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MINIMALITY, CANONICAL FORMS AND STORAGE OF FINITE-HORIZON DISCRETE-TIME COMPENSATORS

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Abstract: Motivated by the objective to further reduce the storage and computation time required by finite-horizon discrete-time optimal reduced-order LQG compensators, this paper introduces the *modified* reachability canonical form of a minimal finite-horizon discrete-time compensator and an algorithm to compute it. Next recursive algorithms for efficient storage and recovery of the compensator matrices in the modified reachability canonical form are presented. Finally the modified reachability grammian, which is associated to the modified reachability canonical form, is used to show and explain why in the finite-horizon case, if the initial compensator state is non-zero, minimality in general does not imply reachability of the compensator. *Copyright © 2002 IFAC*

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1. INTRODUCTION

Minimality of compensators plays a crucial role in solving optimal reduced-order LQG problems (Van Willigenburg and De Koning, 1999, 2000, 2002). One of the reasons is that given a certain input-output behavior of the compensator, due to the desire to reduce the compensator dimensions, minimal compensators are the interesting ones. In the finite-horizon case this forced us to investigate the minimality property of finite-horizon compensators (Van Willigenburg and De Koning, 2002).

It turned out that it is not straightforward to adapt minimality and the related properties reachability and observability (Kalman *et al.*, 1969; Kwakernaak and Sivan, 1972; Kailath 1980) to compensators which are only defined over a finite horizon. For a non-zero initial compensator state, to define minimality, the reachability grammian must be *modified*. As demonstrated in this paper, due to this modification, minimality no longer implies reachability of the compensator. Minimal discrete-time compensators have time-varying dimensions. These are caused by boundary conditions and possibly also by the time-varying dimension of the discrete-time system involved in the LQG problem. The latter occurs in digital LQG problems if the sampling is performed asynchronously (Van Willigenburg and De Koning, 2001)

Given a finite-horizon discrete-time optimal reduced-order LQG compensator, computed from one of the algorithms in Van Willigenburg and De Koning (1999), the use of canonical representations may be exploited to further reduce the computation time and storage needed by this minimal compensator. Therefore in this paper we introduce a *modified* reachability canonical form which is associated to the modified reachability grammian. Then, based on the modified reachability canonical form, recursive algorithms to efficiently store and recover the compensator matrices, forward in time, are presented. Our compensators and their notation comply with the ones in Van Willigenburg and De Koning (1999, 2002)

2. MINIMALITY, REACHABILITY AND OBSERVABILITY

Consider the following deterministic time-varying compensator defined over a finite horizon,

$$\hat{x}_{i+1} = F_i \hat{x}_i + K_i y_i,\ \hat{x}_i \in R^{n_i^c},$$
$$y_i \in R^{l_i},\ i = 0,1,..,N-1 \tag{1}$$

where n_i^c, l_i denote respectively the dimension of the compensator state $\hat{x}_i$ and the compensator input vector (system output vector) y_i at time i. Denote this compensator by $\left(\hat{x}_0, F^N, K^N\right)$ where $F^N = \{F_0, F_1, .., F_{N-1}\}$, $K^N = \{K_0, K_1, .., K_{N-1}\}$.

For the compensator (1) we have,

$$\hat{x}_i = F_{i,0}\hat{x}_0 + \sum_{k=0}^{i-1} F_{i,k+1} K_k y_k,\ i = 1,2,..,N, \tag{2}$$

where,

$$F_{l,m} = F_{l-1}F_{l-2}..F_m,\ l > m,\ F_{l,m} = I_{n_l^c},\ l = m. \tag{3}$$

Definition 1
$\left(0, F^N, K^N\right)$ is called reachable if for $\forall \hat{x} \in R^{n_i^c}$, $\forall i = 1,2,..,N$, $\exists \{y_0, .., y_{i-1}\}$ such that $\hat{x}_i = \hat{x}$ can be reached. •

Consider the following deterministic time-varying compensator defined over a finite horizon.

$$\hat{x}_{i+1} = F_i \hat{x}_i,\ \hat{x}_i \in R^{n_i^c},$$
$$u_i = L_i \hat{x}_i,\ u_i \in R^{m_i},\ i = 0,1,..,N-1. \tag{4}$$

where m_i denotes the dimension of the compensator ouput vector (system input vector) u_i at time i. Denote this compensator by $\left(F^N, L^N\right)$ where $L^N = \{L_0, L_1, .., L_{N-1}\}$.

