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FEEDBACK INVARIANCE AND INJECTION INVARIANCE FOR SYSTEMS OVER RINGS

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Abstract: A straightforward extension to systems over rings of the geometric approach to many control problems is not possible, since the equivalence between invariance properties and existence of feedbacks or output injections no longer holds. In this paper new, geometric, algorithmic characterizations of $(A + BF)$-invariance and of $(A+GC)$-invariance, suitable to symbolic computer algebraic computations, are given for systems over Noetherian rings. *Copyright* ©*2001* IFAC

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1. INTRODUCTION

The geometric approach to linear systems provides solutions to many control problems, by exploiting the equivalence between (A, B)-invariance and feedback invariance and between (A, C)-invariance and invariance by output injection, see (Wonham, 1985), (Basile and Marro, 1992). A crucial point in applying the geometric approach to the solution of specific control problems is the possibility to check practically the solvability conditions and to construct the solution. An extension of the geometric theory to systems over ring (which are useful models to study several interesting classes of systems such as delay-differential systems and parameter depending systems) has been investigated by many authors, (Conte and Perdon, 1998), (Conte and Perdon, 1995*b*),(Conte and Perdon, 1995*a*), (Hautus, 1984), (Inaba and Munaka, 1988), (Sename and Lafay, 1997). In particular, since many classical geometric algorithms no longer work new algorithms have been proposed to compute key geometric objects, such as $\mathcal{V}^*$ and $\mathcal{R}^*$, for system over Principal Ideal Domains (see (Assan and Perdon, 1998), (Assan and Perdon, 1999*b*)). In this paper, extending the results of (Assan and Perdon, 1999*b*) and (Assan and Perdon, 1999*a*) we present, for systems over a Noetherian ring, new algorithmic procedures that allow to check if a submodule is feedback invariant or injection invariant and, in case of positive answer, to compute the corresponding feedback or injection map. These algorithms can be practically implemented by means of symbolic computer algebraic software, such as MapleV, Mathematica and CoCoA (Capani and Robbiano, n.d.).

2. PRELIMINARIES

Let R denote a commutative ring with identity and without zero divisors. Let Σ be a system defined by

$$\begin{cases} x(t+1) = Ax(t) + Bu(t) \\ y(t) \quad\;\; = Cx(t) \end{cases} \tag{1}$$

where $x(\cdot)$ belongs to the free state module $X = R^n$, $u(\cdot)$ belongs to the free input module $U = R^m$, $y(\cdot)$ belongs to the free output module $Y =$

R^p, and A, B, C are matrices of suitable dimensions with entries in R. Let us briefly recall here a number of results and definitions that will be used in extending classical results of the geometric approach to systems over a ring.

Definition 1. Let Σ be a system defined by (1) over a ring R. A submodule $\mathcal{V} \subset R^n$ is said:

i) (A,B)-invariant or controlled invariant if $A\mathcal{V} \subset \mathcal{V} + ImB$;
ii) (A+BF)-invariant or feedback invariant if there exists an R-linear state feedback F:$X \longrightarrow U$ such that $(A+BF)\mathcal{V} \subset \mathcal{V}$.

Any feedback F for which ii) holds is called a *friend* of $\mathcal{V}$.

Definition 2. Let Σ be a system defined by (1) over a ring R. A submodule $\mathcal{S} \subset R^n$ is said:

i) (C,A)-invariant or conditioned invariant if $A(\mathcal{S} \cap KerC) \subset \mathcal{S}$.
ii) (A+GC)-invariant or injection invariant if there exists an R-linear output injection G:$Y \longrightarrow X$ such that $(A+GC)\mathcal{S} \subset \mathcal{S}$.

