15th Triennial World Congress, Barcelona, Spain

ABOUT REDUCING CONSERVATISM IN ROBUST CONTROL DESIGN OF DISCRETE-TIME SYSTEMS: THE NON STATIONARY CASE

Jamal Daafouz [*] **Olivier Bachelier** [**]
Jacques Bernussou [***]

[*] *CRAN - ENSEM, 2 av. de la forêt de Haye, 54516 Vandœuvre Cedex - France*
[**] *LAII-ESIP, Bât. de Mécanique, 40, av. du Recteur Pineau, 86022 Poitiers Cedex- France*
[***] *LAAS-CNRS, 7 av. du colonel Roche, 31077 Toulouse Cedex 4 - France*

Abstract: The paper addresses the problem of stabilizing discrete-time systems subject to time varying polytopic uncertainty. Non stationary quadratic Lyapunov functions are derived for synthesis in a Poly-Quadratic Lyapunov function concept which avoids in a large extent the conservatism linked with the classical single Lyapunov function quadratic approach. The state space feedback synthesis problem is addressed. The results are extended to cope with two particular problems: $\mathcal{H}_\infty$ performance analysis and synthesis problem as well as state feedback design while maximizing the size of the uncertainty domain.

Keywords: Parameter Dependent Lyapunov Functions, Discrete Time Systems, Time Varying Uncertainty, H_∞ performance, $\mathcal{L}$inear $\mathcal{M}$atrix $\mathcal{I}$nequalities ($\mathcal{LMI}$).

1. INTRODUCTION

The interest of Lyapunov functions in both robust analysis and design has been largely proved for systems modelled in the time-domain. Recently, the determination of Parameter-Dependent Lyapunov Functions (PDLF) in order to test the stability of either time-invariant or time-varying uncertain models has been investigated Haddad and Bernstein (1994); Blanchini and Miani (1999); Feron et al. (1996); Geromel et al. (1998); Oliveira et al. (1999); Peaucelle et al. (2000); Trofino and de Souza (1999). The main reason is that quadratic approach, which has been widely used, suffer from conservatism because stability is checked through the use of a single Lyapunov function over the whole uncertainty domain.

In the time varying case, the literature is quite poor as far as the discrete-time case is concerned Amato et al. (1998). In this paper, we largely refer to the work presented in Daafouz and Bernussou (2001) where the concept of Poly-Quadratic stability is introduced. This concept can be seen as an extension to the time varying case of a result proposed in Oliveira et al. (1999). The stability of a discrete-time system subject to polytopic uncertainty is attested owing to the existence of a Lyapunov function that is quadratic with respect to the state vector and that is a convex combination of extreme Lyapunov matrices computed on the vertices. This technique is performed by solving a system of $\mathcal{L}$inear $\mathcal{M}$atrix $\mathcal{I}$nequalities ($\mathcal{LMI}$). It must be noticed that the involved $\mathcal{LMI}$ conditions are necessary and sufficient for Poly-Quadratic stability to be satisfied.

In this paper, discrete-time models with time-varying uncertainty are considered. The uncertainty is either convex polytopic or parametric affine structured Tesi and Vicino (1990). The second case can be seen as a particular case of the first one. In both cases, the problems of robust stabilization and robust $\mathcal{H}_\infty$ control by static state feedback are considered. For the affine type of uncertainty the problem of maximizing the uncertainty domain while preserving stability and possibly achieving an $\mathcal{H}_\infty$ performance level is investigated. All the presented results rely on the $\mathcal{LMI}$ framework. Due to space limitations, all the proofs are omitted and can be found in Daafouz et al. (2001).

Notations : We denote by M', the transpose of M and by $M^\dagger$ the pseudo-inverse of M. The Hadamard product is denoted by $\odot$. $\mathbf{I}$ is the identity matrix and $\mathbf{0}$ is a null matrix of appropriate dimensions. $\mathbb{1}_{u,v}$ is a matrix of dimension $u \times v$ with all entries equaling 1.

