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BIFURCATION PLACEMENT OF HOPF POINTS FOR STABILIZATION OF EQUILIBRIA

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Abstract: Applications of bifurcation control to chemical engineering processes have aimed at rendering subcritical bifurcations supercritical. Beyond affecting the criticality of bifurcation points for control purposes, it is possible to *deliberately introduce* bifurcation points to stabilize parts of the equilibrium manifold of an ODE process model. This can be done without affecting the equilibrium manifold of the uncontrolled process. For brevity this approach is called *bifurcation placement*. The present paper focuses on the introduction of Hopf bifurcations, though the approach is more general. Conditions are stated for the introduction of a Hopf bifurcation at a given location on the equilibrium manifold of a system of ODEs. The proof is constructive in the sense that it allows to calculate the controller needed for the introduction of the desired bifurcation point. A simple motivating example and an application to a model of an industrial continuous polymerization process are presented.

Keywords: bifurcation control, Hopf, saddle-node, stability

1. INTRODUCTION

Abed and Fu were among the first to discuss how to modify the stability properties of bifurcated solutions from a control point of view. They presented results for Hopf and stationary bifurcations (Abed and Fu, 1986; Abed and Fu, 1987). Abed and coworkers suggested the use of dynamic feedback and washout filters which allow to change stability properties of bifurcated solutions without affecting the equilibrium manifold (Abed *et al.*, 1996). Jørgensen and coworkers (Recke *et al.*, 2000) applied these ideas to models of industrial chemical engineering processes to render subcritical Hopf bifurcations supercritical.

To the authors' knowledge, results from bifurcation control theory have so far only been employed to render sharp bifurcations soft in engineering processes. This work goes a step further by showing that Hopf bifurcation points can be *introduced on purpose* to stabilize parts of the equilibrium manifold. The introduction of Hopf bifurcation points may at first sight seem counterintuitive, since Hopf bifurcations are accompanied by limit cycles and an exchange of stability between equilibria and limit cycles. These limit cycles and their effects are generally not acclaimed in chemical engineering processes. However, an exchange of stability need not necessarily imply that steady states of interest *lose* stability to limit cycles, but one may sometimes look at the same phenomenon as unstable steady states *gaining* stability due to an exchange of stability from a limit cycle. In this yet preliminary sense, the exchange of stability need not necessarily be a drawback, but it may be exploited for a stabilization of steady states. It is stressed that Hopf bifurcations are *not* introduced in order to run the process on a limit cycle afterwards, but in order to change the stability properties of steady states. Hopf bifurcations and the accompanying limit cycles are introduced as they have a desired effect on these steady states.

The paper is organized as follows. In Section 2, a motivating example for the deliberate introduction of a Hopf bifurcation is presented. Section 3 states the main theorem, the proof of which is deferred to the Appendix. In Section 4 the approach is applied to an industrial continuous polymerization process example. Section 5 discusses the results and gives a brief outlook.

2. MOTIVATING EXAMPLE

In this section the idea of bifurcation placement is applied to a one dimensional dynamical system with hysteresis. In this example a branch of steady state solutions, which is unstable without the proposed controller, is rendered stable.

The example considered is

$$\dot{x} = -x^3 + 2x + u. \tag{1}$$

A brief analytical analysis or a simple continuation bifurcation analysis reveals that the inner branch is unstable while the outer branches are stable. Figure 1a shows the bifurcation diagram in the bifurcation parameter u.

The dynamic variable z is introduced, and the model is augmented by the scalar dynamic equation

$$\dot{z} = x - dz. \tag{2}$$

Further feedback is introduced by setting

$$u = v + wy, \quad y = x - dz. \tag{3}$$

This yields

$$\begin{aligned} \dot{x} &= -x^3 + 2x + v + wy, \\ \dot{z} &= x - dz, \\ y &= x - dz \end{aligned} \tag{4}$$

where the controller parameters are set to

$$w = -2.36, \quad d = -0.56.$$

The values of w and d have been chosen to introduce a Hopf bifurcation point at $v = 0.5$ using the approach to be presented.