Over a field, the notions of (A,B)-invariance and feedback invariance are equivalent, see (Wonham, 1985), as well as those of (C,A)-invariance and injection invariance, see (Basile and Marro, 1992). A geometric characterization of the $(A+BF)$-invariance property over a PID has been presented in (Assan and Perdon, 1999*b*).
Let $\mathcal{V}$ be a submodule of $X = R^n$, where R is a PID, with $dim(\mathcal{V}) = k$ and let V be a basis-matrix of $\mathcal{V}$. Denote by $S_V = \begin{pmatrix} diag(\alpha_1, \ldots, \alpha_k) \\ 0 \end{pmatrix}$, where α_i divides α_{i+1} for all $i = 1, 2, \ldots, k-1$, the Smith form of the matrix V. Then, there exist unimodular matrices P, Q such that $S_V = PVQ$.

Definition 3. Denoting by $(.)_i$ the i^{th} column of a matrix (.), we will say that the vectors $\tilde{v}_i = V(Q)_i$, $i = 1, \ldots, k$, form an "ordered basis" for the submodule $\mathcal{V}$ with respect to the invariant factors $\alpha_1, \alpha_2, \ldots, \alpha_k$ of any basis-matrix of $\mathcal{V}$.

Proposition 1. (Assan and Perdon, 1999*b*) Let $\mathcal{V}$ be a submodule of R^n, where R is a PID, and let $\{\tilde{v}_i, i = 1, \ldots, k\}$ be an ordered basis for $\mathcal{V}$. Then $\mathcal{V}$ is an (A+BF)-invariant submodule if and only if the vectors $\tilde{v}_i$ for $i = 1, 2, \ldots, k$ are such that $A\tilde{v}_i \subset \mathcal{V} + Im\ \alpha_i B$.

The characterization of feedback invariance given in the above Proposition seems not very transparent, however it can be easily checked by an algorithm and provides directly a "friend of $\mathcal{V}$", as the following example shows.

In fact, let us suppose that each v_i verifies (1). Then, there exist matrices L and K such that for all i $Av_i = V(L)_i + B(M)_i\alpha_i$. It follows that $AVQ = VL + BMdiag(\alpha_1, \alpha_2, \ldots, \alpha_k)$. Since $PVQ = S_V$, writing $P = \begin{pmatrix} P_1 & P_2 \end{pmatrix}^t$, we have $AVQ = VL + BMP_1VQ$, and $F = \begin{pmatrix} -M & G \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$ is a friend of $\mathcal{V}$ for any G in $R^{m \times n-k}$. Practically, the feedback invariance of $\mathcal{V}$ can be checked solving equations $[V\ B] \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}^t = AV$ with respect to matrices X_1 and X_2 If X_2 can be written as $X_2 = \tilde{X}_2 S_V$, we define $L := XZ_1$ and $M := \tilde{X}_2$. Efficient algorithms based on Gröbner basis theory are available for solving equations of this kind over rings of polynomials in several indeterminate over a field.

Example 1. Let us consider the system Σ defined over $\mathcal{R}[\nabla]$, the ring of polynomials in one variable with real coefficients, by (1) with

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \ and\ B = \begin{pmatrix} 1 & 0 \\ \nabla & 0 \\ \nabla & 0 \\ 0 & \nabla^2 \end{pmatrix}.$$

Denote by V and S_V respectively a matrix whose columns span the submodule $\mathcal{V}$ and its Smith form,

$$V = \begin{pmatrix} \nabla & 0 \\ \nabla^2 & 0 \\ 0 & 0 \\ 0 & \nabla \end{pmatrix} \quad S_V = PVQ = \begin{pmatrix} \nabla & 0 \\ 0 & \nabla \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The invariant factors of $\mathcal{V}$ are $\alpha_1 = \alpha_2 = \nabla$. Denoting by $\tilde{v}_1, \tilde{v}_2$ the columns of V, which are already an ordered basis-matrix for $\mathcal{V}$, one can easily verify that $\tilde{v}_i$, for $i = 1, 2$ satisfies the relation $A\tilde{v}_i \subset \mathcal{V} + Im\ \alpha_i\ B$. $\mathcal{V}$ is therefore an $(A+BF)$-invariant submodule. All feedbacks $F \in \mathcal{F}(\mathcal{V})$ can be written as

$$F = \begin{pmatrix} -(1 + g_2\nabla) & g_2 & g_1 & 0 \\ -g_4\nabla & g_4 & g_3 & 0 \end{pmatrix}.$$

All the computations in this example have been performed using the software MapleV.

