (1998). The principle of this approach is to regard the state history as a random parameter vector. The CR bound for the state is obtained as the lower right block of the CR bound for the whole state history. This approach was further elaborated by Šimandl *et al.* (1999), Bergman (1999) and Šimandl *et al.* (2001). In the latter work, the recursive relations for computation of CR bounds for three types of state estimation, i.e. filtering, smoothing and prediction (which could be of a particular interest in econometrics), were presented in a unified form. CR bounds were also derived for combined state and parameter estimation in (Šimandl *et al.*, 2001).

The aim of the paper is to find recursive relations for computation of CR bound for the SV model as an objective gauge for estimator performance evaluation. The derivation of the bound will be based on the general results of Šimandl *et al.* (2001). A special attention will be paid to the regularity conditions for the SV model.

The paper is organized as follows. After defining the SV model and formulating the problem in Section 2, the CR bound is discussed in Section 3. General recursive relations for filtering CR bound are presented and the derivation of the bound for the SV model is performed under regularity conditions, specified previously. The computation of the CR bound is shown in a numerical example in Section 4. The main results of the paper are summarized in Section 5.

2. PROBLEM STATEMENT

2.1 *Discrete-Time SV Model*

Consider a simple SV model

$$\ln \sigma_{k+1} = a + b \ln \sigma_k + w_k \tag{1}$$

$$y_k = \sqrt{\sigma_k} u_k \tag{2}$$

where y_k denotes the demeaned return in the time instant $k = 0, 1, \ldots, f$, the variance σ_k is the volatility of the demeaned return y_k, disturbances u_k and w_k are assumed to be mutually independent Gaussian white noises with zero means and variances 1 and $q > 0$, respectively, and the parameters a, b, q are random and their description will be discussed later in Section 3. The volatility variable σ_k is latent and the only observable variable is y_k. Note that introducing the term $\ln \sigma_k$ ensures positiveness of σ_k.

2.2 *SV Model in State Space Form*

A state space representation of (1), (2) can be given by the following relations

$$x_{k+1} = a + b x_k + w_k \tag{3}$$

$$z_k = x_k + v_k \tag{4}$$

where the new quantities are defined as $z_k = \ln y_k^2$, $x_k = \ln \sigma_k$ and $v_k = \ln u_k^2$. The state equation (3) with the log of the volatility σ_k as the unobserved state variable x_k is an alternative representation of the SV model (1). Note that the observation noise v_k in (4) is not normally distributed and its probability density function (pdf) is given by

$$p_{v_k}(v_k) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}\exp(v_k) + \frac{v_k}{2}\right] \tag{5}$$

where $\exp(v_k) = e^{v_k}$. Relation (5) is obtained by application of the well known formula for transformation of random variable to $v_k = \ln u_k^2$. The state noise w_k is the same as in (1),

$$p_{w_k}(w_k) = N(w_k : 0, q) \,. \tag{6}$$

2.3 *Estimation Problem and CR Bound*

The vector of observations $\mathbf{z}^k = [z_0 \; z_1 \; \ldots \; z_k]^{\mathrm{T}}$ is given. The intention is to find the conditional posterior pdf $p(\boldsymbol{\theta}|\mathbf{z}^k)$ of the time-invariant parameter $\boldsymbol{\theta} = [a \; b \; q]^{\mathrm{T}}$ and the filtering pdf $p(x_k|\mathbf{z}^k)$ of the unobserved state variable x_k for $k = 0, 1, \ldots, f$. It is the problem of nonlinear state estimation and an exact solution is impossible. The general solution of the state estimation problem, based on the Bayesian approach, is given by the functional relations for conditional pdf's of the state.

In this case the extended state vector is defined as $\boldsymbol{\xi}_k = [x_k, \boldsymbol{\theta}^{\mathrm{T}}]^{\mathrm{T}}$ and the corresponding state equation is obviously nonlinear.

The closed-form solution of the state estimation problem is known only for linear Gaussian systems (Anderson and Moore, 1979) and a few special cases (Šimandl, 1996). For that reason many analytical and numerical approximations of the system or pdf's have been developed since the beginning of the 1970's (Sorenson, 1974; Sorenson, 1988; Kulhavý, 1996). The most significant representatives of such estimation techniques are the Gaussian sum method and the point mass method. Their application to the SV model is infeasible because of the unknown variance q, or computationally demanding. Another alternative approach to the numerical solution of the Bayesian recursive relations is based on the statistical simulation methods, namely the algorithms of Gibbs sampler (Šimandl and Soukup, 2001) and particle filter (Nagahara and Kitagawa, 1999) appear to be powerful tools for SV model estimation.