Definition 2
$\left(F^N, L^N\right)$ is called observable if for $\forall i = 0,1,..,N-1$, $u_i = 0,\ u_{i+1} = 0,\ ..,\ u_{N-1} = 0$ implies $\hat{x}_i = 0$. •

Consider the compensator

$$\hat{x}_{i+1} = F_i \hat{x}_i + K_i y_i,\ \hat{x}_i \in R^{n_i^c},\ y_i \in R^{l_i}$$
$$u_i = L_i \hat{x}_i,\ u_i \in R^{m_i},\ i = 0,1,..,N-1 \tag{5}$$

Denote this compensator by $\left(\hat{x}_0, F^N, K^N, L^N\right)$ •

A non-zero initial condition $\hat{x}_0$ and the boundary condition $\hat{x}_N$ complicate the definition of a minimality property over a finite horizon. From equation (5) observe that $\hat{x}_N$ does not influence the input-output behavior of the compensator $\left(\hat{x}_0, F^N, K^N, L^N\right)$ so its minimal dimension $n_N^c = 0$. Since $\hat{x}_0$ is deterministic, at time $i = 0$ a basis transformation exists such that at most one compensator state variable of the transformed $\hat{x}_0$ is unequal to zero. Therefore $n_0^c = 1$ is the minimal dimension of $\hat{x}_0$ that preserves the input-output behavior.

Definition 3
$\left(0, F^N, K^N, L^N\right)$ is called minimal if $\left(0, F^N, K^N\right)$ is reachable and $\left(F^N, L^N\right)$ is observable and if in addition $n_0^c = 1$ and $n_N^c = 0$. •

The following analysis explains why definition 3 must be *generalised* for compensators with $\hat{x}_0 \neq 0$. Consider the sets $\left\{\hat{x}_i^r \,\middle|\, \hat{x}_i = \hat{x}_i^r\right\}$ of states that can be reached by the compensator $\left(\hat{x}_0, F^N, K^N\right)$ at each time $i = 1,2,..,N$ using $y_0, y_1, .., y_{i-1}$. These sets are determined by equation (2). The first term on the right in equation (2) is a constant term while the second term, through the variation of $y_0, y_1, .., y_{i-1}$ either represents the full compensator state-space at time i, i.e. $R^{n_i^c}$, or it represents a hyperplane with dimension $n_i^{cr} < n_i^c$ inside the state-space $R^{n_i^c}$. In the latter case, since the hyperplane represented by the second term contains the origin, a basis transformation exists such that n_i^{cr} unit vectors of the new basis span this hyperplane. If the first term is part of this hyperplane, which always is the case if $\hat{x}_0 = 0$, then it does not change the hyperplane. If not, the first term shifts the hyperplane away from the origin. Then, to represent the hyperplane, *one additional unit vector* i.e. $n_i^{cr} + 1$ state variables are needed.

Let $W_{0,i} \in R^{n_i^c \times n_i^c}$ denote the reachability grammian of the compensator $\left(0, F^N, K^N\right)$ associated to the state transition $\hat{x}_0 = 0$ to $\hat{x}_i = \hat{x}$, $i \in [1, N]$, i.e.,

$$W_{0,i} = \sum_{k=0}^{i-1} F_{i,k+1} K_k K_k^T F_{i,k+1}^T,\ i = 1,2,..,N. \tag{6}$$

Based on equation (2) define the *modified* reachability grammian,

$$W'_{0,i} = F_{i,0}\hat{x}_0\hat{x}_0^T F_{i,0}^T + \sum_{k=0}^{i-1} F_{i,k+1}K_k K_k^T F_{i,k+1}^T ,$$
$$i = 1,2,..,N . \qquad (7)$$

This grammian is associated with the compensator state transition from $\hat{x}_0$ to $\hat{x}_i = \hat{x}$, $i \in [1, N]$. Dual to the reachability grammian (6) consider the observability grammian $M_i^N \in R^{n_i^c \times n_i^c}$ given by,

$$M_{i,N} = \sum_{k=i}^{N-1} F_{k,i}^T L_k^T L_k F_{k,i} ,\ i = 0,1,..,N-1 . \qquad (8)$$

Then from definitions 1-3 and equations (2), (6), (7), (8) the following two lemmas are immediate.