3. GEOMETRIC CHARACTERIZATION OF INJECTION INVARIANCE OVER A PID

In this section, we will characterize the property of injection invariance following the same lines of Proposition 1. Let R be a PID, $\mathcal{S}$ a submodule of $R^n$of dimension k, $\mathcal{S}_c = Ker\ C \bigcap \mathcal{S}$ and denote by $\mathcal{S}_1$ its direct summand, i.e. $\mathcal{S} = \mathcal{S}_1 \bigoplus \mathcal{S}_c$. Let S_1 and S be, respectively, a basis-matrix for $\mathcal{S}$ and

$\mathcal{S}_1$. Then we have $S = [S_1|S_c]$ and $CS = [CS_1|0]$. Assume $dim(\mathcal{S}_1) = k_1$ and compute the Smith form of CS,

$$S_{CS} = PCSQ = \begin{pmatrix} diag(\beta_1, \beta_2, .., \beta_{k_1}) & 0 \\ 0 & 0 \end{pmatrix}, (2)$$

where β_{i+1} divides β_i for all $i = 1, 2, \ldots, k_1 - 1$, $\beta_i = 0$ for $i = k_1+1, .., k$, $P \in R^{p \times p}$ and $Q \in R^{k \times k}$ are unimodular matrices. Remark that $k_1 \leq p$ and $k_1 \leq k$. Let us call the columns of SQ, denoted by $\{\tilde{s}_i,\ i = 1, \ldots, k\}$, an *ordered basis* for $\mathcal{S}$ with respect to β_i, $i = 1, \ldots, k$. The following technical result will be used in the sequel.

Lemma 1. Given (2), there exists a $k \times p$ matrix P_1 such that $P_1 CSQ = diag(\beta_1, \beta_2, .., \beta_k)$ and a $p \times k$ matrix $\tilde{P}_1$ such that $[CSQ = \tilde{P}_1\, diag(\beta_1, \beta_2, .., \beta_k)$.

Proof Being P unimodular, there exists a $\tilde{P}$ such that $P\tilde{P} = I_p$. When $k \leq p$ write $P = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$ and $\tilde{P} = \begin{pmatrix} \tilde{P}_1 & \tilde{P}_2 \end{pmatrix}$, where $P_1 \in R^{k \times p}$, $P_2 \in R^{(p-k_1) \times p}$, $\tilde{P}_1 \in R^{p \times k}$ and $\tilde{P}_2 \in R^{p \times (p-k_1)}$. From $\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} \begin{pmatrix} \tilde{P}_1 & \tilde{P}_2 \end{pmatrix} = I_p$, we have $P_1 CSQ = diag(\beta_1, \beta_2, \ldots, \beta_k)$ and $CSQ = \tilde{P}_1 diag(\beta_1, \beta_2, \ldots, \beta_k)$. When $k > p$, write $P_1 = \begin{pmatrix} P \\ 0_{(k-p) \times p} \end{pmatrix}$ and $\tilde{P}_1 = \begin{pmatrix} \tilde{P} & 0_{p \times (k-p)} \end{pmatrix}$. Then the result follows.
We can now characterize the injection invariance property.