Anyway, the evaluation of the results of any nonlinear filter is necessary. Therefore the aim of this paper is to find the filtering CR bound $\mathbf{C}_{k|k}$ for the SV model, which is the lower bound for mean-square estimate error matrix $\mathbf{P}_{k|k}$, i.e.

$$\mathbf{C}_{k|k} \leq \mathbf{P}_{k|k} \tag{7}$$

where

$$\mathbf{P}_{k|k} = \mathsf{E}\{(\boldsymbol{\xi}_k - \hat{\boldsymbol{\xi}}_{k|k})(\boldsymbol{\xi}_k - \hat{\boldsymbol{\xi}}_{k|k})^{\mathrm{T}}\} \tag{8}$$

and $\hat{\boldsymbol{\xi}}_{k|k}$ is an arbitrary point estimate of $\boldsymbol{\xi}_k$ given measurements $\mathbf{z}^k$. The inequality (7) means that $\mathbf{P}_{k|k} - \mathbf{C}_{k|k}$ is a positive semidefinite matrix.

3. CRAMÉR-RAO BOUND IN STATE ESTIMATION

In this section, recursive relations for the filtering CR bound for the general discrete-time nonlinear stochastic system with unknown parameters will be presented according to (Šimandl *et al.*, 2001) and these relations will be used for derivation of the CR bound for the SV model (3), (4).

3.1 *General Relations for CR Bound*

Consider the general discrete-time nonlinear stochastic system

$$\mathbf{x}_{k+1} = \mathbf{f}_k(\mathbf{x}_k, \boldsymbol{\theta}, \mathbf{w}_k) \tag{9}$$
$$\mathbf{z}_k = \mathbf{g}_k(\mathbf{x}_k, \boldsymbol{\theta}, \mathbf{v}_k) \tag{10}$$

where $\mathbf{x}_k$ and $\mathbf{z}_k$ with $\dim(\mathbf{x}_k) = n$ and $\dim(\mathbf{z}_k) = r$ represent the state and measurement vectors, respectively, $\boldsymbol{\theta} = [\theta_1\, \theta_2\, \ldots\, \theta_m]^{\mathrm{T}}$ is a random vector parameter given by the known twice differentiable pdf $p(\boldsymbol{\theta})$, $\mathbf{f}_k(\mathbf{x}_k, \boldsymbol{\theta}, \mathbf{w}_k)$ and $\mathbf{g}_k(\mathbf{x}_k, \boldsymbol{\theta}, \mathbf{v}_k)$ are known vector functions, $\{\mathbf{w}_k\}$ and $\{\mathbf{v}_k\}$ are mutually independent white noise sequences, with $\dim(\mathbf{w}_k) = n$ and $\dim(\mathbf{v}_k) = r$, which are described by known pdf's $p_{\mathbf{w}_k}(\mathbf{w}_k)$ and $p_{\mathbf{v}_k}(\mathbf{v}_k)$, respectively. The noises and the parameter are independent of the initial state $\mathbf{x}_0$ which is described by the known pdf $p(\mathbf{x}_0)$.

Suppose that the state transition pdf $p(\mathbf{x}_{k+1}|\mathbf{x}_k, \boldsymbol{\theta})$ exists and is twice differentiable with respect to its arguments. Similarly, suppose that the measurement pdf $p(\mathbf{z}_k|\mathbf{x}_k, \boldsymbol{\theta})$ exists and is twice differentiable with respect to $\mathbf{x}_k$ and $\boldsymbol{\theta}$.

The derivation of the recursive relations for the filtering CR bound for the pair $(\mathbf{x}_k, \boldsymbol{\theta})$ starts with the expression for the logarithm of the joint pdf of state history $\mathbf{x}^k = [\mathbf{x}_0^{\mathrm{T}}\, \mathbf{x}_1^{\mathrm{T}}\, \ldots\, \mathbf{x}_k^{\mathrm{T}}]^{\mathrm{T}}$, parameter vector $\boldsymbol{\theta}$, and measurement history $\mathbf{z}^k = [\mathbf{z}_0^{\mathrm{T}}\, \mathbf{z}_1^{\mathrm{T}}\, \ldots\, \mathbf{z}_k^{\mathrm{T}}]^{\mathrm{T}}$,

$$\ln p(\mathbf{x}^k, \boldsymbol{\theta}, \mathbf{z}^k) = \textstyle\sum_{i=0}^{k} \ln p(\mathbf{z}_i|\mathbf{x}_i, \boldsymbol{\theta}) + \ln p(\mathbf{x}_0) + \ln p(\boldsymbol{\theta}) + \textstyle\sum_{i=1}^{k} \ln p(\mathbf{x}_i|\mathbf{x}_{i-1}, \boldsymbol{\theta})\,. \tag{11}$$