The bifurcation diagram of the controlled process (4) is shown in Figure 1b. The controlled model now has unstable outer branches of equilibria, while the formerly unstable steady states on the middle branch are stable for $v \in (0.5, 1.1)$. Note that equilibria in the range $v \in (-1.1, -0.5)$ have also been stabilized. It is stressed that formerly unstable steady states have been stabilized without affecting the form, size, shape, etc. of the manifold of steady state solutions.

A two-parameter continuation of the Hopf bifurcations in the parameters v and d, cf. Fig. 2b, reveals that values of d exist for which the Hopf bifurcations disappear in the bifurcation diagram. Figure 2a shows such a bifurcation diagram in which no Hopf bifurcations exist on the stable branch. Thus, the proposed approach allows us to go from the bifurcation diagram in Fig. 1a to the bifurcation diagram in Fig. 2a. Note, however, that this paper only discusses the introduction of Hopf bifurcations. The concurrent disappearing of Hopf bifurcations on the inner branch after a variation of controller parameters is the result of an analysis.

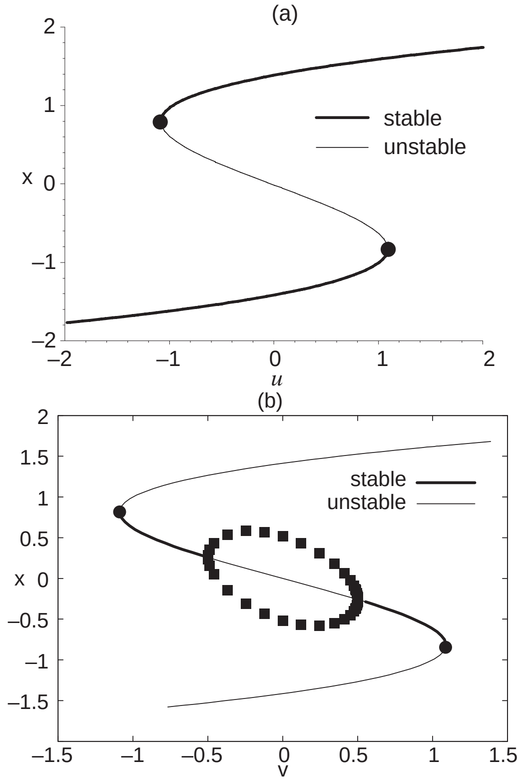

Fig. 1. (a) Bifurcation diagram of the uncontrolled system (1), (b) bifurcation diagram of the controlled system (4). The filled boxes represent the extrema of stable limit cycles after projection to the (x, v)-plane. The filled circles mark saddle-node bifurcations.

3. BIFURCATION PLACEMENT

3.1 *Notation*

Consider the ODE system

$$\dot{x} = f(x, u), \tag{5}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is assumed to be sufficiently smooth.

The system of equations

$$\begin{aligned} \dot{x} &= f(x, u) \\ \dot{z} &= x_k - dz, \end{aligned} \tag{6}$$

where

$$y = x_k - dz, \tag{7}$$

$y, d, z \in \mathbb{R}$, for some $k \in \{1, \ldots, n\}$ is referred to as the *augmented system*.

