

MATCHING OF EULER-LAGRANGE AND HAMILTONIAN SYSTEMS

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Abstract: This paper discusses the matching conditions as introduced in two recently developed methods for stabilization of underactuated mechanical systems. It is shown that the controlled Lagrangians method is naturally embedded in the IDA-PBC method. The integrability of the latter method is studied in general.

Keywords: underactuated mechanical systems, Euler-Lagrange systems, Hamiltonian systems, stabilization

1. INTRODUCTION

In a number of recent papers a new method has been introduced for stabilizing underactuated mechanical systems. The key idea of the method is to look for a stabilizing feedback law which renders the closed loop system into another mechanical system, that is, which preserves the physical structure of the system. Such a method is obviously desirable since physical motivations and knowledge can be used to design the feedback law according to the desired properties of the closed loop system. Since the mechanical systems under consideration are underactuated, stabilizing a desired equilibrium point of the system cannot be done by shaping of the potential energy only (as in the case for fully actuated systems). In general one also needs to adjust the kinetic energy of the system, leading to a closed loop mechanical system with a modified total energy.

The method has been developed using the Euler-Lagrange formalism in (Bloch *et al.*, 1997; Bloch *et al.*, 1998; Bloch *et al.*, 1999; Bloch *et al.*, 2000),

see also (Auckly *et al.*, 2000; Auckly and Kapitan-ski, 2000; Hamberg, 1999; Hamberg, 2000), and was called the *controlled Lagrangians method*. The existence of a structure preserving feedback law is determined by the so-called matching conditions. These matching conditions constitute a set of partial differential equations (PDEs), which have to be solved for the closed loop systems' modified total energy function (under the constraint of stabilizing the original underactuated system). If a stabilizing total energy function which solves the PDEs is found, then the corresponding feedback law can be immediately obtained as a result from these data.

Independently, the analogy of the method using the port-controlled Hamiltonian formalism has been developed in (Ortega *et al.*, 2001a; Ortega *et al.*, 2001b; Ortega *et al.*, 2001c; Gómez-Estern *et al.*, 2001), and was called the *interconnection and damping assignment passivity based control method (IDA-PBC)*. The existence of a structure preserving feedback law is again described by a set of PDEs. However, since the class of port-

controlled Hamiltonian systems is strictly larger than class of Euler-Lagrange systems, more freedom is obtained in finding solutions of these PDEs. In fact, next to modifying the total energy function of the system, also the internal interconnection structure (corresponding to the Poisson bracket of the system) is allowed to be changed.

In this paper, we describe and compare the matching conditions of both methods. It is shown that for a particular choice of the internal interconnection structure the IDA-PBC method effectively results in the controlled Lagrangians method. This leads to the study of integrable Hamiltonian systems (i.e. transformable to EL systems) which can result from the IDA-PBC method. It is shown that the ‘integrable’ IDA-PBC method allows to introduce gyroscopic terms in the closed loop system. Finally, some remarks are given on how to translate and extend certain interesting results on the controlled Lagrangians side (in particular the so-called λ -method of (Auckly *et al.*, 2000)) into the IDA-PBC method. More details on the results of this paper can be found in (Blankenstein *et al.*, 2001).

2. MATCHING OF EULER-LAGRANGE SYSTEMS

Consider a forced Euler-Lagrange system with n -dimensional configuration space \mathcal{Q} , described by a Lagrangian $L : T\mathcal{Q} \rightarrow \mathbb{R}$,

$$\frac{d}{dt}\nabla_{\dot{q}}L(q, \dot{q}) - \nabla_q L(q, \dot{q}) = G(q)u \quad (1)$$