To simplify the notations, the nabla operator will be defined for a vector $\boldsymbol{\kappa} = [\kappa_1\, \kappa_2\, \ldots\, \kappa_\ell]^{\mathrm{T}}$ as

$$\nabla_{\boldsymbol{\kappa}} = \left[\frac{\partial}{\partial \kappa_1} \quad \frac{\partial}{\partial \kappa_2} \quad \ldots \quad \frac{\partial}{\partial \kappa_\ell}\right]$$

and for two vectors $\boldsymbol{\kappa}$, $\boldsymbol{\lambda}$ the notation $\nabla_{\boldsymbol{\kappa},\boldsymbol{\lambda}} = [\nabla_{\boldsymbol{\kappa}} \quad \nabla_{\boldsymbol{\lambda}}]$ will be used.

The Fisher information matrix (FIM) for the state history $\mathbf{x}^k$ and the parameter $\boldsymbol{\theta}$ is defined as

$$\mathbf{J}^{k|k}(\mathbf{x}^k, \boldsymbol{\theta}) = -\mathsf{E}\{\nabla_{\mathbf{x}^k,\boldsymbol{\theta}}[\nabla_{\mathbf{x}^k,\boldsymbol{\theta}} \ln p(\mathbf{x}^k, \boldsymbol{\theta}, \mathbf{z}^k)]^{\mathrm{T}}\}, \tag{12}$$

provided that the expectation and the derivatives exist. The CR bound $\mathbf{C}^{k|k}(\mathbf{x}^k, \boldsymbol{\theta})$ for the state history $\mathbf{x}^k$ and the parameter $\boldsymbol{\theta}$ is defined as

$$\mathbf{C}^{k|k}(\mathbf{x}^k, \boldsymbol{\theta}) = [\mathbf{J}^{k|k}(\mathbf{x}^k, \boldsymbol{\theta})]^{-1}\,. \tag{13}$$

The size of the matrices $\mathbf{J}^{k|k}(\mathbf{x}^k, \boldsymbol{\theta})$, $\mathbf{C}^{k|k}(\mathbf{x}^k, \boldsymbol{\theta})$ is $[m + n(k+1)] \times [m + n(k+1)]$.

The filtering CR bound $\mathbf{C}_{k|k}$ for $(\mathbf{x}_k, \boldsymbol{\theta})$ is found as the lower-right $(n+m) \times (n+m)$ block of $[\mathbf{J}^{k|k}(\mathbf{x}^k, \boldsymbol{\theta})]^{-1}$. The detailed derivation is shown in (Šimandl *et al.*, 2001). For computation of $\mathbf{C}_{k|k}$ the following $n \times n$ matrices are introduced

$$\mathbf{K}_{k+1}^{k} = \mathsf{E}\{-\nabla_{\mathbf{x}_k}[\nabla_{\mathbf{x}_k} \ln p(\mathbf{x}_{k+1}|\mathbf{x}_k, \boldsymbol{\theta})]^{\mathrm{T}}\} \tag{14}$$
$$\mathbf{K}_{k+1}^{k,k+1} = \mathsf{E}\{-\nabla_{\mathbf{x}_{k+1}}[\nabla_{\mathbf{x}_k} \ln p(\mathbf{x}_{k+1}|\mathbf{x}_k, \boldsymbol{\theta})]^{\mathrm{T}}\} \tag{15}$$
$$\mathbf{K}_{k+1}^{k+1} = \mathsf{E}\{-\nabla_{\mathbf{x}_{k+1}}[\nabla_{\mathbf{x}_{k+1}} \ln p(\mathbf{x}_{k+1}|\mathbf{x}_k, \boldsymbol{\theta})]^{\mathrm{T}}\} \tag{16}$$
$$\mathbf{L}_{k}^{k} = \mathsf{E}\{-\nabla_{\mathbf{x}_k}[\nabla_{\mathbf{x}_k} \ln p(\mathbf{z}_k|\mathbf{x}_k, \boldsymbol{\theta})]^{\mathrm{T}}\} \tag{17}$$

with $\mathbf{K}_{k+1}^{k+1,k} = [\mathbf{K}_{k+1}^{k,k+1}]^{\mathrm{T}}$,

$$\mathbf{K}_0^0 = \mathsf{E}\{-\nabla_{\mathbf{x}_0}[\nabla_{\mathbf{x}_0} \ln p(\mathbf{x}_0)]^{\mathrm{T}}\}, \tag{18}$$