2. PRELIMINARIES

In this section, we introduce the uncertain model that is considered in the paper and give useful preliminary results concerning robust stability analysis for time-varying discrete-time systems. We largely refer to Daafouz and Bernussou (2001). The following discrete-time model is considered:

$$x(k+1) = \mathcal{A}(\xi(k))x(k) + \mathcal{B}(\xi(k))u(k), \qquad (1)$$

where $x(k) \in \mathbb{R}^n$ is the state, $u(k) \in \mathbb{R}^m$ is the control. The state space matrices are given by

$$\mathcal{A}(\xi(k)) = \sum_{i=1}^{N} \xi_i(k) A_i, \quad \mathcal{B}(\xi(k)) = \sum_{i=1}^{N} \xi_i(k) B_i \qquad (2)$$

with A_i and B_i, $i = 1, ..., N$, being constant matrices. The time varying parameter vector $\xi(k)$ is such that

$$\xi(k) = \begin{bmatrix} \xi_1(k) & \xi_2(k) & ... & \xi_N(k) \end{bmatrix}' \qquad (3)$$

with

$$\xi_i \geq 0 \ \forall i \in \{1, ..., N\} \ \text{ and } \ \sum_{i=1}^{N} \xi_i = 1$$

A state feedback control design problem is to find

$$u(k) = Kx(k) \qquad (4)$$

such that the closed system

$$x(k+1) = \mathcal{A}_{cl}(\xi(k))x(k) \qquad (5)$$

with

$$\mathcal{A}_{cl}(\xi(k)) = \mathcal{A}(\xi(k)) + \mathcal{B}(\xi(k))K = \sum_{i=1}^{N} \xi_i(k) \underbrace{(A_i + B_i K)}_{A_{cl_i}}$$

is asymptotically stable. For simplicity reasons, from now $\xi(k)$ will be denoted ξ. Following Daafouz and Bernussou (2001), we propose parameter-dependent Lyapunov functions (PDLF) of the form:

$$V(x(k), \xi) = x'(k)\mathcal{P}(\xi)x(k) \ \text{ with } \ \mathcal{P}(\xi) = \sum_{i=1}^{N} \xi_i(k) P_i \qquad (6)$$

where the various P_i, $i = 1, ..., N$, are $n \times n$ symmetric positive definite (SPD) matrices.

Definition 1. Daafouz and Bernussou (2001): System (5) is said to be Poly-Quadratically stable if and only if there exists a PDLF of the form (6) that is negative definite decrescent.

Poly-Quadratic stability is sufficient for asymptotic stability. Assessing the Poly-Quadratic stability of polytopic uncertain closed-loop model is equivalent to find N SPD matrices P_i, $i = 1, ...N$ such that (6) and

$$\mathcal{A}'_{cl}(\xi)\mathcal{P}_+(\xi)\mathcal{A}_{cl}(\xi) - \mathcal{P}(\xi) < 0 \qquad (7)$$

with:

$$\begin{cases} \mathcal{P}(\xi) = \sum_{i=1}^{N} \xi_i(k) P_i \\ \mathcal{P}_+(\xi) = \sum_{i=1}^{N} \xi_i(k+1) P_i \end{cases} \qquad (8)$$

In this part, we propose $\mathcal{LMI}$ conditions for a time-varying polytopic system such as (5) to be Poly-Quadratically stable.

Theorem 1. Daafouz et al. (2001): System (5) is Poly-Quadratically stable if and only if there exist SPD matrices $P_i \in \mathbb{R}^{n \times n}$, $i = 1, ...N$ as well as matrices $G_i \in \mathbb{R}^{n \times n}$ such that $\forall \{i; j\} \in \{1, ..., N\}^2$:

$$\begin{bmatrix} -P_i & A'_{cl_i} G'_j \\ G_j A_{cl_i} & Pj - Gj - G'_j \end{bmatrix} < 0 \qquad (9)$$

Now, we give a dual condition which was already proposed in Daafouz and Bernussou (2001).