($\nabla_q L$ stands for the partial derivative of $L(q, \dot{q})$ with respect to q , etc.). The matrix $G(q) : \mathbb{R}^m \rightarrow T_q^* \mathcal{Q} \simeq \mathbb{R}^n$, with $\text{rank } G = m$, defines the force fields corresponding to the input $u \in \mathbb{R}^m$. The system is called underactuated if $m < n$. Suppose the objective is to stabilize a desired equilibrium point (q^*, \dot{q}^*) of this system. In the method of controlled Lagrangians this is pursued by searching for a possible closed loop Euler-Lagrange system, defined by a Lagrangian $L_c : T\mathcal{Q} \rightarrow \mathbb{R}$, such that (q^*, \dot{q}^*) is a stable equilibrium point of the closed loop dynamics

$$\frac{d}{dt}\nabla_{\dot{q}}L_c(q, \dot{q}) - \nabla_q L_c(q, \dot{q}) = 0. \quad (2)$$

The existence of a feedback law $u(q, \dot{q})$ which transforms the system (1) into the closed loop system (2) is given by the so-called matching conditions, which can be described as follows:

Let $G^\perp(q) : (\mathbb{R}^{n-m})^T \rightarrow (\mathbb{R}^n)^T$ denote a full rank left annihilator of $G(q)$, i.e. $G^\perp(q)G(q) = 0, \forall q \in \mathcal{Q}$. From (1) it follows that

$$G^\perp(q) \left(\frac{d}{dt}\nabla_{\dot{q}}L(q, \dot{q}) - \nabla_q L(q, \dot{q}) \right) = 0. \quad (3)$$

Furthermore, since $(\mathbb{R}^n)^T = \text{Im } G^T(q) \oplus \text{Im } G^\perp(q)$, (2) can be equivalently written as the following set of equations

$$G^T(q) \left(\frac{d}{dt}\nabla_{\dot{q}}L_c(q, \dot{q}) - \nabla_q L_c(q, \dot{q}) \right) = 0, \quad (4)$$

$$G^\perp(q) \left(\frac{d}{dt}\nabla_{\dot{q}}L_c(q, \dot{q}) - \nabla_q L_c(q, \dot{q}) \right) = 0. \quad (5)$$

The matching conditions are described in the following proposition.

Proposition 1. The systems (1) and (2) match if and only if equation (5) holds along solutions of the system (3, 4).

In that case the feedback law is explicitly given by

$$u = (G^T G)^{-1} G^T \left[\left(\frac{d}{dt}\nabla_{\dot{q}}L - \nabla_q L \right) - \left(\frac{d}{dt}\nabla_{\dot{q}}L_c - \nabla_q L_c \right) \right]. \quad (6)$$

Taking into account the regularity of the Lagrangians L and L_c , the matching conditions can be written as a set of nonlinear partial differential equations, to be satisfied for all (q, \dot{q}) . Therefore write out (1) as

$$\ddot{q} = -(\nabla_{\dot{q}\dot{q}}L)^{-1}((\nabla_{q\dot{q}}L)\dot{q} - \nabla_q L - Gu). \quad (7)$$

Analogously, (2) can be written as

$$\ddot{q} = -(\nabla_{\dot{q}\dot{q}}L_c)^{-1}((\nabla_{q\dot{q}}L_c)\dot{q} - \nabla_q L_c). \quad (8)$$

Then the matching conditions can be translated into the following set of $n - m$ nonlinear PDEs (which can be obtained by equating (7) with (8))

$$G^\perp \left[\{ \nabla_{q\dot{q}}L - (\nabla_{\dot{q}\dot{q}}L)(\nabla_{\dot{q}\dot{q}}L_c)^{-1}(\nabla_{q\dot{q}}L_c) \} \dot{q} - \{ \nabla_q L - (\nabla_{\dot{q}\dot{q}}L)(\nabla_{\dot{q}\dot{q}}L_c)^{-1}\nabla_q L_c \} \right] = 0. \quad (9)$$

The PDEs (9) have to be solved for the closed loop Lagrangian L_c , constrained to the condition of stabilizing the desired equilibrium point (q^*, \dot{q}^*) . Once a (stabilizing) solution L_c is found, the feedback law is explicitly given by

$$u = (G^T G)^{-1} G^T w(q, \dot{q}), \quad (10)$$

where $w(q, \dot{q})$ denotes the terms between square brackets in (9). This feedback law is equal to the one in (6). The PDEs (9) are equivalent to the ones obtained in (Hamberg, 2000).