and

$$\mathbf{K}_{k+1}^{\theta k} = \mathsf{E}\{-\nabla_{\mathbf{x}_k}[\nabla_{\boldsymbol{\theta}} \ln p(\mathbf{x}_{k+1}|\mathbf{x}_k, \boldsymbol{\theta})]^{\mathrm{T}}\} \quad (19)$$
$$\mathbf{K}_{k+1}^{\theta,k+1} = \mathsf{E}\{-\nabla_{\mathbf{x}_{k+1}}[\nabla_{\boldsymbol{\theta}} \ln p(\mathbf{x}_{k+1}|\mathbf{x}_k, \boldsymbol{\theta})]^{\mathrm{T}}\} \quad (20)$$
$$\mathbf{K}_{k+1}^{\theta} = \mathsf{E}\{-\nabla_{\boldsymbol{\theta}}[\nabla_{\boldsymbol{\theta}} \ln p(\mathbf{x}_{k+1}|\mathbf{x}_k, \boldsymbol{\theta})]^{\mathrm{T}}\} \quad (21)$$
$$\mathbf{L}_{k}^{\theta k} = \mathsf{E}\{-\nabla_{\mathbf{x}_k}[\nabla_{\boldsymbol{\theta}} \ln p(\mathbf{z}_k|\mathbf{x}_k, \boldsymbol{\theta})]^{\mathrm{T}}\} \quad (22)$$
$$\mathbf{L}_{k}^{\theta} = \mathsf{E}\{-\nabla_{\boldsymbol{\theta}}[\nabla_{\boldsymbol{\theta}} \ln p(\mathbf{z}_k|\mathbf{x}_k, \boldsymbol{\theta})]^{\mathrm{T}}\} \quad (23)$$
$$\mathbf{A} = \mathsf{E}\{-\nabla_{\boldsymbol{\theta}}[\nabla_{\boldsymbol{\theta}} \ln p(\boldsymbol{\theta})]^{\mathrm{T}}\} \quad (24)$$

where $\mathbf{L}_k^{k\theta} = [\mathbf{L}_k^{\theta k}]^{\mathrm{T}}$, $\mathbf{K}_k^{k\theta} = [\mathbf{K}_k^{\theta k}]^{\mathrm{T}}$, $\mathbf{K}_{k+1}^{k\theta} = [\mathbf{K}_{k+1}^{\theta k}]^{\mathrm{T}}$. The matrices $\mathbf{K}_k^{\theta k}$, $\mathbf{K}_{k+1}^{\theta k}$, and $\mathbf{L}_k^{\theta k}$ have the size $m \times n$, $\mathbf{K}_{k+1}^{\theta}$, $\mathbf{L}_k^{\theta}$, and $\mathbf{A}$ are $m \times m$ matrices, and $\mathbf{K}_0^{\theta 0}$ is an $m \times n$ zero matrix.

The indexes in the $\mathbf{K}$ matrices have the following meaning: The lower index $k + 1$ is the time instant of the state described by the transition pdf $p(\mathbf{x}_{k+1}|\mathbf{x}_k, \boldsymbol{\theta})$. The upper index expresses the states or parameters for which the derivatives of the transition pdf are performed.

Let the inverse of the filtering CR bound be decomposed in blocks as

$$\mathbf{C}_{k|k}^{-1} = \begin{bmatrix} \mathbf{J}_{k|k}^{xx} & \mathbf{J}_{k|k}^{x\theta} \\ \mathbf{J}_{k|k}^{\theta x} & \mathbf{J}_{k|k}^{\theta\theta} \end{bmatrix}. \quad (25)$$

The block elements of $\mathbf{C}_{k|k}^{-1}$, as denoted in (25), obey the following recursive relations

$$\mathbf{J}_{k|k}^{xx} = \mathbf{J}_{k|k-1}^{xx} + \mathbf{L}_k^{k} \quad (26)$$
$$\mathbf{J}_{k|k}^{x\theta} = \mathbf{J}_{k|k-1}^{x\theta} + \mathbf{L}_k^{k\theta} \quad (27)$$
$$\mathbf{J}_{k|k}^{\theta\theta} = \mathbf{J}_{k|k-1}^{\theta\theta} + \mathbf{L}_k^{\theta} \quad (28)$$