Proposition 2. Let $\mathcal{S}$ be a submodule of R^n, where R is a PID, and let $S_{CS} = PCSQ = diag(\beta_1, \beta_2, \ldots, \beta_k)$, with $\beta_i = 0$ for $i = k_1+1, .., k$ be the Smith form of CS and $\{\tilde{s}_i,\ i = 1, \ldots, k\}$ be an ordered basis for $\mathcal{S}$ with respect to β_i, $i = 1, \ldots, k$. Then, $\mathcal{S}$ is an (A+GC)-invariant submodule if and only if the vectors $\tilde{s}_i$ for $i = 1, 2, \ldots, k$ are such that $A\tilde{s}_i \subset \mathcal{S} + Im\ \beta i \mathcal{X}$.

Proof (ii) $\Rightarrow$ (i) Let us suppose that $A\tilde{s}_i \subset \mathcal{S} + Im\ \beta_i X$ holds for all $i = 1, \ldots, k$. Then, there exist $A\tilde{s}_i = S(L)_i + (K)_i \beta_i$. As a consequence $ASQ = SL + K diag(\beta_1, \beta_2, \ldots, \beta_k)$. By Lemma 1 there exists a $k \times p$ matrix P_1 such that $P_1 CSQ = diag(\beta_1, \ldots, \beta_k)$, then $ASQ = SL + KP_1 CSQ$. Therefore $(A + GC)\mathcal{S} \subseteq \mathcal{S}$ for $G = -KP_1$ and $\mathcal{S}$ is an injection-invariant submodule.
(i) $\Rightarrow$ (ii) Assume that $\mathcal{S}$ is $(A + GC)$-invariant and that $\tilde{s}_1, \tilde{s}_2, \ldots,\ \tilde{s}_k$ is an ordered basis of $\mathcal{S}$ with respect to the invariant factors of the matrix CS. Then, by Lemma 1 there exists a matrix L such that for all $i = 1, \ldots, k$,

$$A\tilde{s}_i = S(L)_i - GC\tilde{s}_i. \qquad (3)$$

As a consequence, $ASQ = SL - GCSQ$ and, by Lemma 1, for a suitable $p \times k$ matrix $\tilde{P}_1$ we have $ASQ = SL - G\tilde{P}_1 diag(\beta_1, \beta_2, \ldots, \beta_k)$. Therefore $A\tilde{s}_i \subset \mathcal{S} + Im \beta_i \mathcal{X}$ for all $i = 1, \ldots, k$.

Example 2. Let Σ be the system defined over the ring $\mathcal{R}[\nabla]$ by (1) with

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ C = \begin{pmatrix} 1 & 0 \end{pmatrix}.$$

The submodule $\mathcal{S} = span\{\begin{pmatrix} \nabla \\ 0 \end{pmatrix}\}$ is (C, A)-invariant, since $Ker\ C \cap \mathcal{S} = 0$. Moreover, $C\mathcal{S} = [\nabla]$, hence $P = 1$, $Q = 1$ and $\beta_1 = \nabla$ is the invariant factor associated to $s_1 = \begin{pmatrix} \nabla \\ 0 \end{pmatrix}$, the vector that generates $\mathcal{S}$. As a consequence $As_1 = \begin{pmatrix} 0 \\ \nabla \end{pmatrix} = \begin{pmatrix} \nabla \\ 0 \end{pmatrix}(g_1) + \begin{pmatrix} -g_1 \\ 1 \end{pmatrix}(\nabla) \subset \mathcal{S} + \nabla \mathcal{X}$. for every $g_1 \in \mathcal{R}[\nabla]$, therefore $\mathcal{S}$ is $(A + GC)$-invariant for all matrices $G = \begin{pmatrix} g_1 & -1 \end{pmatrix}$. In fact, $(A + GC) = \begin{pmatrix} -g_1 & 0 \\ 0 & 0 \end{pmatrix}$ and $(A + GC)\mathcal{S} \subseteq \mathcal{S}$.