Now, suppose (1) describes an underactuated mechanical system with a Lagrangian defined as the difference between the kinetic and the potential energy

$$L(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} - V(q). \quad (11)$$

Here $M = M^T$ describes the generalized mass matrix of the system. Assume that M is invertible, this is equivalent to L being regular. Following the

basic idea of the method, we consider (stabilizing) feedback laws which preserve the mechanical structure of the system. That is, the closed loop system (2) has a Lagrangian of the form

$$L_c(q, \dot{q}) = \frac{1}{2} \dot{q}^T M_c(q) \dot{q} - V_c(q), \quad (12)$$

for some modified generalized mass matrix $M_c = M_c^T$ (assumed to be invertible) and potential energy function V_c . In this case, the matching conditions (9) split into a set of two coupled nonlinear PDEs (corresponding to the terms dependent, respectively independent, of the velocities \dot{q})

$$G^\perp \left[\{\nabla_q(M\dot{q}) - MM_c^{-1}\nabla_q(M_c\dot{q})\}\dot{q} - \left\{ \nabla_q\left(\frac{1}{2}\dot{q}^T M\dot{q}\right) - MM_c^{-1}\nabla_q\left(\frac{1}{2}\dot{q}^T M_c\dot{q}\right) \right\} \right] = 0 \quad (13)$$

and

$$G^\perp \left[\nabla_q V - MM_c^{-1}\nabla_q V_c \right] = 0. \quad (14)$$

Equation (13) describes the PDEs that have to be satisfied by the closed loop kinetic energy, and is independent of the potential energy V_c . Equation (14) describes the PDEs for the potential energy, and depends on the kinetic energy described by M_c . A solution (M_c, V_c) of these PDEs has to be found which stabilizes the desired equilibrium point (q^*, \dot{q}^*) of the system. These matching conditions have appeared earlier in (Auckly *et al.*, 2000; Auckly and Kapitanski, 2000; Hamberg, 1999).

Remark 1. The method of controlled Lagrangians was first introduced in (Bloch *et al.*, 1997; Bloch *et al.*, 1998; Bloch *et al.*, 2000) within the context of mechanical systems with symmetry. Next to preserving the mechanical structure of the system, the feedback law is designed to preserve the symmetries of the system. In particular, the potential energy is left unchanged. Extensive computations lead to matching conditions again described by a set of nonlinear PDEs. These PDEs can be very nicely interpreted in terms of the PDEs (13, 14) describing the matching of kinetic and potential energy, see (Blankenstein *et al.*, 2001) for more details.

3. MATCHING OF HAMILTONIAN SYSTEMS

Consider a port-controlled Hamiltonian system of the form

$$\dot{z} = J(z)\nabla_z H(z) + g(z)u, \quad (15)$$

where $z \in \mathcal{M}$ (a manifold), $J(z) = -J^T(z)$ is a skew-symmetric matrix describing the internal interconnection structure of the system, $g(z) : \mathbb{R}^m \rightarrow T_z\mathcal{M}$ is a full rank matrix describing the input vector fields corresponding to the input $u \in \mathbb{R}^m$ and $H(z)$ is the Hamiltonian (or energy)

function of the system. Analogously to the method of controlled Lagrangians, the IDA-PBC method uses the idea of stabilizing a desired equilibrium point z^* of the system by considering structure preserving feedback laws. That is, the closed-loop system is described by the equations

$$\dot{z} = J_d(z)\nabla_z H_d(z), \quad (16)$$

where $J_d(z) = -J_d^T(z)$ denotes the closed-loop interconnection matrix and $H_d(z)$ the closed-loop Hamiltonian function. The existence of a feedback law which transforms the system (15) into the closed loop system (16) is determined by the solvability of following matching conditions