where the one-step predictive FIMs are given as

$$\mathbf{J}_{k+1|k}^{xx} = \mathbf{K}_{k+1}^{k+1} - \mathbf{K}_{k+1}^{k+1,k} \boldsymbol{\Delta}_k^{-1} \mathbf{K}_{k+1}^{k,k+1} \quad (29)$$
$$\mathbf{J}_{k+1|k}^{x\theta} = \mathbf{K}_{k+1}^{k+1,\theta} - \mathbf{K}_{k+1}^{k+1,k} \boldsymbol{\Delta}_k^{-1} \boldsymbol{\Delta}_k^{x\theta} \quad (30)$$
$$\mathbf{J}_{k+1|k}^{\theta\theta} = \mathbf{J}_{k|k}^{\theta\theta} + \mathbf{K}_{k+1}^{\theta} - \boldsymbol{\Delta}_k^{\theta x} \boldsymbol{\Delta}_k^{-1} \boldsymbol{\Delta}_k^{x\theta} \quad (31)$$

with

$$\boldsymbol{\Delta}_k = \mathbf{J}_{k|k}^{xx} + \mathbf{K}_{k+1}^{k}, \quad \boldsymbol{\Delta}_k^{\theta x} = \mathbf{J}_{k|k}^{\theta x} + \mathbf{K}_{k+1}^{\theta k} \quad (32)$$

and $\boldsymbol{\Delta}_k^{x\theta} = [\boldsymbol{\Delta}_k^{\theta x}]^{\mathrm{T}}$.

The initial conditions for the recursive relations (26)–(31) are $\mathbf{J}_{0|-1}^{xx} = \mathbf{K}_0^0$, $\mathbf{J}_{0|-1}^{x\theta} = \mathbf{K}_0^{0\theta} = 0$, and $\mathbf{J}_{0|-1}^{\theta\theta} = \mathbf{A}$. The relations describe a recursive block-wise computation of the filtering CR bound for state and parameter estimation.

3.2 *Regularity Conditions for SV Model*

Before the derivation of the filtering CR bound for the SV model (3), (4), it is necessary to verify the regularity conditions, i.e. to examine the existence of the expectation and derivatives in (12) by specifying the pdf's on the right-hand side of (11). Note that because mean value and derivative are linear operators, the regularity conditions for (12) must be fulfilled for each summand in (11).

The computation of transition and measurement pdf's is straightforward from (6) with (3) and (5) with (4), respectively, and the pdf's have the following form

$$p(x_{k+1}|x_k, \boldsymbol{\theta}) = N(x_{k+1} : a + bx_k, q) \quad (33)$$
$$p(z_k|x_k, \boldsymbol{\theta}) = p_{v_k}(z_k - x_k). \quad (34)$$

Since σ_k is a positive volatility, its initial pdf is considered to be χ^2 distributed. Then the pdf of the initial state x_0 is

$$p(x_0) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}\exp(x_0) + \frac{x_0}{2}\right] \quad (35)$$

because $x_0 = \ln \sigma_0$. Pdf's of the parameters a and b are considered as

$$p(a) = N(a \colon \mu_a, c_a) \quad (36)$$
$$p(b) = N(b \colon \mu_b, c_b) \quad (37)$$

where $c_a > 0$, $c_b > 0$. It is easy to see that partial derivatives up to second order of logarithms of pdf's (33)–(37) and their expectations exist.

Similarly to volatility σ_0, the description of parameter q by the χ^2 distribution seems realistic, i.e. $p(q) = \frac{1}{\sqrt{2\pi q}} \exp(-\frac{q}{2})$. The second derivative of $\ln p(q)$ is $0.5q^{-2}$ which exists for the defined domain of q. However, the expectation of the second derivative does not exist, because

$$\mathsf{E}\{q^{-2}\} = \int_0^{\infty} q^{-2} \frac{1}{\sqrt{2\pi q}} \exp\left(-\frac{q}{2}\right) \mathrm{d}q$$

and after substitution $t = \frac{q}{2}$ it is obvious that the integral

$$\mathsf{E}\{q^{-2}\} = \frac{1}{4\sqrt{\pi}} \int_0^{\infty} t^{-2.5} \exp(-t)\, \mathrm{d}t \quad (38)$$

diverges. Hence, another description of q must be chosen. As the χ^2 distribution is a special case of the Γ distribution, the Γ distribution with parameters $\alpha > 0$, $\beta > 0$ will be examined now; thus

$$p(q) = \Gamma(q \colon \alpha, \beta) = \frac{1}{\beta^{\alpha}\Gamma(\alpha)} q^{\alpha-1} \exp\left(-\frac{q}{\beta}\right) \quad (39)$$

where $\Gamma(\alpha)$ is the gamma function of α. In this case, the second derivative of $\ln p(q)$ is $(1-\alpha)q^{-2}$ and the integral analogous to (38) is

$$\mathsf{E}\{q^{-2}\} = \left[\beta^2\Gamma(\alpha)\right]^{-1} \int_0^{\infty} t^{\alpha-3} \exp(-t)\, \mathrm{d}t \quad (40)$$

which exists for $\alpha > 2$, and then $\mathsf{E}\{q^{-2}\} = [(\alpha-1)(\alpha-2)\beta^2]^{-1}$.