Choosing u to be a function of y as defined in equation (7), $u = U(y)$ with

$$U(y) = v + wy + \text{h.o.t.}, \tag{8}$$

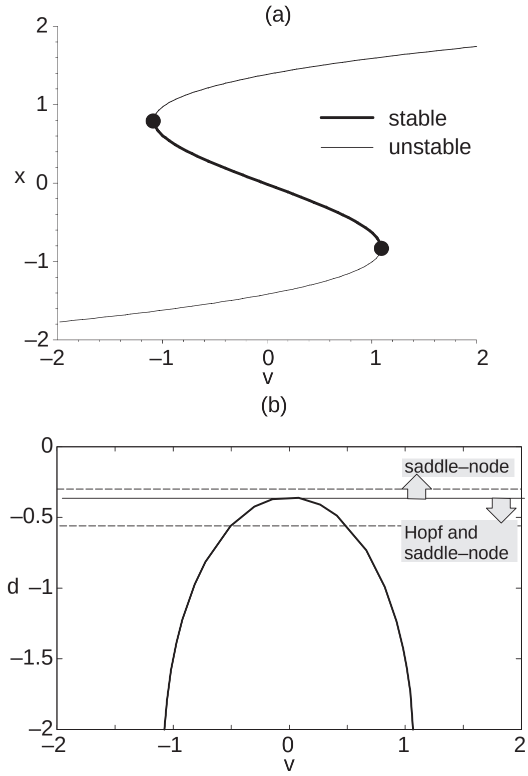

Fig. 2. (a) Bifurcation diagram for the augmented model where d was set to $d = -0.3$. (b) Two-parameter continuation of Hopf bifurcations. The dashed horizontal lines mark the values of d used in the bifurcation diagrams Fig. 1a ($d = -0.56$) and Fig. 2a ($d = -0.3$).

the augmented system has an equilibrium solution

$$(x,u,z) = (x_0, U(y), x_{k0}/d) \tag{9}$$

if and only if the system of ODEs (5) has an equilibrium solution

$$(x,u) = (x_0, v). \tag{10}$$

This can be shown directly by setting $y = 0$. Note that this relation between open loop and closed loop equilibria persists if higher order terms in (8) are allowed for.

Assume that (5) has an equilibrium at $(x,u) = (0,0)$. Linearizing system (5) at this equilibrium yields

$$\dot{x} = A(0,0)x + B(0,0)u + \text{h.o.t.} \tag{11}$$

with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$,

$$A_{\mu\nu}(x,u) = \frac{\partial f_\mu}{\partial x_\nu}(x,u), \tag{12}$$

$$B_{\mu\nu}(x,u) = \frac{\partial f_\mu}{\partial u_\nu}(x,u). \tag{13}$$

3.2 *Objective*

Loosely speaking, we intent to stabilize a branch of steady state solutions which is unstable due to a single eigenvalue λ_n in the right half complex plane. Branches of this type arise, for example, when the process model undergoes a saddle-node bifurcation while the input or bifurcation parameter u is varied. Given an unstable steady state on this branch, an additional dynamic degree of freedom is introduced, resulting in an additional eigenvalue. This eigenvalue is intentionally placed in the open *right* half of the complex plane. In the next step this eigenvalue is forced to become the complex conjugate of λ_n (rendering the previously real λ_n complex). Finally, the complex conjugate pair is forced to cross over to the open left half complex plane by varying the controller parameters d and w in equations (6)-(8) appropriately, resulting in a Hopf bifurcation of the augmented system. Since the other stable eigenvalues are not affected, this will stabilize the formerly unstable steady state. By continuity, the steady states in some neighborhood of the discussed equilibrium will be stable as well.

3.3 *Theorem: Controller for single unstable mode*

Assume that at the equilibrium $(x,u) = (0,0)$, matrix A from (12) has eigenvalues $\lambda_1,\ldots,\lambda_n$, where

$$\mathrm{Re}(\lambda_i) < 0, \quad i = 1,\ldots,n-1, \tag{14}$$

whereas for the remaining eigenvalue

$$\lambda_n \in \mathbb{R}, \quad \lambda_n > 0 \tag{15}$$

holds. Furthermore, let v_n denote the vector that spans the eigenspace which corresponds to λ_n.

Let k be an arbitrary but fixed value $1 \leq k \leq n$. If the vectors ∇f_k and v_n are not normal, i.e.

$$v_n^T \nabla f_k \neq 0, \tag{16}$$

and if further

$$v_n \in \mathrm{range}(B), \tag{17}$$

where B is defined as in (13), then $w \in \mathbb{R}^n$ and $d \in \mathbb{R}$ in the augmented system can be chosen such that the linearization of the augmented system and (8) has the eigenvalues

$$\lambda_1,\ldots,\lambda_{n-1} \tag{18}$$

of the system (5), and arbitrary eigenvalues

$$\tilde{\lambda}_n, \tilde{\lambda}_{n+1} \tag{19}$$

either real and $\lambda_n \neq \tilde{\lambda}_n + \tilde{\lambda}_{n+1}$, or complex conjugate and not zero.