Lemma 1

$(0, F^N, K^N)$ reachable $\Leftrightarrow W_{0,i}$ full rank $\forall i \in [1, N]$.

Dually (F^N, L^N) observable $\Leftrightarrow M_{i,N}$ full rank $\forall i \in [0, N-1]$ ●

Lemma 2

1) The first term on the right in equation (2) lies inside the hyperplane with dimension $n_i^{cr} < n_i^c$ determined by the second term on the right in equation (2) $\Rightarrow rank(W'_{0,i}) = n_i^{cr} < n_i^c$, $i \in [1, N]$.
2) The first term on the right in equation (2) lies outside the hyperplane with dimension $n_i^{cr} < n_i^c$ determined by the second term on the right in equation (2) $\Rightarrow rank(W'_{0,i}) = n_i^{cr} + 1 \le n_i^c$, $i \in [1, N]$.
3) The second term on the right in equation (2) spans the full state-space $R^{n_i^c}$ $\Rightarrow rank(W'_{0,i}) = n_i^c$, $i \in [1, N]$. ●

From lemma 2 and the analysis below definition 3 $rank(W'_{0,i})$ represents precisely the minimum number of compensator state variables needed to describe the reachable space at time $i \in [1, N]$.

Definition 4

$(\hat{x}_0, F^N, K^N, L^N)$ is called minimal if $\forall i \in [0, N-1]$, $M_{i,N}$ full rank and if $\forall i \in [1, N]$, $W'_{0,i}$ full rank and if in addition $n_o^c = 1$ and $n_N^c = 0$ ●

It is well known that the reachability and observability grammian (6), (8) can be given in recursive form as follows,

$$W_{0,i+1} = F_i W_{0,i} F_i^T + K_i K_i^T ,\ i = 0,1,..,N-1,$$
$$W_{0,0} = 0 \in R^{n_0^c \times n_0^c} \qquad (9)$$

$$M_{i,N} = F_i^T M_{i+1,N} F_i + L_i^T L_i ,\ i = 0,1,..,N-1,$$
$$M_{N,N} = 0 \in R^{n_N^c \times n_N^c} . \qquad (10)$$

Similar to (9) the recursive form of (7) is given by,

$$W'_{0,i+1} = F_i W'_{0,i} F_i^T + K_i K_i^T ,\ i = 0,1,..,N-1,$$
$$W'_{0,0} = \hat{x}_0\hat{x}_0^T \in R^{n_0^c \times n_0^c} . \qquad (11)$$

Equations (9) and (11) are identical except for the initial value which from 0 is changed into $\hat{x}_0\hat{x}_0^T$. This constitutes the generalization. Introduce,

$$r_i^c = \min\left(rank(W'_{0,i}), rank(M_{i,N})\right),$$
$$i = 0,1,..,N . \qquad (12)$$

Then from equations (10), (11), (12),

$$\hat{x}_0 \ne 0 \Rightarrow r_0^c = 1,\ \hat{x}_0 = 0 \Rightarrow r_0^c = 0,\ r_N^c = 0 ,(13.1)$$

$$r_i^c - m_i \le r_{i+1}^c \le r_i^c + l_i ,\quad i \in [0, N-1] . \qquad (13.2)$$

From equation (13) and definition 4 the dimensions of a minimal compensator satisfy,

$$n_i^c = r_i^c ,\ i = 1,2,..,N-1 ,\ n_0^c = 1 ,\ n_N^c = 0 . \quad (14)$$

On the other hand if $(\hat{x}_0, F^N, K^N, L^N)$ has dimensions n_i^c satisfying (14) then one can always choose the compensator such that it is minimal.