$$g^\perp(z) [J_d(z)\nabla_z H_d(z) - J(z)\nabla_z H(z)] = 0, \quad (17)$$

where $g^\perp(z)$ denotes a full rank left annihilator of $g(z)$. These matching conditions have appeared in (Ortega *et al.*, 2001a; Ortega *et al.*, 2001b). The matching conditions (17) constitute a set of $n - m$ nonlinear PDEs which have to be solved for H_d and J_d such that z^* is a stable equilibrium point of the closed loop system (16). Once a stabilizing solution (H_d, J_d) is found, the corresponding feedback law is explicitly given by

$$u = (g^T g)^{-1} g^T [J_d \nabla_z H_d - J \nabla_z H]. \quad (18)$$

In the following the method is applied to the class of underactuated mechanical systems, see (Ortega *et al.*, 2001c). A mechanical system can be described by a port-controlled Hamiltonian system of the form (15),

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \begin{bmatrix} \nabla_q H \\ \nabla_p H \end{bmatrix} + \begin{bmatrix} 0 \\ G(q) \end{bmatrix} u, \quad (19)$$

where (q, p) (consisting of configuration coordinates q and impulses p) denote local coordinates for the state space $\mathcal{M} = T^*\mathcal{Q}$, with $\mathcal{Q} \simeq \mathbb{R}^n$ denoting the configuration space of the mechanical system. The matrix $G(q) : \mathbb{R}^m \rightarrow T_q^*\mathcal{Q} \simeq \mathbb{R}^n$ defines the force fields corresponding to the input $u \in \mathbb{R}^m$. The Hamiltonian function $H(q, p)$ is given by the total, i.e. kinetic plus potential, energy in the system

$$H(q, p) = \frac{1}{2} p^T M^{-1}(q) p + V(q), \quad (20)$$

where $M = M^T$ describes the generalized mass matrix of the system, and is assumed to be invertible. Since we are interested in preserving the structure of the system, we propose the shaped Hamiltonian function $H_d(q, p)$ to be again of the form (20),

$$H_d(q, p) = \frac{1}{2} p^T M_d^{-1}(q) p + V_d(q), \quad (21)$$

for some shaped generalized mass matrix $M_d = M_d^T$ (assumed to be invertible) and potential energy function $V_d(q)$. On the other hand, the

internal interconnection structure of the system is allowed to be modified into the form

$$J_d(q, p) = \begin{bmatrix} 0 & M^{-1}(q)M_d(q) \\ -M_d(q)M^{-1}(q) & J_2(q, p) \end{bmatrix} \quad (22)$$

for some skew-symmetric matrix $J_2(q, p)$ (acting as an extra design parameter). Notice that the first row of J_d is determined by the fact that the relation $\dot{q} = M^{-1}(q)p$ should also hold in closed loop, since \dot{q} is a nonactuated coordinate. Then, system (16) becomes

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & M^{-1}M_d \\ -M_dM^{-1} & J_2 \end{bmatrix} \begin{bmatrix} \nabla_q H_d \\ \nabla_p H_d \end{bmatrix}. \quad (23)$$

The matching conditions (17) yield

$$G^\perp [\nabla_q H - M_d M^{-1} \nabla_q H_d + J_2 M_d^{-1} p] = 0. \quad (24)$$