The analysis has shown that for the pdf's (33)–(37) and (39) with $\alpha > 2$ the regularity conditions for the SV model are fulfilled.

3.3 *CR Bound for SV Model*

In order to use the relations for CR bound computation (26)–(31) for the SV model (3) and (4), the $\mathbf{K}$, $\mathbf{L}$ and $\mathbf{A}$ matrices (14)–(24) must be specified. The transition and measurement pdf's (33), (34), and the pdf's of the initial state and parameters (35)–(37), (39) are considered in (14)–(24). After tedious calculations the following expressions are obtained

$$K_{k+1}^{k} = (c_b + \mu_b^2)\mathsf{E}\{q^{-1}\} \tag{41}$$

$$K_{k+1}^{k,k+1} = -\mu_b \mathsf{E}\{q^{-1}\} \tag{42}$$

$$K_{k+1}^{k+1} = \mathsf{E}\{q^{-1}\} \tag{43}$$

$$L_k^k = 0.5\mathsf{E}\{\exp(z_k - x_k)\} \tag{44}$$

$$K_0^0 = 0.5\mathsf{E}\{\exp(x_0)\} \tag{45}$$

$$\mathbf{K}_{k+1}^{k\theta} = [\mu_b \mathsf{E}\{q^{-1}\} \quad \mathsf{E}\{bq^{-1}x_k\} \quad 0] \tag{46}$$

$$\mathbf{K}_{k+1}^{k+1,\theta} = [-\mathsf{E}\{q^{-1}\} \quad -\mathsf{E}\{q^{-1}x_k\} \quad 0] \tag{47}$$

$$\mathbf{K}_{k+1}^{\theta} = \begin{bmatrix} \mathsf{E}\{q^{-1}\} & \mathsf{E}\{q^{-1}x_k\} & 0 \\ \mathsf{E}\{q^{-1}x_k\} & \mathsf{E}\{q^{-1}x_k^2\} & 0 \\ 0 & 0 & 0.5\mathsf{E}\{q^{-2}\} \end{bmatrix} \tag{48}$$

$$\mathbf{A} = \mathrm{diag}(c_a^{-1}, c_b^{-1}, -0.5\mathsf{E}\{q^{-2}\}) \tag{49}$$

and $\mathbf{L}_k^{k\theta}$, $\mathbf{L}_k^{\theta}$ are zero matrices of sizes 1×3 and 3×3, respectively. The following expectations in (41)–(49) can be enumerated analytically:

$$\mathsf{E}\{\exp(z_k - x_k)\} = \mathsf{E}\{\exp(v_k)\} = \mathsf{E}\{u_k^2\} = 1 \tag{50}$$

$$\mathsf{E}\{\exp(x_0)\} = \mathsf{E}\{\exp(\ln \sigma_0)\} = \mathsf{E}\{\sigma_0\} = 1 \tag{51}$$

$$\begin{aligned} \mathsf{E}\{q^{-1}\} &= \int_0^\infty q^{-1} \frac{q^{\alpha-1}\exp(-q\beta^{-1})}{\beta^\alpha \Gamma(\alpha)}\,\mathrm{d}q \\ &= \frac{1}{\beta\Gamma(\alpha)} \int_0^\infty t^{\alpha-2}\exp(-t)\,\mathrm{d}t = \frac{\Gamma(\alpha-1)}{\beta\Gamma(\alpha)} \\ &= [(\alpha-1)\beta]^{-1} \end{aligned} \tag{52}$$

$$\begin{aligned} \mathsf{E}\{q^{-2}\} &= \int_0^\infty q^{-2} \frac{q^{\alpha-1}\exp(-q\beta^{-1})}{\beta^\alpha \Gamma(\alpha)}\,\mathrm{d}q \\ &= \frac{1}{\beta^2\Gamma(\alpha)} \int_0^\infty t^{\alpha-3}\exp(-t)\,\mathrm{d}t = \frac{\Gamma(\alpha-2)}{\beta^2\Gamma(\alpha)} \\ &= [(\alpha-1)(\alpha-2)\beta^2]^{-1} \end{aligned} \tag{53}$$

where in (52), (53) the substitution $t = q/\beta$ was used. The other expectations, i.e. $\mathsf{E}\{q^{-1}x_k\}$, $\mathsf{E}\{bq^{-1}x_k\}$ and $\mathsf{E}\{q^{-1}x_k^2\}$, must be generated numerically. Their approximations are obtained by Monte Carlo simulations of the plant (3).