Theorem 2. Daafouz and Bernussou (2001): System (5) is Poly-Quadratically stable if and only if there exist SPD matrices $X_i \in \mathbb{R}^{n \times n}$ as well as matrices $G_i \in \mathbb{R}^{n \times n}$, $i = 1, ...N$, such that $\forall \{i; j\} \in \{1, ..., N\}^2$:

$$\begin{bmatrix} X_i - G_i - G'_i & G'_i A'_{cl_i} \\ A_{cl_i} G_i & -X_j \end{bmatrix} < 0 \qquad (10)$$

Remark 1. : Abviously the Poly-Quadratic stability encompasses the former ones on quadratic stability with $G_i = P_i = P \ \ \forall i \in \{1, ..., N\}$ in (9).

A state feedback control that makes the closed loop system (5) Poly-Quadratically stable can be obtained using the following result.

Theorem 3. Daafouz et al. (2001): System (1) is Poly-Quadratically stabilizable by a state feedback control if there exist SPD matrices $X_i \in \mathbb{R}^{n\times n}$, matrices $G \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{m\times n}$, $i = 1, ..., N$, such that $\forall\{i; j\} \in \{1, ..., N\}^2$:

$$\begin{bmatrix} X_i - G - G' & (A_iG + B_iR)' \\ A_iG + B_iR & -X_j \end{bmatrix} < \mathbf{0}, \qquad (11)$$

The state feedback control law is then given by (4) with $K = RG^{-1}$.

One can notice that imposing $G_i = G$, $\forall i = 1, ..., N$ to derive the result of Theorem 3 introduces some conservatism, but still less pessimistic than the single Lyapunov function approaches.

3. ROBUST H_∞ PERFORMANCE

Consider the following discrete time system

$$\begin{cases} x(k+1) = \mathcal{A}(\xi)x(k) + \mathcal{B}_1(\xi)w(k) + \mathcal{B}_2(\xi)u(k), \\ z(k) = \mathcal{C}_1(\xi)x(k) + \mathcal{D}_1(\xi)w(k) + \mathcal{D}_2(\xi(k))u(k) \end{cases} \qquad (12)$$

where $x(k) \in \mathbb{R}^n$ is the state, $u(k) \in \mathbb{R}^m$ is the control vector $w(k) \in \mathbb{R}^q$ is the disturbance of the system and $z(k) \in \mathbb{R}^p$ is the controlled output. The state space matrices are given by,

$$\begin{array}{llll} \mathcal{A}(\xi) = \sum_{i=1}^N \xi_i A_i, & \mathcal{B}_1(\xi) = \sum_{i=1}^N \xi_i B_{1i}, \\ \mathcal{B}_2(\xi) = \sum_{i=1}^N \xi_i B_{2i}, & \mathcal{C}_1(\xi) = \sum_{i=1}^N \xi_i C_{1i} \\ \mathcal{D}_1(\xi(k)) = \sum_{i=1}^N \xi_i D_{1i}, & \mathcal{D}_2(\xi) = \sum_{i=1}^N \xi_i D_{2i} \end{array} \qquad (13)$$

with

$$\xi = \left[\xi_1,\ \xi_2,\ ...,\ \xi_N\right]', \quad \xi_i \geq 0, \quad \sum_{i=1}^N \xi_i(k) = 1$$

The matrices A_i, B_{1i}, B_{2i}, C_{1i}, D_{1i} and D_{2i}, $i = 1, ..., N$, are constant matrices of appropriate dimensions. Given $\gamma > 0$, the well known $\mathcal{H}_\infty$ state feedback control problem is to find

$$u(k) = Kx(k) \qquad (14)$$

making the closed system

$$\begin{cases} x(k+1) = \mathcal{A}_{cl}(\xi)x(k) + \mathcal{B}_1(\xi)w(k) \\ z(k) = \mathcal{C}_{cl}(\xi)x(k) + \mathcal{D}_1(\xi)w(k) \end{cases} \qquad (15)$$

with

$$\mathcal{A}_{cl}(\xi) = \mathcal{A}(\xi) + \mathcal{B}_2(\xi)K, \quad \mathcal{C}_{cl}(\xi) = \mathcal{C}_1(\xi) + \mathcal{D}_2(\xi)K,$$

asymptotically stable and enforcing the γ-gain condition

$$\gamma^{-1}\sum_{k=0}^{\infty} z(k)^2 < \gamma \sum_{k=0}^{\infty} w(k)^2, \quad \forall \sum_{k=0}^{\infty} w(k)^2 > 0 \qquad (16)$$