Example 3. Consider now the system Σ_1 defined over the ring $\mathcal{R}[\nabla]$ by (1) with

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad and\ C = \begin{pmatrix} \nabla & 1 \end{pmatrix}.$$

The submodule $\mathcal{S} = span\{\begin{pmatrix} \nabla \\ 0 \end{pmatrix}\}$ is still an (C, A)-invariant submodule for Σ_1 but now $\beta_1 = \nabla^2$, and the conditions of Proposition 2 are no longer verified for s_1. In this case $\mathcal{S}$ is not an $(A + GC)$-invariant submodule .

4. GEOMETRIC CHARACTERIZATION OF FEEDBACK INVARIANT SUBMODULES OVER A RING

Let us now introduce a more general characterization of feedback invariant submodules that holds also when the Smith form is no longer available. Such characterization may appear very technical, but it is practically computable, for instance, over $\mathcal{R}[z_1, z_2, \ldots, z_k]$, the ring of polynomial in several indeterminates over the reals, a very important case in applications.

Definition 4. Let Σ a system defined by (1) over a commutative ring R, let $\mathcal{V}$ be a submodule of R^n and v an element of $\mathcal{V}$. We will denote by $\mathcal{F}_V(v)$ the set of all matrix $F \in R^{m \times n}$ such that $(A + BF)v \in \mathcal{V}$.

The following technical results will be used in the following. The proof, which is straightforward will be omitted.

Lemma 2. Let $\{v_i, i \in \mathcal{I}\}$ a set of generators for a submodule $\mathcal{V}$ of R^n. Then $\mathcal{V}$ is (A+BF)-invariant if and only if $\bigcap_{i\in\mathcal{I}} \mathcal{F}_V(v_i) \neq \emptyset$.

Lemma 3. Let Σ a system defined by (1) over a commutative ring R and $\mathcal{V}$ be a submodule of R^n. Denoting by $v_1, v_2, .., v_n$ the n components of v we have that $\mathcal{F}_V(v) \neq \emptyset$ if and only if $Av \in \mathcal{V} + \sum_{j=1}^{n} Imv_jB$.

4.1 *The Noetherian case*

Noetherian rings, in particular rings of polynomials in a finite number of unknown over a field are used to model, for instance, systems whose defining matrices depend polynomially on a vector of parameters or delay differential systems with a finite number of incommensurable delays. A crucial property of a Noetherian ring R is that any submodule of R^n is finitely generated. As a consequence, the following result holds true.

Proposition 3. Let $\mathcal{V}$ be a submodule of R^n, where R is a Noetherian ring, and let $\{v_i,\ i = 1..s\}$ be a set of generators for $\mathcal{V}$. Then, $\mathcal{V}$ is an (A+BF)-invariant submodule if and only if $\bigcap_{i=0}^{s} \mathcal{F}_V(v_i) \neq \emptyset$.

The intersection of a finite number of submodules can be practically computed, using the theory of *Gröebner* bases over rings of the form $R = K[z_1, \ldots, z_n]$, where K is a field (or an integral domain, or a Unique Factorization Domain (UFD) or a Principal Ideal Domain (PID)). Roughly speaking, given any suitable notion of division in the ring, a *Gröebner* basis for an ideal I of R is a set of generators for I with the property that an element $f \in R$, a polynomial, belongs to I if and only if the remainder of f divided by each element of the *Gröebner* basis is zero. The important fact is that *Gröebner* bases can be computed by an algorithm.
A crucial technical point in improving the efficiency of the algorithm giving the *Gröebner* basis of an ideal consists in the computation of the *syzygy module* of a set of polynomials.

Definition 5. (Adams and Loustaunau, 1996) Let R be the Noetherian ring $R = K[z_1, \ldots, z_n]$ and let $f_1, \ldots, f_s$ be polynomials in R. The *syzygy module* of the matrix $[f_1 \ \ldots \ f_s]$, denoted by $Syz(f_1, \ldots, f_s)$ is the set of all the solutions of the single linear equation with polynomial coefficients (the f_i's) $f_1X_1 + f_2X_2 + \ldots + f_sX_s = 0$, where the solutions X_i are also polynomials in R.