Using (20) and (21) these PDEs spilt into the following set of two coupled nonlinear PDEs

$$G^\perp \left[\nabla_q \left(\frac{1}{2} p^T M^{-1} p \right) - M_d M^{-1} \nabla_q \left(\frac{1}{2} p^T M_d^{-1} p \right) + J_2 M_d^{-1} p \right] = 0, \quad (25)$$

and

$$G^\perp [\nabla_q V - M_d M^{-1} \nabla_q V_d] = 0. \quad (26)$$

Analogously to the controlled Lagrangians method, equation (25) describes the PDEs to be satisfied by the closed loop kinetic energy, and is independent of the potential energy V_d . Equation (26) describes the PDEs to be satisfied by V_d , and depends on M_d . An important difference with the controlled Lagrangians method however is the presence of the matrix J_2 , which acts as a design parameter which can be suitably chosen to allow the PDEs to be solvable for specific choices of M_d and V_d (directed by the stabilizability objective). Exploiting this extra degree of freedom might simplify the search for solutions of the matching conditions (25, 26) and help in the design of a suitable feedback law.

4. INTEGRABILITY OF THE IDA-PBC DESIGN

Since the class of port-controlled Hamiltonian systems is strictly larger than the class of Euler-Lagrange systems, the matching of Euler-Lagrange systems is a special case of the matching of Hamiltonian systems. That is, the controlled Lagrangians method is embedded in the IDA-PBC method. In this section it is shown that, with respect to mechanical systems, for a particular choice of the design parameter J_2 , the IDA-PBC method effectively results in the controlled Lagrangians method. Furthermore, the integrability of the IDA-PBC design is studied in general.

Consider an underactuated mechanical system described by the Euler-Lagrange equations (1) together with a Lagrangian of the form (11). The

system (1, 11) can be equivalently written as the port-controlled Hamiltonian system (19, 20), with the impulse variables defined by $p = M(q)\dot{q}$. Next, consider the closed loop Euler-Lagrange system (2) together with the closed loop Lagrangian (12). Analogously, this system can be written as the following port-controlled Hamiltonian system, where the closed loop impulse variables are defined by $p_c = M_c(q)\dot{q}$:

$$\begin{bmatrix} \dot{q} \\ \dot{p}_c \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}}_{J_c} \begin{bmatrix} \nabla_q H_c \\ \nabla_{p_c} H_c \end{bmatrix}, \quad (27)$$

together with the closed loop Hamiltonian function

$$H_c(q, p_c) = \frac{1}{2} p_c^T M_c^{-1}(q) p_c + V_c(q). \quad (28)$$

It is now clear that the IDA-PBC method results in the same closed loop system (and therefore the same feedback law) as the controlled Lagrangians method if and only if the closed loop Hamiltonian system (21, 23) is equivalent to the Hamiltonian system (27, 28).

The systems (21, 23) and (27, 28) are equivalent up to a coordinate transformation if and only if the Hamiltonians H_c and H_d are equivalent and in addition the internal interconnection structures defined by the matrices J_c and J_d are equivalent. Notice that $p_c = M_c M^{-1} p$, and calculate H_c in the coordinates (q, p) to obtain

$$H_c(q, p) = \frac{1}{2} p^T M^{-1}(q) M_c(q) M^{-1}(q) p + V_c(q). \quad (29)$$

Therefore the Hamiltonians H_c and H_d are equivalent if and only if

$$M_c(q) = M(q) M_d^{-1}(q) M(q), \quad V_c(q) = V_d(q) \quad (30)$$

(notice that there is a one-to-one relation between M_c and M_d). The structure matrices J_c and J_d are equivalent if and only if J_d becomes in the coordinates (q, p_c) the matrix J_c . This means that the coordinates (q, p_c) should satisfy the relations

$$\{q, q\}_d = 0, \quad \{q, p_c\}_d = I_n, \quad \{p_c, p_c\}_d = 0, \quad (31)$$

where $\{\cdot, \cdot\}_d$ denotes the Poisson bracket corresponding to the structure matrix J_d , defined by

$$\{F_1, F_2\}_d = [\nabla_q F_1 \quad \nabla_{p_c} F_1] J_d [\nabla_q F_2 \quad \nabla_{p_c} F_2]^T \quad (32)$$

for any two smooth functions $F_1(q, p), F_2(q, p)$. It is easy to check that the first two conditions in (31) are satisfied, while the last one is satisfied if and only if J_2 is defined as follows

$$J_2(q, p) = M_d M^{-1} \left[[\nabla_q (M M_d^{-1} p)]^T - \nabla_q (M M_d^{-1} p) \right] M^{-1} M_d \quad (33)$$