Such a control law is said γ-gain state feedback controller. The following definition associates Poly-Quadratic stability with the well known $\mathcal{H}_\infty$ performance criterion for discrete time systems.

Definition 2. The autonomous system (12) is said Poly-Quadratically stable with an H_∞ performance γ if it is Poly-Quadratically stable and

$$\gamma^{-1}\sum_{k=0}^{\infty} z(k)^2 < \gamma \sum_{k=0}^{\infty} w(k)^2, \quad \forall \sum_{k=0}^{\infty} w(k)^2 > 0 \qquad (17)$$

Consider the autonomous discrete time system (12) where $u(k) = 0$. Given $\gamma > 0$, we are interested in answering the question: Is the autonomous system (12) Poly-Quadratically stable with an H_∞ performance γ ?

Theorem 4. Daafouz et al. (2001): The system (12) is Poly-Quadratically stable with an $\mathcal{H}_\infty$ performance γ if and only if there exist SPD matrices $X_i \in \mathbb{R}^{n\times n}$ and matrices $G_i \in \mathbb{R}^{n\times n}$, $i = 1, ..., N$, such that $\forall\{i; j\} \in \{1, ..., N\}^2$:

$$\begin{bmatrix} X_i - G_i - G_i' & (\bullet)' & (\bullet)' & (\bullet)' \\ \mathbf{0} & -\gamma\mathbf{I} & (\bullet)' & (\bullet)' \\ A_iG_i & B_{1i} & -X_j & (\bullet)' \\ C_{1i}G_i & D_{1i} & \mathbf{0} & -\gamma\mathbf{I} \end{bmatrix} < 0 \qquad (18)$$

A γ-gain state feedback controller for the system given by (12) can be obtained using the following result.

Theorem 5. System (12) is Poly-Quadratically stabilizable with an $\mathcal{H}_\infty$ performance γ by a state feedback control if there exist SPD matrices $X_i \in \mathbb{R}^{n\times n}$, matrices $G \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{m\times n}$, $i = 1, ..., N$, such that $\forall\{i; j\} \in \{1, ..., N\}^2$:

$$\begin{bmatrix} X_i - G - G' & (\bullet)' & (\bullet)' & (\bullet)' \\ \mathbf{0} & -\gamma\mathbf{I} & (\bullet)' & (\bullet)' \\ A_iG + B_{2i}R & B_{1i} & -X_j & (\bullet)' \\ C_{1i}G + D_{2i}R & D_{1i} & \mathbf{0} & -\gamma\mathbf{I} \end{bmatrix} < 0, \qquad (19)$$

The γ-gain state feedback control law is then given by (14) with $K = RG^{-1}$.

4. ROBUST STABILIZATION

First, the classical problem of robust stabilization is handled. Then, the same problem is addressed while ensuring an $\mathcal{H}_\infty$ gain lower than γ.