Lemma 3

For the compensator $(\hat{x}_0, F^N, K^N, L^N)$ let k, $0 \le k \le N$ denote the first time instant for which $rank(W'_{0,k}) = rank(W_{0,k})$ holds.

Then $rank(W'_{0,i}) = rank(W_{0,i})$, $k \le i \le N$ ●

Proof

Follows directly from equations (9) and (11) ●

Before time k there is a significant difference between the reachability grammian $W_{0,i}$ and the *modified* reachability grammian $W'_{0,i}$. At and after time k there is no significant difference.

Theorem 1

For a finite-horizon compensator minimality only implies reachability if $\hat{x}_0 = 0$ or $k = 1$ in lemma 3 ●

Proof

From definitions 3, 4 and the analysis after definition 3 minimality implies reachability only if $\hat{x}_0 = 0$ or $k = 0 \vee k = 1$ in lemma 3. But $k = 0 \Leftrightarrow \hat{x}_0 = 0$ ●

Remark 1

Equation (13.2) and (14) imply that the *change* of the dimension of the state of a minimal compensator, from one discrete time instant to the next, is bounded from above and below by respectively the number of outputs and inputs of the compensator. Equation (13) and (14) imply that, at the initial and final time, the dimension of the compensator state of a minimal compensator drops in one or several time steps to one and zero respectively ●

3. RECURSIVE COMPUTATION OF THE MODIFIED REACHABILITY CANONICAL FORM

Since the compensator matrices of finite horizon optimal full and reduced-order LQG compensators are time-varying, to store them a serious amount of computer memory is required. Therefore saving storage may be especially important. In the infinite horizon case the reachability and observability canonical forms of a minimal compensator, in general, require less storage than other representations of minimal compensators. Therefore an interesting issue is the computation of similar canonical forms of minimal finite-horizon compensator. Due to the time-varying dimensions of a finite-horizon compensator, and due to the modification of the reachability grammian, needed to establish minimality, this issue is not straightforward.

In this section, the so called scheme II, presented by Kailath (1980, page 427) which computes the reachability canonical form for minimal linear time-invariant systems (compensators) defined over an infinite horizon is modified to apply to finite-horizon time-varying discrete-time compensators with time-varying dimensions. The canonical form that results will be called the *modified* reachability canonical form for reasons to be explained.

Consider basis transformations of the compensator state-space,

$$\hat{x}_i' = P_i^{-1}\hat{x}_i\,,\; i = 0,1,\ldots,N\,, \qquad (15)$$

where $\hat{x}_i',\; i = 0,1,\ldots,N$ represents the compensator state in the new basis. $P_i,\; i = 0,1,\ldots,N$ are square matrices of dimension n_i^c the columns of which contain the basis vectors of the new basis at each time $i = 0,1,\ldots,N$ (Kailath 1980, page 334). Denote the n_i^c columns of P_i by $p_i^k,\; k = 1,\ldots,n_i^c$. The compensator matrices represented in the new basis are,

$$\hat{x}_0' = P_0^{-1}\hat{x}_0,\;\; F_i' = P_{i+1}^{-1}F_iP_i,\; L_i' = L_iP_i,$$

$$K_i' = P_{i+1}^{-1}K_i\,,\; i = 0,1,\ldots,N \qquad (16)$$

Note that since the final compensator state has dimension zero P_N is irrelevant. Consider the following recursive algorithm that determines the matrices $P_i,\; i = 0,1,\ldots,N-1$.