The filtering CR bound for state and parameters of the SV model can be compared with the mean-square error matrix $\mathbf{P}_{k|k}$ of an estimate $\hat{\boldsymbol{\xi}}_{k|k}$. The mean-square error matrix must be enumerated numerically by Monte Carlo simulations.

4. NUMERICAL EXAMPLE

Consider the SV model (3), (4) with the following specification of its random variables:

$$p(a) = N(a\colon \mu_a, c_a), \text{ where } \mu_a = -1, c_a = 0.1$$

$$p(b) = N(b\colon \mu_b, c_b), \text{ where } \mu_b = 0.9, c_b = 5 \cdot 10^{-4}$$

$$p(q) = \Gamma(q\colon \alpha, \beta), \text{ where } \alpha = 3, \beta = 1/9$$

and $p(x_0)$ is given by (35). The unknown expectations $\mathsf{E}\{q^{-1}x_k\}$, $\mathsf{E}\{bq^{-1}x_k\}$ and $\mathsf{E}\{q^{-1}x_k^2\}$ in (46)–(48), where enumerated numerically using 3000 Monte Carlo simulation runs for (3); i.e. the estimate of $\mathsf{E}\{bq^{-1}x_k\}$ is computed as

$$\mathsf{E}\{bq^{-1}x_k\} \approx \frac{1}{3000}\sum_{i=1}^{3000} \frac{b(i)x_k(i)}{q(i)}, \; k = 0, 1, \ldots, 50,$$

where i is an ith simulation run.

Substituting (41)–(49) to (14)–(24), the FIMs (26)–(31) for state x_k and parameters a, b, q given measurements $\mathbf{z}^k$, $k = 0, 1, \ldots, 50$, were computed. Using (25), the CR bound matrix $\mathbf{C}_{k|k}$ is obtained. The diagonal elements of $\mathbf{C}_{k|k}$ represent scalar CR bounds for x_k, a, b and q, respectively. The time development of these bounds is shown in Figures 1, 2.

5. CONCLUSION

The SV model can be estimated by a variety of nonlinear estimation methods. However, quality evaluation of the methods has not been treated sufficiently. In this paper, the tool for a solution of the problem is designed. Recursive relations for the CR bound of the SV model's state and three parameters has been derived and regularity conditions for CR bound computation were applied to the SV model. The random variables of the model had to be properly specified to ensure the regularity conditions. The CR bound in state and parameter estimation of dynamic systems is a lower bound for the mean-square error matrix of an arbitrary point estimate of state and parameters. Thus, it may be taken for an objective limit of cognizability of unknown variables of the system and it is possible to evaluate the quality of estimators by their proximity to the bound. To illustrate the computation of the CR bound for SV model, a numerical example has been included.

ACKNOWLEDGEMENTS

The work was supported by the Grant Agency of the Czech Republic, project 102/01/0021, and by the Ministry of Education, Youth and Sports of the Czech Republic, project 2352 00004.

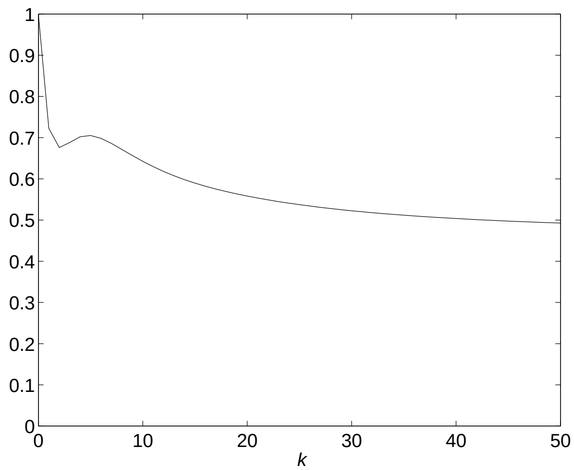

Fig. 1. CR bound for state x_k.