4.1 *Classical robust stabilization*

The following discrete-time model is considered:

$$x(k+1) = A(\delta(k))x(k) + B(\delta(k))u(k) \qquad (20)$$

$x(k) \in \mathbb{R}^n$ is the state vector at time k and $u(k) \in \mathbb{R}^m$ is the input vector at the same time. In the following we use the notation:

$$M = M(\delta(k)) = \left[\, A(\delta(k))\ B(\delta(k)) \,\right] \qquad (21)$$

$$M = M_0 + \sum_{i=1}^{p}(\delta_{[i]}(k)M_{[i]})$$

In the above expression $M_0 = [A_0\ B_0]$ corresponds to the nominal plant. $\delta(k)$ is a vector time function corresponding to uncertain but bounded parameters and matrices $M_{[i]} \in \mathbb{R}^{(n\times(n+m))}$, $i = 1, ..., p$ are precisely known and specify which entries of M are affected by parameter variations. δ is assumed to belong to an hyper-rectangular set Δ *i. e.* $\delta(k) \in \Delta \; \forall k$ where:

$$\Delta = \Delta(\tilde{\delta}) = \{v \in \mathbb{R}^p \mid (-\underline{\alpha}_{[i]}\tilde{\delta} \le v_{[i]} \le \overline{\alpha}_{[i]}\tilde{\delta}, \; \forall i \in \{1, ..., p\})\} \tag{22}$$

where $v = \begin{bmatrix} v_{[1]} & \dots & v_{[p]} \end{bmatrix}'$ and the positive scalar numbers $\underline{\alpha}_{[i]}$ and $\overline{\alpha}_{[i]}$ are introduced to define the form of Δ in the $\mathbb{R}^p$-space. $\tilde{\delta}$ is some sort of "size" of Δ to be derived. The nominal value of δ is $\mathbf{0}$ so that $M(\mathbf{0}) = M_0$.

Define Φ by $\{0; 1\}^p$, *i.e.* the set of the $N = 2^p$ distinct elements of $\mathbb{R}^p$, ϕ_j, $j = 1, ..., N$, with entries only equaling either 0 or 1. Then, Δ is actually a convex hull that can be defined through its vertices

$$\delta_j = (-(\phi_j \odot \underline{\alpha}) + ((\mathbb{1}_{p,1} - \phi_j) \odot \overline{\alpha}))\tilde{\delta} \tag{23}$$

where all matrices

$$\phi_j = \begin{bmatrix} \phi_{j_{[1]}} \\ \vdots \\ \phi_{j_{[p]}} \end{bmatrix}, \quad j = 1, ..., N, \tag{24}$$

make the whole set Φ up and where $\overline{\alpha}$ and $\underline{\alpha}$ are vectors defined by:

$$\underline{\alpha} = \begin{bmatrix} \underline{\alpha}_{[1]} \\ \vdots \\ \underline{\alpha}_{[p]} \end{bmatrix} \quad ; \quad \overline{\alpha} = \begin{bmatrix} \overline{\alpha}_{[1]} \\ \vdots \\ \overline{\alpha}_{[p]} \end{bmatrix} \tag{25}$$

Using these notations, when $\delta(k)$ describes Δ, $M(\delta(k))$ describes a polytope $\mathbb{M}(\tilde{\delta})$ that reads the following description:

$$\mathbb{M}(\tilde{\delta}) = \{M(\xi) \in \mathbb{R}^{n\times n} \mid M(\xi) = \sum_{j=1}^{N} (\xi_j(k) M_j) \, ; \, \xi \in \Xi\} \tag{26}$$

where Ξ, the set of all suitable functions ξ, is defined by:

$$\Xi = \{\xi = \begin{bmatrix} \xi_1(k) \\ \vdots \\ \xi_N(k) \end{bmatrix} \in \{\mathbb{R}^+\}^N \mid \sum_{j=1}^{N} \xi_j(k) = 1\} \tag{27}$$

Extreme matrices $M_j \in \mathbb{R}^{(n\times(n+m))}$, $j = 1, ..., N$ are the vertices of polytope $\mathbb{M}(\tilde{\delta})$ and can be detailed as follows:

$$M_j = M(\delta_j) = M_0 + \tilde{\delta}\bar{M}_j \quad \forall j \in \{1, ..., N\} \tag{28}$$

and $\forall j \in \{1, ..., N\}$

$$\bar{M}_j = \begin{bmatrix} \bar{A}_j & \bar{B}_j \end{bmatrix} = \sum_{i=1}^{p} ((-\phi_{j_{[i]}}\underline{\alpha}_{[i]} + (1 - \phi_{j_{[i]}})\overline{\alpha}_{[i]})M_{[i]}) \tag{29}$$