The proof is deferred to the Appendix. The proof is constructive, i.e., d in the augmented model and w in the feedback (8) can be determined. In particular, $\tilde{\lambda}_n$ and $\tilde{\lambda}_{n+1}$ can be chosen to have negative real parts to result in a (locally exponentially) stable equilibrium of the augmented system, whereas system (5) is unstable at the corresponding equilibrium. Similarly, $\tilde{\lambda}_n$ and $\tilde{\lambda}_{n+1}$ can be chosen to be complex conjugate with real part zero. Note that this is necessary but not sufficient for a Hopf bifurcation to exist at the equilibrium of interest.

4. APPLICATION TO POLYMERIZATION PROCESS

Process models for continuous homopolymerization have been presented, analyzed and discussed in a series of papers by Ray and coworkers, see Ray and Villa (2000) and references therein. The process treated here is the solution free radical homopolymerization of vinyl acetate, which has been analyzed thoroughly with respect to its nonlinear dynamic behavior by Teymour and Ray (1992*b*). The model consists of four ODEs $k = 1,\ldots,4$ for the monomer, solvent and initiator concentration and the temperature T, and 18 algebraic equations which describe reaction rates, densities, heat capacities and the gel effect. The residence time θ, initiator concentration, feed temperature and cooling temperature are considered to be manipulated inputs. For details on the process model refer to Teymour and Ray (1992*a*; 1992*b*). A good summary of the model can be found in DeCicco (2000).

A typical bifurcation diagram of the process is shown in Fig. 3a. The diagram shows that between about $T = 60^o$C and $T = 100^o$C, no stable steady states exist, since the middle branch is unstable and the upper branch loses stability due to a Hopf bifurcation in this temperature range.

A Hopf bifurcation is introduced on the middle branch of steady states in Fig. 3 at a residence time of $\theta = 40$ min. Applying the proposed approach yields

$$w = (-21.13, 0.02197, 1.373, -24.31)^T, \quad (20)$$
$$d = -0.4205. \quad (21)$$

for $k = 4$. The bifurcation diagram for the controlled model is shown in Figure (3b).

As in the illustrative example in Sect. 2, the stabilization of equilibria on the inner branch can only be achieved at the cost of destabilizing the outer branches. This will in general be the case, since saddle-node bifurcations link a stable and an unstable branch of solutions in the bifurcation diagram.

The result is analyzed further by a two-parameter Hopf continuation in the bifurcation parameter v and the controller parameter d. The result of the two-parameter continuation is shown in Fig. 4b. The figure

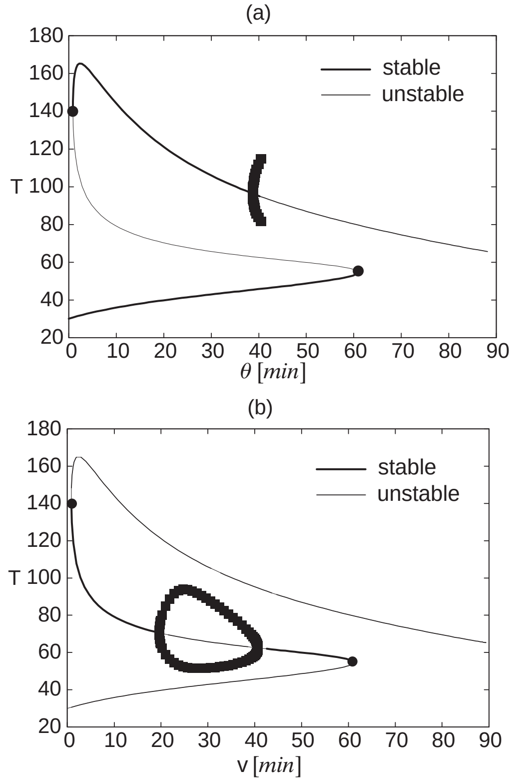

Fig. 3. Bifurcation diagram of the polymerization process, (a) uncontrolled process, (b) controlled process. The filled circles mark saddle-node bifurcations. The filled squares represent the amplitudes of the stable oscillations arising from the two limiting Hopf bifurcation points.

implies that there are values of the controller parameter d for which the Hopf bifurcations disappear on the inner branch. Such a bifurcation diagram is shown in Fig. 4a. Note that similar to the motivating example, the proposed approach allowed us to turn open-loop unstable branches into closed-loop stable branches of equilibria, cf. Figs. 3a and 4a.