Algorithm 1

1) Initialisation : $i := 0,\; R_0 := \hat{x}_0$
2) While $i \le N-1$
   search R_i for n_i^c independent columns $p_i^j,\; j = 1,..,n_i^c$
3) $P_i := \left[\, p_i^1 \;\; p_i^2 \;\; .. \;\; p_i^{n_i^c} \,\right] \in R^{n_i^c \times n_i^c}$
4) $R_{i+1} := \left[K_i \;\; F_iP_i\right] \in R^{n_{i+1}^c \times \left(n_i^c + l_i\right)}$
5) $i := i+1$ ●

Theorem 2
When algorithm 1 is applied to a minimal compensator $\left(\hat{x}_0, F^N, K^N, L^N\right)$ step 3 of algorithm 1 produces matrices $P_i,\; i = 0,1,\ldots,N-1$ that transform this compensator into $\left(\hat{x}_0', F'^N, K'^N, L'^N\right)$ as determined by equation (16). The latter compensator is represented in a canonical form that we will call the *modified* reachability canonical form ●

Proof
From Kailath (1980, page 334) the j[th] column of $F_i' = P_{i+1}^{-1}F_iP_i$ may be interpreted as the coefficients of the representation of the column vector $F_ip_i^j$ in the basis $p_{i+1}^k,\; k = 1,..,n_{i+1}^c$. Similarly the j[th] column of $K_i' = P_{i+1}^{-1}K_i$ may be interpreted as the coefficients of the representation of the j[th] column of K_i in the basis $p_{i+1}^k,\; k = 1,\ldots,n_{i+1}^c$. By selecting $p_{i+1}^k,\; k = 1,\ldots,n_{i+1}^c$ partly equal to the columns of K_i and the column vectors $F_ip_i^j,\; j = 1,..,n_i^c$, as algorithm 1 does, the associated columns of $K_i',\; F_i'$ will be standard basis vectors (unit column vectors) and thus result in a canonical form. Observe that these results apply regardless of whether the dimension of the compensator state and input, i.e. $n_i^c,\; l_i$, vary with the discrete time i. Due to the minimality of the compensator $W_{0,i}' > 0$. Then from equation (11) and the recursive definition of R_i in algorithm 1 observe that there always are precisely n_i^c independent columns present in R_i. Finally observe that if $\hat{x}_0 = 0$ algorithm 1 generates the reachability canonical form. From theorem 1 a non-zero initial condition may destroy reachability and consequently we refer to our canonical form as the *modified* reachability canonical form. ●

Remark 2
The observability canonical form is dual to the reachability canonical form. Thus the algorithm to obtain the observability canonical form is the dual of our algorithm 1 when we set $\hat{x}_0 = 0$ ●

Remark 3
Although the usual terminology is to speak of *the* (modified) reachability and observability canonical form these forms are by no means unique (Kailath 1980). Firstly they depend on the order in which R_i is searched for independent columns. Secondly they depend on the order in which the independent columns are put into P_i. It would therefore be better to speak of *a* reachability and observability canonical form. Although they are not unique the computer storage they all require (save) is roughly the same ●

Theorem 3

Let r_i^k, $k = 1,..,n_{i-1}^c + l_{i-1}$ denote the columns of R_i. Assume that step 3 of algorithm 1 searches the columns of R_i from left to right. Also assume that the order in which the independent columns p_i^j, $j = 1,..,n_i^c$ are found is identical to the order in which they are put into P_i. Then the modified reachability canonical form generated by algorithm 1 can be represented as follows,

$$\hat{x}_0' = 0 \vee \hat{x}_0' = 1,$$

$$\begin{bmatrix} K_i' & F_i' \end{bmatrix} = \begin{bmatrix} v_i^1 & v_i^2 & .. & v_i^{l_i+n_i^c} \end{bmatrix}, i = 0,1,..,N-1 \quad (17)$$

If $p_i^j = r_i^k$ i.e. the k[th] column of R_i is the j[th] independent column found in R_i then v_i^k is a standard basis vector (unit column vector) with 1 appearing at the k[th] position. If not then v_i^k is a column vector of which only the first *n* elements may be unequal to zero where *n* is the number of independent columns found so far ●

Proof

Follows from the first part of the proof of theorem 2 ●

4. ALGORITHMS FOR EFFICIENT STORAGE AND RECOVERY

Algorithms that exploit the modified reachability canonical structure to reduce, as much as possible, the computer memory needed for storage of the compensator, are presented in this section.