(note that J_2 is clearly skew-symmetric). It can also be calculated that under the conditions (30, 33) the matching conditions (13, 14) and

(25, 26) are equal, as well as the feedback laws (10) and (18). In conclusion, we have the following proposition:

Proposition 2. The IDA-PBC method results in the controlled Lagrangians method if and only if the internal interconnection structure of the closed loop system is chosen as in (33). The controlled Lagrangian L_c and the shaped Hamiltonian H_d are related by (30).

Remark 2. The coordinates (q, p_c) , transforming J_d into the constant matrix J_c , are called canonical coordinates for J_d . According to the well known Darboux Theorem, the existence of canonical coordinates is equivalent to the Poisson bracket $\{\cdot, \cdot\}_d$ satisfying the integrability (i.e. the Jacobi) identities.

Remark 3. In a recent publication (Chang *et al.*, 2001) have extended the controlled Lagrangians method by allowing the presence of “uncontrollable” external forces, modeling the non-integrable part of the closed loop dynamics. In this way, the controlled Lagrangians and IDA-PBC method become essentially equivalent.

More generally, let us state the conditions under which the internal interconnection structure matrix J_d (22) can be transformed into the constant matrix J_c . Without loss of generality, see (Blankenstein *et al.*, 2001), we can assume that the canonical coordinates have the form $(q, p_c(q, p))$. In order to transform J_d into J_c , these coordinates should satisfy the relations (31). The first relation is trivially satisfied, while it can be calculated that the last two are satisfied if and only if the canonical coordinates have the form

$$(q, p_c(q, p)) = (q, M(q)M_d^{-1}(q)p + Q(q)), \quad (34)$$

with $Q(q)$ any smooth vector-valued function of the coordinates q , and

$$\begin{aligned} J_2(q, p) = & M_d M^{-1} [[\nabla_q(MM_d^{-1}p)]^T - \\ & \nabla_q(MM_d^{-1}p)] M^{-1} M_d + \\ & M_d M^{-1} [[\nabla_q Q]^T - \nabla_q Q] M^{-1} M_d. \end{aligned} \quad (35)$$

It is easy to check that the closed loop Hamiltonian system (21, 23, 35) corresponds to the Euler-Lagrange system (2) with a closed loop Lagrangian L_c given by

$$L_c(q, \dot{q}) = \frac{1}{2} \dot{q}^T M_c(q) \dot{q} + \dot{q}^T Q(q) - V_c(q), \quad (36)$$

where (M_c, V_c) and (M_d, V_d) are related by (30). Comparing (36) to (12), it is noticed that the IDA-PBC method introduces the gyroscopic terms $\dot{q}^T Q(q)$ (i.e., linear in the velocities \dot{q}) in the closed loop Lagrangian function. Thus, we have the following proposition.

Proposition 3. The IDA-PBC method results in a closed loop Euler-Lagrange system if and only if the internal interconnection structure of the closed loop system is chosen as in (35). In general, gyroscopic terms are introduced in the closed loop Lagrangian.

Notice that taking $Q = 0$ (i.e. no gyroscopic terms) yields proposition 2. Finally, consider the matching conditions (25). Plugging J_2 (35) into (25) and separating the terms which are quadratic and linear in the p variables, yields the following two matching conditions:

$$G^\perp M_d M^{-1} [[\nabla_q Q]^T - \nabla_q Q] M^{-1} = 0, \quad (37)$$

and secondly the condition (25) with J_2 substituted by the expression (33). This shows that the gyroscopic forces should independently satisfy the matching condition (37).