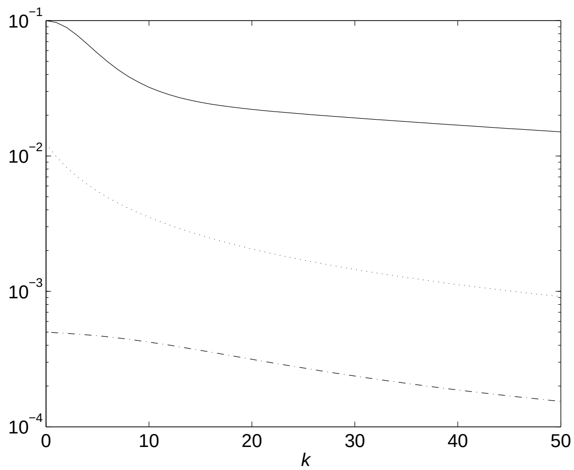

Fig. 2. CR bounds for parameters a (solid line), b (dot-and-dash line), and q (dotted line).

6. REFERENCES

Andersen, T.G. and B.E. Sørensen (1996). GMM estimation of a stochastic volatility model: a Monte Carlo study. *Journal of Business and Economic Statistics* **14**, 328–352.

Anderson, B.D.O. and S.B. Moore (1979). *Optimal Filtering*. Prentice Hall. New Jersey.

Bergman, N. (1999). Recursive Bayesian estimation: navigation and tracking applications. PhD thesis. Linköping University. Sweden.

Bobrovsky, B.Z. and M. Zakai (1975). A lower bound on the estimation error for Markov processes. *IEEE Trans. Automat. Control* **20**(6), 785–788.

Danielsson, J. (1994). Stochastic volatility in asset prices: estimation with simulated maximum likelihood. *Journal of Econometrics* **64**, 375–400.

Engle, R.F. (1982). Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom inflation. *Econometrica* **50**, 987–1007.

Galdos, J.I. (1980). A Cramer-Rao bound for multi-dimensional discrete-time dynamical systems. *IEEE Trans. Automat. Control* **25**(1), 117–119.

Jacquier, E., N.G. Polson and P.E. Rossi (1994). Bayesian analysis of stochastic volatility models. *Journal of Business and Economic Statistics* **12**, 371–389.

Kerr, T.H. (1989). Status of CR-like lower bounds for nonlinear filtering. *IEEE Trans. on Aerospace and Electronic Systems* **25**(5), 590–601.

Kulhavý, R. (1996). *Recursive Nonlinear Estimation*. Springer-Verlag. London.

Nagahara, Y. and G. Kitagawa (1999). A non-Gaussian stochastic volatility model. *Journal of Computational Finance* **2**(2), 33–47.

Ruiz, E. (1994). Quasi-maximum likelihood estimation of stochastic volatility models. *Journal of Econometrics* **63**, 289–306.

Šimandl, M. (1996). State estimation for non-Gaussian models. In: *Proc. 13th IFAC World Congress*. Vol. H. San Francisco. pp. 463–468.

Šimandl, M. and T. Soukup (2001). Gibbs sampler to stochastic volatility models. In: *Preprints of European Control Conference ECC'01*. Porto. September 4–7.

Šimandl, M., J. Královec and P. Tichavský (1999). Predictive and filtering lower bounds for nonlinear filters. In: *Preprints of 14th World Congress of IFAC*. Vol. H. Beijing, China. pp. 43–48. July 5–9.

Šimandl, M., J. Královec and P. Tichavský (2001). Filtering, predictive, and smoothing Cramér-Rao bounds for discrete-time nonlinear dynamic systems. *Automatica* **37**(11), 1703–1716.

Sorenson, H.W. (1974). On the development of practical nonlinear filters. *Inf. Sci.* **7**, 230–270.

Sorenson, H.W. (1988). Recursive estimation for nonlinear dynamic systems. In: *Bayesian Analysis of Time Series and Dynamic Models*. Marcel Dekker. New York.

Tanner, M.A. (1996). *Tools for Statistical Inference*. Springer-Verlag. New York.

Taylor, S.J. (1986). *Modeling Financial Time Series*. John Wiley. Chichester.

Tichavský, P., C. Muravchik and A. Nehorai (1998). Posterior Cramér-Rao bounds for discrete-time nonlinear filtering. *IEEE Trans. on Signal Processing* **46**(5), 1386–1396.

Van Trees, H.L. (1968). *Detection, Estimation and Modulation Theory*. John Wiley. New York.

Watanabe, T. (1999). A non-linear filtering approach to stochastic volatility models with application to daily stock return. *Journal of Applied Econometrics* **14**, 101–121.