From the last part of theorem 3 note that, to recover the modified reachability canonical form from only the possibly non-zero and non-one elements of the compensator matrices, during the search for independent columns it is important to store the indices of these independent columns. Alternatively one might store the indices of the dependent columns. Due to the minimality of the compensator there are at each time $i = 1,..,N-1$ precisely n_i^c independent and $n_{i-1}^c + l_{i-1} - n_i^c$ dependent columns in R_i . Since usually the dimension of consecutive states are equal and larger then the number of compensator inputs the latter are fewer numbers. Denote these numbers by q_i^j, $j = 1,..,n_{i-1}^c + l_{i-1} - n_i^c$.

Algorithm 2

Encode a compensator $\left(\hat{x}_0', F'^N, K'^N, L'^N\right)$, obtained from algorithm 1 and theorem 3, into the following array of numbers:

$$N, \hat{x}_0', m_0, l_0, n_1^c, Q_1, C_0,$$
$$m_1, l_1, n_2^c, Q_2, C_1,$$
$$.....$$
$$m_{N-2}, l_{N-2}, n_{N-1}^c, Q_{N-1}, C_{N-2},$$
$$m_{N-1}, C_{N-1}$$

where $Q_i = q_i^1, q_i^2, .., q_i^{n_{i-1}^c + l_{i-1} - n_i^c}$, $i = 1,2,..,N-1$. Furthermore C_i, $i = 0,1,..,N-1$ are arrays of numbers containing the possibly non-zero elements of the columns of the compensator matrices at time i that are not standard basis vectors (unit column vectors). These arrays are generated recursively as follows :

$C_i = L_i'(:)$, $n = 1$

for $k = 2,3,..,n_i^c + l_i$

  if $k \in \left\{q_{i+1}^j,\ j = 1,..,n_i^c + l_i - n_{i+1}^c\right\}$ then

  $C_i := \left[C_i \ v_i^k(1:n)\right]$

  else $n = n + 1$

Here $L_i'(:)$ denotes an array of numbers containing the elements of the column vector which stacks all the columns of the compensator matrix L_i'. Furthermore $v_i^k(1:n)$ denotes an array of numbers equal to the first n elements of the column vector v_i^k as defined by theorem 3 in the previous section.

Theorem 4

By working the array of numbers generated by algorithm 2 from left to right the compensator matrices of $\left(\hat{x}_0', F'^N, K'^N, L'^N\right)$ can be recovered, forward in time. ●

Proof / Algorithm 3

Note that since the compensator is minimal $n_0^c = 1$ and therefore need not be present in the array. Then the first number in the array equals N and the second the scalar $\hat{x}_0' \in R^{1\times1}$. From the next three numbers m_0, l_0, n_1^c in the array the dimensions of $F_0' \in R^{n_1^c \times n_0^c}, K_0' \in R^{n_1^c \times l_0}, L_0' \in R^{m_0 \times n_0^c}$ follow. Next from the array we obtain $Q_1 = q_1^1, q_1^2, .., q_1^{n_0^c + l_0 - n_1^c}$ and $L_0'(:)$, the stacked columns of L_0', that appear first in C_0, as can be seen from algorithm 2. From theorem 3 the indices $Q_1 = q_1^1, q_1^2, .., q_1^{n_0^c + l_0 - n_1^c}$ determine respectively which columns of $\begin{bmatrix} K_0' & F_0' \end{bmatrix}$ are not standard basis vectors and also the number of possibly non-zero elements of these columns. These possibly non-zero elements are all stacked in C_0 after $L_0'(:)$, as can be seen from algorithm 2. As a result from Q_1, C_0 we can recover the matrices F_0', K_0', L_0' using the known structure of $\begin{bmatrix} K_0' & F_0' \end{bmatrix}$ mentioned in theorem 3. Similar arguments apply to $m_i, l_i, n_{i+1}^c, Q_{i+1}, C_i$ from which F_i', K_i', L_i', $i = 1,2,..,N-2$ can be recovered.