The syzygy module of a matrix V whose columns $v_i, \ldots v_s$ belongs to R^n, $Syz(V) = Syz(v_1, \ldots, v_s)$, is the set of all polynomial solutions $\mathbf{X} \in R^n$ of the system of homogeneous linear equations $V\mathbf{X} = 0$, i.e. the set of all polynomial elements in the nullspace of V.

The computation of the syzygy module of a polynomial matrix requires the solution of a number of diophantine equations over polynomial rings.
We can now introduce an algorithm to check if a submodule $\mathcal{V}$ is $(A + BF)$-invariant, based on the following technical characterization of the state feedbacks which are *friends* of $\mathcal{V}$.

Proposition 4. Let Σ be a system defined by (1) over a Noetherian ring $R = K[z_1, \ldots, z_n]$. Let $\mathcal{V}$ be an (A,B)-invariant submodule of R^n, $V \in R^{n\times s}$ a basis matrix for $\mathcal{V}$ and $v_1,..,v_n$ the coordinates of a vector $v \in \mathcal{V}$. Denote by $\begin{pmatrix} x_0 \\ Y_0 \\ Y_1 \\ \vdots \\ Y_n \end{pmatrix}$, with $x_0 \in R^{1\times t}$, $Y_0 \in R^{s\times t}$, $Y_i \in R^{m\times t}$ for $i = 1, .., n$, the $(1 + s + nm) \times t$ matrix whose columns span the syzygy module of $[Av| - V|v_1B| \ldots |v_nB]$, i.e. such that

$$[Av| - V|v_1B| \ldots |v_nB] \begin{pmatrix} x_0 \\ Y_0 \\ Y_1 \\ \vdots \\ Y_n \end{pmatrix} = 0.$$

Then,

- if $\mathcal{F}_V(v) \neq \emptyset$, there exists a row vector $k_0 \in R^{t\times 1}$ such that $x_0k_0 = 1$ and the matrix $[Y_1k_0|Y_2k_0| \ldots |Y_nk_0]$, shortly denoted by Yk_0, belongs to $\mathcal{F}_V(v)$.
- Moreover, $\mathcal{F}_V(v)$ consists of Yk_0 and of all the matrices that can be written as $[Y_1k|Y_2k| \ldots |Y_nk]$, where $k \in R^{t\times 1}$ is such that $k - k_0$ is a syzygy for x_0.

Proof 1. Suppose $\mathcal{F}_V(v) \neq \emptyset$ then, $Av \in \mathcal{V} + \sum_{j=1}^{n} Imv_jB$ and there exist column vectors l and F_i, $i = 1, .., n$ such that $Av = Vl + \sum_{j=1}^{n} v_jBF_j$. Hence the vector $w := \begin{pmatrix} 1 \ l \ F_1 \ \ldots \ F_n \end{pmatrix}^t$ is a syzygy for $[Av| - V|v_1B| \ldots |v_nB]$ and there exists $k_0 \in R^{t\times 1}$ such that $x_0k_0 = 1$, and $Yk_0 = [Y_1k_0|Y_2k_0| \ldots |Y_nk_0] \in \mathcal{F}_V(v)$.
Now, let $F \in \mathcal{F}_V(v)$ and write $F = [F_1|..|F_n]$ for an $F \in \mathcal{F}_V(v)$. Then, there exists a column vector l, such that $Av = Vl - \sum_{j=1}^{n} v_jBF_j$ and the vector $w := \begin{pmatrix} 1 \ l \ F_1 \ \ldots \ F_n \end{pmatrix}^t$ being a syzygy for $[Av| - V|v_1B| \ldots |v_nB]$ is contained in the submodule spanned by $\begin{pmatrix} x_0 \ Y_0 \ Y_1 \ \ldots \ Y_n \end{pmatrix}^t$. Hence

there exists k such that $x_0 k = 1 = x_0 k_0$ and $x_0(k - k_0) = 0$. So, for all $i = 1,..,n$ we have $F_i = Y_i k$ and $k - k_0$ is a syzygy for x_0.