In summary, the inner branch of steady state solutions of the process was stabilized. In particular this introduces stable equilibria into a range of residence times versus temperature in which no stable points of continuous operation exist for the open-loop process.

5. DISCUSSION

The present paper followed Abed *et al.* (1996) in introducing the augmented equation and feedback of the type (2), (3) in order to retain the equilibrium manifold of the open-loop process. This is crucial to our approach, since it allows to, roughly speaking, decouple equilibria location and stability boundary location. Abed (1995) discusses the relocation of Hopf bifurcation for extending stability margins in a thought experiment. The present work goes a step further by introducing Hopf bifurcations deliberately, in order to exploit the exchange of stability which accompanies

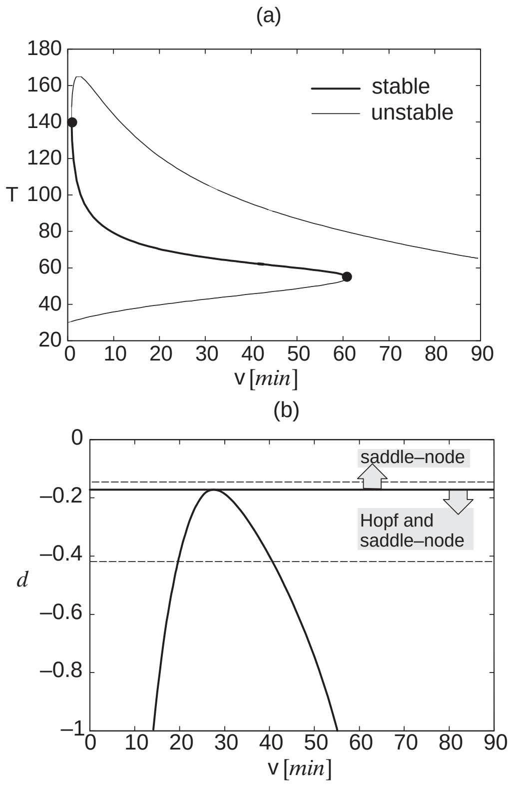

Fig. 4. (a) Bifurcation diagrams of the controlled polymerization process for $d = -0.15$, (b) result of the two-parameter continuation of Hopf points. The upper dashed line marks $d = -0.15$, the lower dashed line corresponds to $d = -0.42$ which is the value at which the bifurcation diagram 3b was obtained.

the Hopf bifurcation for rendering previously unstable equilibria stable.

Future work must aim at extending theorem 3.3 with respect to several aspects. Most importantly, the restrictive condition (17) has to be weakened and its relation to the controllability condition rank$[B\ AB \cdots A^{n-1}B] = n$ has to be investigated. Secondly, eigenvalues $\lambda_1, \ldots, \lambda_{n-1}$ need not remain at their open-loop locations in the complex plane, but it is sufficient to restrict them to the left half complex plane or to bound their real and imaginary parts to obtain the desired dynamic behavior of the process to first order. In this context, the relation of the approach presented to pole placement, eigenstructure assignment and bifurcation point relocation by static state feedback has to be discussed. Thirdly, it remains to be investigated whether theorem 3.3 can be extended to allow for a treatment of equilibria which are unstable due to more than one eigenvalue in the open right half of the complex plane.