In this paper, we aim to derive a state feedback control law $u(k) = Kx(k)$ that stabilizes the model (20) . The closed-loop system behaviour is described by:

$$x(k+1) = A_{cl}(\delta(k))x(k) = (A(\delta(k)) + B(\delta(k))K)x(k) \tag{30}$$

It is clear that the closed-loop dynamic matrix A_{cl} can be written:

$$A_{cl}(k) = A_{cl_0} + \sum_{i=1}^{N} (\delta_{[i]}(k) A_{cl_{[i]}}) \tag{31}$$

where $A_{cl_0} = A_0 + B_0 K$ and where the various matrices $A_{cl_{[i]}}$ are defined by:

$$A_{cl_{[i]}} = A_{[i]} + B_{[i]}K \quad \forall i \in \{1, ...N\} \tag{32}$$

Following the same reasonning as above, it is clear that when $\delta(k)$ describes $\Delta(\tilde{\delta})$, then the closed-loop state matrix describes a polytope $\mathbb{A}_{cl}(\tilde{\delta})$ defined by:

$$\mathbb{A}_{cl}(\tilde{\delta}) = \{A_{cl}(\xi) \in \mathbb{R}^{n\times n} \mid A_{cl}(\xi) = \sum_{j=1}^{N} (\xi_j(k) A_{cl_j}) \, ; \, \xi \in \Xi\} \tag{33}$$

Besides, extreme matrices A_{cl_j} are defined by $\forall j \in \{1, ..., N\}$:

$$A_{cl_j} = A_{cl_0} + \tilde{\delta}\bar{A}_{cl_j} = A_{cl_0} + \tilde{\delta}(\bar{A}_j + \bar{B}_j K) \tag{34}$$

This structure is useful to achieve our goal that is to compute a matrix K which makes (30) be stable for all function $\delta(k)$ varying in $\Delta(\tilde{\delta})$. While computing a suitable K, we look for $\tilde{\delta}^\star$, the maximal value of $\tilde{\delta}$ such that stability is ensured. Hence, $\tilde{\delta}^\star$ is a robust stability bound. We state the following theorem:

Theorem 6. Daafouz et al. (2001): Let an uncertain discrete-time system be described by (20) where $\delta(k)$ varies in $\Delta(\tilde{\delta})$ defined by (22). There exists a static state feedback control law $u(k) = Kx(k) \, \forall k \in \mathbb{N}$ such that closed-loop system described by (30) is Poly-Quadratically stable **if** $\tilde{\delta} \le \tilde{\delta}^\star$ with:

$$\tilde{\delta}^\star = \lambda^{\star -1} \tag{35}$$

$\lambda^\star \in \mathbb{R}$ being the solution to the following optimization problem:

$$\min_{X_0, \bar{X}_1, ..., \bar{X}_N, G, R, \lambda} \lambda \tag{36}$$

where $X_0 = X_0' > 0 \in \mathbb{R}^{n\times n}$, $\bar{X}_i = \bar{X}_i' \in \mathbb{R}^{n\times n}$, $i = 1, ..., N$, $G \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{m\times n}$, are variables satisfying the following $\mathcal{LMI}$ constraints :

$$\begin{bmatrix} \bar{X}_i & G'\bar{A}_i + R'\bar{B}_i \\ \bar{A}_i G + \bar{B}_i R & -\bar{X}_j \end{bmatrix} < \lambda \begin{bmatrix} -X_0 + G + G' & -G'A_0' - R'B_0 \\ -A_0 G - B_0 R & X_0 \end{bmatrix} \tag{37}$$

for all $\{i; j\} \in \{1, ..., N\}^2$. Then, the feedback matrix is given by $K = RG^{-1}$.