Conversely, if $k - k_0$ is a syzygy for x_0, we have that $x_0 k = 1$ and $\begin{pmatrix} 1 & Y_0 k & Y_1 k & \dots & Y_n k \end{pmatrix}^t$ is a syzygy for $[Av| - V|v_1 B| \dots |v_n B]$, i.e. $Av = VYk_0 - (v_1 BY_1 k + \dots + v_n BY_n k)$. Therefore $Av \in (\mathcal{V} - B\sum_{j=1}^n Y_i k v_i)$, which proves that $Av + v_1 BY_1 k + \dots + v_n BY^n k = VY_0 k \in \mathcal{V}$ and $[Y_1 k|..|Y_n k] \in \mathcal{F}_V(v)$.

An analogous procedure, concerning the injection invariance property over Noetherian rings, based on Proposition 2 is actually being developed. The procedure described in the above Proposition can be practically implemented using software which performs formal computations, for instance MapleV and CoCoa (Capani and Robbiano, n.d.).

Let us summarize the different steps required to compute $\mathcal{F}_V(v)$ by means of Proposition 4.

(1) Compute x_0 and Y from the matrix whose columns generate the syzygy module of the matrix $[Av| - V|v_1 B| \dots |v_n B]$;
(2) A vector k_0 such that $x_0 k_0 = 1$ exists if and only if the reduced *Gröebner* basis for the ideal generated by its components $\{x_{0i},\ i = 1, 2, \dots, t\}$, is $\{1\}$ (see (Adams and Loustaunau, 1996));
(3) Compute $\mathcal{F}_V(v)$ as kernel of x_0.

We shall now apply the above procedure to two systems over $R = \mathcal{R}[x, y]$, the ring of polynomials in two variables with real coefficients.

Example 4. Let us consider the system Σ defined by (1) over $\mathcal{R}[x, y]$ with

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ B = \begin{pmatrix} 1 \\ x \\ 0 \end{pmatrix}.$$

The submodule

$$\mathcal{V} = Im \begin{pmatrix} 1 & 0 \\ 0 & x \\ 0 & y \end{pmatrix} = Im(v_1|v_2).$$

is (A,B)-invariant. To check if it is also feedback invariant, using Propositions 2 and Proposition 4 let us first compute the syzygy module

$$Syz[Av_1| - V|B|0|0] = \begin{pmatrix} 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -x & x & 0 & 0 \\ 0 & 0 & -y & 0 & 0 & 0 \end{pmatrix}.$$

We obtain $Syz[Av_1| - V|B|0|0] =$

$$column\ span\ of \begin{pmatrix} x_0 \\ Y_0 \\ Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = \begin{pmatrix} 1\ 0\ 0 \\ \hline 0\ 0\ 0 \\ \hline 0\ 0\ 0 \\ \hline 0\ 0\ 0 \\ \hline 0\ 1\ 0 \\ \hline 0\ 0\ 1 \end{pmatrix}$$

so that $x_0 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$ and we can choose, for instance, $k_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

The kernel of x_0 is spanned by the columns of the matrix $\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$ and the set of vectors $K = \{k\ such\ that\ k - k_0 \in Ker(x_0)\}$ can be described as $K = span \begin{pmatrix} 1 \\ k_1 \\ k_2 \end{pmatrix}$, where k_1, k_2 are arbitrary elements of the ring R. We then deduce that the feedbacks "friends" of the vetor v_1 are of the following type.