It is stressed that the introduction of a complex conjugate pair of eigenvalues on the imaginary axis is only necessary for a Hopf bifurcation to occur. While this necessary condition can be fulfilled using linear feedback (3), third order terms in the feedback law can determine whether the Hopf bifurcation is sub- or supercritical (Abed and Fu, 1986). The introduction of a Hopf bifurcation as suggested here, and the control of its criticality type can therefore be done subsequently and independently. The influence of the second order of the feedback law on the nondegeneracy of the Hopf bifurcation remains to be investigated. Furthermore, it remains to be studied whether conditions for the introduction of a degenerate Hopf bifurcation related to the maxima in d in Figs. 2b and 4b can be stated. For the examples shown, this would have allowed to directly design the controller to turn diagram 1a into 2a, and 3a into 4b.

6. APPENDIX: PROOF OF 3.3

Change of coordinates of augmented system. Differentiating $y = x_k - dz$ and using $\dot{z} = y$, $\dot{x}_k = f_k(x,u)$ yields $\dot{y} = f_k(x,u) - dy$. System (6) can therefore be rewritten as

$$\dot{x} = f(x,u) \tag{22}$$

$$\dot{y} = f_k(x,u) - dy. \tag{23}$$

The Jacobian of this system is

$$\tilde{A} = \begin{pmatrix} A & b \\ A_{k\cdot} & b_k - d \end{pmatrix} \tag{24}$$

where $A_{ij} = \frac{\partial f_i}{\partial x_j}$, $A \in \mathbb{R}^{n\times n}$, $b = Bw \in \mathbb{R}^n$, b_k is the kth element of b, and $A_{k\cdot}$ denotes the kth row of A. After substitution of $u = U(y)$, $U(y) = v + wy + \text{h.o.t.}$, cf. (8), the linearization of system (6) in (x,z) is

$$\hat{A} := \begin{pmatrix} A + \hat{B} & -db \\ e_k^T & -d \end{pmatrix}, \tag{25}$$

where $\hat{B}$ has $b = Bw$ on its kth column and entries zero otherwise. Matrix (25) is similar to (24) with the transformation matrix

$$R^{(k)} := \left(\begin{array}{c|c} I^{(n)} & 0_n \\ \hline e_k^T & -d \end{array}\right). \tag{26}$$

Characteristic Polynomial of (25). Let $V \in \mathbb{R}^{n\times n}$, rank $V = n$, be the transformation that yields the upper Jordan normal form

$$V^{-1}AV = \Lambda = \begin{pmatrix} B_1 & & \\ & \ddots & \\ & & B_r \end{pmatrix}, \tag{27}$$

where B_i, $i = 1, \ldots, r \le n$ are Jordan block matrices and the last block matrix is $B_r = \lambda_n \in \mathbb{R}$. Setting

$$\tilde{V} := \begin{pmatrix} V & 0_n \\ 0_n^T & 1 \end{pmatrix} \tag{28}$$

implies that $\tilde{V}$ has full rank, furthermore

$$\tilde{V}^{-1}\tilde{A}\tilde{V} = \begin{pmatrix} \Lambda & b^\star \\ a^{\star T} & b_k - d \end{pmatrix} =: \tilde{\Lambda}, \tag{29}$$

with $b^\star = V^{-1}b \in \mathbb{R}^n$, $a^{\star T} = A_{k\cdot}V \in \mathbb{R}^{1\times n}$. Choosing $b^\star = (0,\dots,0,b_n^\star)^T$ with $b_n^\star \neq 0$, the characteristic polynomial P_c of $\tilde{A}$ as defined in (24) can be determined by expanding $\det(\tilde{\Lambda} - \lambda I^{(n+1)})$ along its last column,

$$\begin{aligned} P_c &= \det(\tilde{\Lambda} - \lambda I^{(n+1)}) \qquad (30)\\ &= \left((b_k - d - \lambda)(\lambda - \lambda_n) - b_n^\star a_n^\star\right)\prod_{i=1}^{n-1}(\lambda_i - \lambda)\,. \end{aligned}$$