It is important to notice that if K exists, it stabilizes the nominal plant and then there exits $X_0 = X_0' > 0$, G and R such that:

$$\begin{bmatrix} X_0 - G - G' & G'A_0' + R'B_0' \\ A_0 G + B_0 R & -X_0 \end{bmatrix} < 0 \tag{38}$$

As a consequence, problem (36) appears as a typical generalized eigenvalue problem which can be solved owing to $\mathcal{LMI}$ tools. In practice, constraint (38) must be clearly added to the $\mathcal{LMI}$ system.

4.2 *Robust stabilization with $\mathcal{H}_\infty$ performance*

The following system is now considered:

$$\begin{cases} x(k+1) = \mathcal{A}(\delta(k))x(k) + \mathcal{B}_1(\delta(k))w(k) + \mathcal{B}_2(\delta(k))u(k) \\ z(k) = \mathcal{C}_1(\delta(k))x(k) + \mathcal{D}_1(\delta(k))w(k) + \mathcal{D}_2(\delta(k))u(k) \end{cases} \quad (39)$$

where various signals are defined in paragraph 3 and where δ is a vector varying in $\Delta(\tilde{\delta})$ defined in (22). We define a global matrix $\mathcal{S}$ by:

$$\mathcal{S} = \begin{bmatrix} \mathcal{A}(\delta(k)) & \mathcal{B}_1(\delta(k)) & \mathcal{B}_2(\delta(k)) \\ \mathcal{C}1(\delta(k)) & \mathcal{D}_1(\delta(k)) & \mathcal{D}_2(\delta(k)) \end{bmatrix} = S_0 + \sum_{i=1}^{p}(\delta_{[i]}(k)S_{[i]})$$

$$= \begin{bmatrix} A_0 & B_{10} & B_{20} \\ C_{10} & D_{10} & D_{20} \end{bmatrix} + \sum_{i}^{p}(\delta_{[i]}(k) \begin{bmatrix} A_{[i]} & B_{1_{[i]}} & B_{2_{[i]}} \\ C_{1_{[i]}} & D_{1_{[i]}} & D_{2_{[i]}} \end{bmatrix}) \quad (40)$$

It is aimed to find a control law $u(k) = Kx(k)$ such that:

- the closed-loop system is stable
- the γ-gain condition defined in (16) is enforced for a given value of γ.
- the size $\tilde{\delta}$ of hyperrectangular set $\Delta(\tilde{\delta})$ is maximized.

The closed loop-system, considering w as the single closed-loop input vector, is described by:

$$\mathcal{S}_{cl} = S_{cl_0} + \sum_{i=1}^{p}(\delta_{[i]}(k)S_{cl_{[i]}}) =$$

$$\begin{bmatrix} A_0 + B_{20}K & B_{10} \\ C_{10} + D_{20}K & D_{10} \end{bmatrix} + \sum_{i}^{p}\left(\delta_{[i]}(k) \begin{bmatrix} A_{[i]} + B_{2_{[i]}}K & B_{1_{[i]}} \\ C_{1_{[i]}} + D_{2_{[i]}}K & D_{1_{[i]}} \end{bmatrix} \right) \quad (41)$$

Hence, following the same reasonning as in the above paragraph, it is clear that $\mathcal{S}_{cl}$ varies in a polytope which is defined by:

$$\mathbb{S}_{cl}(\tilde{\delta}) = \{\mathcal{S}_{cl}(\xi) \in \mathbb{R}^{n\times n} \mid \mathcal{S}_{cl}(\xi) = \sum_{j=1}^{N}(\xi_j(k)S_{cl_j})\,;\, \xi \in \Xi\} \quad (42)$$

where the extreme matrices read:

$$S_{cl_j} = S_{cl_0} + \tilde{\delta}\bar{S}_{cl_j} = S_{cl_0} + \tilde{\delta} \begin{bmatrix} \bar{A}_j + \bar{B}_{2j}K & \bar{B}_{1j} \\ \bar{C}_{1j} + \bar{D}_{2j}K & \bar{D}_{1j} \end{bmatrix} \quad (43)$$

and:

$$\bar{S}_j = \begin{bmatrix} \bar{A}_j & \bar{B}_{1j} & \bar{B}_{1j} \\ \bar{C}_{1j} & \bar{D}_{1j} & \bar{D}_{2j} \end{bmatrix} =$$

$$\sum_{i=1}^{p}((-\phi_{j_{[i]}}\underline{\alpha}_{[i]} + (1-\phi_{j_{[i]}})\overline{\alpha}_{[i]})S_{[i]}) \quad \forall j \in \{1,...,N\} \quad (44)$$