Note that since the compensator is minimal $n_N^c = 0$. Due to this F'_{N-1}, K'_{N-1} are empty and from $m_{N-1}, C_{N-1} = L'_{N-1}(:)$ we recover L'_{N-1} ●

Example 1: Encoding of a minimal compensator in the modified reachability canonical form

$N = 4,\ n_i^c = 1,3,3,2,0,\ l_i = 2,2,2,2,0,$

$m_i = 2,2,2,2,0,\ i = 0,1,..,N$

$$x'_0 = 1,\ F'_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ K'_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},\ L'_0 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$F'_1 = \begin{bmatrix} 1 & 0 & 6 \\ 2 & 0 & 0 \\ 0 & 1 & 5 \end{bmatrix},\ K'_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},\ L'_1 = \begin{bmatrix} 7 & 8 & 9 \\ 6 & 5 & 3 \end{bmatrix},$$

$$F'_2 = \begin{bmatrix} 2 & 3 & 5 \\ -1 & 4 & -3 \end{bmatrix},\ K'_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$L'_2 = \begin{bmatrix} 7 & -6 & 5 \\ 3 & -1 & 2 \end{bmatrix},$$

$$F'_3 = [\ \ ],\ K'_3 = [\ \ ],\ L'_3 = \begin{bmatrix} 2 & -4 \\ 3 & -6 \end{bmatrix}$$

The associated array generated by algorithm 2 :

$N = 4,\ \hat{x}'_0 = 1,\ \ m_0 = 2,\ \ l_0 = 2,\ \ n_1^c = 2,$

$Q_1 = [],\ \ C_0 = [2,\ 3]$

$m_1 = 2,\ \ l_1 = 2, n_2^c = 3,$

$Q_2 = [3, 5],\ \ C_1 = [7, 6, 8, 5, 9, 3, 1, 2, 6, 0, 5],$

$m_2 = 2,\ \ l_2 = 2,\ \ n_3^c = 2,\ Q_3 = [3, 4, 5],$

$C_2 = [7, 3, -6, -1, 5, 2, 2, -1, 3, 4, 5, -3],$

$m_3 = 2,\ C_3 = [2, 3, -4 , -6]$ ●

Theorem 5

Compared to ordinary storage of a minimal compensator (i.e. storing all the dimensions and all the elements of the compensator matrices) algorithm 2 reduces the number of numbers needed for storage by at least

$$\sum_{i=1}^{N-1} n_i^c n_i^c - \left(n_{i-1}^c + l_{i-1} - n_i^c\right)$$ ●

Proof

Ordinary storage, excluding $n_i^c,\ l_i,\ m_i,\ i = 0,1,..,N$ requires the storage of

$$\underset{N,\hat{x}'_0}{2} + \underset{L'_0}{m_0} + \sum_{i=1}^{N-1} n_i^c \underbrace{\left(n_{i-1}^c + l_{i-1} + m_i\right)}_{F'_{i-1}, K'_{i-1}, L'_i}$$

numbers. The storage required by algorithm 2 is the largest if the non-standard basis vectors in equation (17) are the last columns of $\begin{bmatrix} K'_i & F'_i \end{bmatrix}$. Excluding $n_i^c,\ l_i,\ m_i,\ i = 0,1,..,N$, in this case

$$\underset{N,\hat{x}'_0}{2} + \underset{L'_0}{m_0} + \sum_{i=1}^{N-1} n_i^c \underbrace{\left(n_{i-1}^c + l_{i-1} - n_i^c + m_i\right)}_{F'_{i-1}, K'_{i-1}, L'_i} + \underbrace{\left(n_{i-1}^c + l_{i-1} - n_i^c\right)}_{Q_i}$$

numbers are needed for storage ●

5. CONCLUSIONS

Having computed a finite-horizon discrete-time optimal reduced-order LQG compensator, with one of the algorithms presented by Van Willigenburg and De Koning (1999), the results of this paper enable further reduction of the storage and computation time required by this minimal compensator. Especially when the dimensions of the compensator state are high, from theorem 5, the modified reachability canonical form and the associated algorithms 1-3 for efficient storage and recovery, developed in this paper, significantly reduce the computer memory needed for storage. Besides having time-varying dimensions theorem 1 revealed that minimal finite-horizon discrete-time compensators are usually not reachable if the initial compensator state is non-zero.

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