Eigenvalues of (25). Comparing coefficients of the first factor of (30) and

$$Q := \left(\tilde{\lambda}_n - \lambda\right)\left(\tilde{\lambda}_{n-1} - \lambda\right) \qquad (31)$$

and substituting $b_k = b_n^\star V_{kn}$ yields the linear equations

$$\begin{pmatrix} V_{kn}\lambda_n - a_n^\star & -\lambda_n \\ -V_{kn} & 1 \end{pmatrix}\begin{pmatrix} b_n^\star \\ d \end{pmatrix} = \qquad (32)$$
$$\begin{pmatrix} \tilde{\lambda}_n\tilde{\lambda}_{n+1} \\ \lambda_n - \left(\tilde{\lambda}_n + \tilde{\lambda}_{n+1}\right) \end{pmatrix}.$$

Since the determinant of the matrix on the l.h.s. of (32) equals $-a_n^\star = -A_{k\cdot}v_n$ and since further by requirements on $\tilde{\lambda}_n$, $\tilde{\lambda}_{n+1}$ the r.h.s. of (32) is not equal to 0, $b_n^\star$ and d can be chosen such that P and Q have the same roots $\tilde{\lambda}_n$ and $\tilde{\lambda}_{n+1}$ if and only if

$$A_{k\cdot}v_n \neq 0 \qquad (33)$$

which is the non-normality condition (16).

Eigenvalues of (24). In summary, the characteristic polynomial P_c from (30) has the roots (18), (19). Therefore $\tilde{\Lambda}$ has the eigenvalues (18), (19). Since $\tilde{A}$, $\tilde{\Lambda}$ and the linearization of (6), equation (25), are similar, they have the desired eigenvalues (18), (19).

Note that $b_n^\star$ and d can be determined from linear system (32), b can then be determined by transforming it according to $b = Vb^\star$, $b^\star = (0,\dots,0,b_n^\star)^T$. Furthermore $b = Vb^\star$, $b^\star = (0,\dots,0,b_n^\star)^T$ implies $b = b_n^\star v_n$, and since $v_n \in \mathrm{range}(B)$, w can be determined from $b = Bw$.

7. REFERENCES

Abed, E. H. (1995). Bifurcation-theoretic issues in the control of voltage collapse. In: *Systems and Control Theory For Power Systems* (J. H. Chow, P. V. Kokotovic and R. J. Thomas, Eds.). Vol. 64. Chap. 1, pp. 1–21. IMA.

Abed, E. H. and J.-H. Fu (1986). Local feedback stabilization and bifurcation control, I. Hopf bifurcation. *Systems & Control Letters* **7**, 11–17.

Abed, E. H. and J.-H. Fu (1987). Local feedback stabilization and bifurcation control, II. stationary bifurcation. *Systems & Control Letters* **8**, 467–473.

Abed, E. H., H. O. Wang and A. Tesi (1996). Control of bifurcations and chaos. In: *The Control Handbook* (W. S. Levine, Ed.). Chap. 57.6, pp. 951–966. The electrical engineering handbook series. CRC Press, IEEE Press.

DeCicco, J. (2000). Simulation of an industrial polyvinyl acetate CSTR and semi-batch reactor utilizing MATLAB and SIMULINK: Version 1.0. http://www.chee.iit.edu/ jdecicco/vinylhtml/VRCTMAN1.html.

Golubitsky, M. and D.G. Schaeffer (1985). *Singularities and Groups in Bifurcation Theory, Volume I*. Springer-Verlag. New York.

Ray, W. H. and C. M. Villa (2000). Nonlinear dynamics found in polymerisation processes. *Chem. Eng. Sci.* **55**, 275–290.

Recke, Andersen and Jorgensen (2000). Bifurcation control of sample chemical reaction systems. In: *Proceedings of the international symposium on advanced control of chemical processes*. IFAC. Pisa, Italy. p. 398.

Teymour, F. and W. H. Ray (1992*a*). The dynamic behaviour of continuous polymerization reactors – V. Experimental investigation of limit–cycle behavior for vinyl acetate polymerization. *Chem. Eng. Sci.* **47**, 4121–4132.

Teymour, F. and W. H. Ray (1992*b*). The dynamic behaviour of continuous polymerization reactors – VI. Complex dynamics in full–scale reactors. *Chem. Eng. Sci.* **47**, 4133–4140.