$$\mathcal{F}_V(v_1) = \begin{pmatrix} 0 & k_1 & k_2 \end{pmatrix}.$$

Let us now compute $\mathcal{F}_V(v_2)$. To this aim we must compute the syzygy module of the matrix $[Av_2| - V|0|xB|yB]$. $Syz([Av_2| - V|0|xB|yB]) =$ $column\ span\ of \begin{pmatrix} 0\ -y\ -x \\ \hline 0\ -y\ -x \\ \hline 0\ -y\ -x \\ \hline 1\ \ 0\ \ \ 0 \\ \hline 0\ \ 0\ -1 \\ \hline 0\ -1\ \ 0 \end{pmatrix}$. In this case $x_0 = \begin{pmatrix} 0 & -y & -x \end{pmatrix}$ and there does not exist a k_0 in $\mathcal{R}^3$ such that $x_0 k_0 = 1$. Hence $\mathcal{F}_V(v_2) = \emptyset$ and we can conclude that the submodule $\mathcal{V}$ is not (A+BF)-invariant.

Example 5. Let us now slightly modify the dynamic matrix of the previous example and consider the system Σ_1 defined by (1) over $\mathcal{R}[x, y]$ with

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & y \end{pmatrix}\ B = \begin{pmatrix} 1 \\ x \\ 0 \end{pmatrix}.$$

Computations analogous to those described in more details in the previous example give

$$\mathcal{F}_V(v_1) = column\ span\ of\ \begin{pmatrix} 0 & k_1 & k_2 \end{pmatrix}$$

as before. However now we have

$$Syz[Av_2| - V|0|xB|yB] =$$
$$column\ span\ of \begin{pmatrix} 0 & -1 & -x \\ 0 & -y & -xy \\ 0 & -y & -xy \\ 1 & 0 & 0 \\ 0 & 0 & -y \\ 0 & -1 & 0 \end{pmatrix}$$

where $x_0 = \begin{pmatrix} 0 & -1 & -x \end{pmatrix}$. We can chose, for instance.

$$k_0 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}.$$

Any element in the kernel of x_0 is of the form $k = \begin{pmatrix} k_1 \\ -1 - xk_2 \\ k_2 \end{pmatrix}$, with k_1 and k_2 in R. Then, any feedback F of the form

$$F = \begin{pmatrix} k_1 & -yk_2 & 1 + xk_2 \end{pmatrix},$$

belongs to $\mathcal{F}_V(v_2)$ for all pair (k_1, k_2) in R^2. Finally we have that

$$\mathcal{F}_V(v_1) \cap \mathcal{F}_V(v_2) = \\ = \begin{pmatrix} 0 & -yk_2 & 1 + xk_2 \end{pmatrix}, \ \forall k_2 \in R$$

Therefore, $\mathcal{V}$ is an $(A+BF)$-invariant submodule. In fact we have tha, $\forall k_2 \in R$

$$A + BF = \begin{pmatrix} 0 & -yk_2 & 1 + xk_2 \\ 0 & -xyk_2 & x + x^2k_2 \\ 0 & 0 & y \end{pmatrix}$$

and

$$(A + BF)V = \begin{pmatrix} 0 & y \\ 0 & xy \\ 0 & y^2 \end{pmatrix} \subset \mathcal{V}.$$

Remark 1. In order to compute $\bigcap_i \mathcal{F}_V(v_i)$ we have only to solve a set of linear equations, since the parametrization of $\mathcal{F}_V(v_i)$ is linear. Consequently, it is always possible and relatively simple to compute it.

Since conditions for the solvability of many control problems are formulated in terms of set theoretic relations concerning controlled invariant subspaces which are feedbasck invariant (see, for instance (Assan and Perdon, 1999*a*), (Conte and Perdon, 1998)), the above results widen the practical applications of these results.
If R is a Noetherian ring, condition (4) can be checked following the procedure described in Proposition 4.1, which, in case of positive answer, allows to compute the friends of $\mathcal{V}$.

5. CONCLUSIONS

In this paper new characterizations have been proposed for the invariant feedback property and the injection invariant property for systems over rings. Their use allows to develop practical design methodologies based on the geometric approach for systems over Noetherian rings, in particular for delay differential systems.

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