5. FINDING SOLUTIONS OF THE MATCHING CONDITIONS

Since the matching conditions (13, 14) and (25, 26) constitute a set of nonlinear PDEs, they are not easy to solve in general. In this section we shall briefly describe two methods which can help in the problem of finding solutions of these PDEs.

First, in (Gómez-Estern *et al.*, 2001) it is shown that for a special class of port-controlled Hamiltonian systems of the form (19, 20) the PDEs (25, 26) can be transformed into a set of nonlinear ordinary differential equations (ODEs). Obviously, such a set of equations is much easier to solve. The class of systems for which this transformation is possible is defined by the following assumptions: i) the system is assumed to have n degrees of freedom and $n - 1$ actuators (i.e. there is only one unactuated coordinate), and ii) the kinetic energy matrix M is assumed only to depend on the unactuated coordinate. This class of systems is quite common in underactuated mechanical systems and includes for instance the well known example of a cart and pendulum. By choosing the shaped kinetic energy matrix M_c to only depend on the unactuated coordinate, it can be shown that the set of PDEs (25, 26) can be transformed into an equivalent set of ODEs. In (Gómez-Estern *et al.*, 2001) the method is applied to the examples of a cart and pendulum system and a ball and beam system.

Secondly, in (Auckly *et al.*, 2000; Auckly and Kapitanski, 2000) a method called the λ -method is described to solve the PDEs (13, 14) by recursively solving a set of three *linear* PDEs. This greatly reduces the complexity of finding solutions. Basically the method proceeds as follows: Define \tilde{G}^\perp to be an orthogonal projection matrix

of rank $n - m$ such that $\bar{G}^\perp G = 0$. Then the matching conditions remain unaltered if we substitute \bar{G}^\perp for G^\perp in (13, 14). Next observe that (13) defines a quadratic expression in the variables \dot{q} . Hence, we can polarize this expression to obtain an equivalent expression which is bilinear in the new ('velocity') variables v_1, v_2 . Introduce the matrix $\lambda = M_c^{-1}M$. Then taking $v_1 = \lambda \bar{G}^\perp M v'_1$ and premultiplying (13) with $(v'_1)^T M$ yields after some extensive algebra a linear PDE to be solved for $\lambda \bar{G}^\perp M$. Given a solution $\lambda \bar{G}^\perp M$ of this linear PDE, a second linear PDE can be obtained from (13) which is to be solved for λ . Knowing λ we have found a solution M_c of the nonlinear PDE (13). Finally, given M_c , (14) becomes a linear PDE to be solved for V_c . In (Auckly *et al.*, 2000; Auckly and Kapitanski, 2000) this method is applied to obtain a stabilizing control law in the cart and pendulum example.

In (Blankenstein *et al.*, 2001) it is shown that the λ -method can also be applied to the matching conditions (25, 26) obtained in the IDA-PBC method. However, instead of resulting in a set of three linear PDEs, the method results in one quadratic PDE and two linear PDEs. In fact, the first PDE becomes a quadratic PDE in $\lambda \bar{G}^\perp M$ (where $\lambda = M^{-1}M_d$). It is quadratic in the sense that there appear terms which are quadratic in the components of $\lambda \bar{G}^\perp M$, however, the derivatives of $\lambda \bar{G}^\perp M$ still appear linearly in the equation. Furthermore, this PDE (linearly) contains the matrix J_2 , which acts as a design parameter. Exploiting the freedom in J_2 might further simplify the search for solutions of this PDE. Having found a solution $\lambda \bar{G}^\perp M$ for this quadratic PDE, this results in a linear PDE for λ , which again results in a linear PDE for V_d .

6. CONCLUSIONS

In this paper two recently developed methods for stabilizing underactuated mechanical systems are described and compared. It is shown that the controlled Lagrangians method is embedded in the IDA-PBC method. The integrability of the latter method is studied in general, yielding explicit expressions for the internal interconnection structure of the closed loop system.

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