Thus, using arguments detailed in the proof of Theorem 5 given in Daafouz et al. (2001) we get the next theorem:

Theorem 7. Daafouz et al. (2001): Let a system be described by (39) where $\delta(k)$ varies in $\Delta(\tilde{\delta})$ defined by (22). Let γ be a scalar positive number. There exists a static state feedback control law $u(k) = Kx(k)$, such that the obtained closed-loop system is Poly-Quadratically stable with a $\mathcal{H}_\infty$ performance γ **if** $\tilde{\delta} \le \tilde{\delta}^\star$ with:

$$\tilde{\delta}^\star = \lambda^{\star -1} \quad (45)$$

$\lambda^\star \in \mathbb{R}$ being the solution to the following optimization problem:

$$\min_{X_0,\bar{X}_1,...,\bar{X}_N,G,R,\lambda} \lambda \quad (46)$$

where $X_0 = X_0' > 0 \in \mathbb{R}^{n\times n}$, $\bar{X}_i = \bar{X}_i' \in \mathbb{R}^{n\times n}$, $i = 1,...,N$, $G \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{m\times n}$, are variables satisfying the following $\mathcal{LMI}$ constraints, $\forall\{i;j\} \in \{1,...,N\}^2$:

$$\begin{bmatrix} \bar{X}_i & (\bullet)' & (\bullet)' & (\bullet)' \\ \mathbf{0} & \mathbf{0} & (\bullet)' & (\bullet)' \\ \bar{A}_iG + \bar{B}_{2i}R & \bar{B}_{1i} & -\bar{X}_j & (\bullet)' \\ \bar{C}_{1i}G + \bar{D}_{2i}R & \bar{D}_{1i} & \mathbf{0} & \mathbf{0} \end{bmatrix} < \lambda \begin{bmatrix} -X_0 + G + G' & (\bullet)' & (\bullet)' & (\bullet)' \\ \mathbf{0} & \gamma\mathbf{I} & (\bullet)' & (\bullet)' \\ -A0G - B_{20}R & -B_{10} & -X_0 & (\bullet)' \\ -C_{10}G - D_{20}R & -D_{10} & \mathbf{0} & \gamma\mathbf{I} \end{bmatrix} \quad (47)$$

Then, the feedback matrix is given by $K = RG^{-1}$.

Notice that the problem introduced above is a classical generalized eigenvalue problem.

5. NUMERICAL EXAMPLES

5.1 *Example 1*

Consider a system given by (12) with

$$A_1 = \begin{bmatrix} -0.06 & -0.25 & 0.10 & -0.47 \\ 0.09 & -0.50 & -0.63 & 0.52 \\ 0.55 & 0.47 & -0.59 & -0.50 \\ 0.03 & 0.29 & 0.87 & 0.56 \end{bmatrix}, \quad B_{21} = \begin{bmatrix} -0.06 \\ -0.59 \\ -0.32 \\ -0.53 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -0.19 & 0.28 & -0.12 & 0.66 \\ 0.34 & -0.32 & -0.32 & 0.54 \\ -0.06 & 0.29 & 0.38 & 0.39 \\ -0.03 & 0.36 & 0.52 & -0.28 \end{bmatrix}, \quad B_{21} = \begin{bmatrix} 0.04 \\ 0.39 \\ 0.04 \\ -.11 \end{bmatrix}$$

$$D_{11} = D_{12} = 0, \quad C_{11} = C_{12} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix},$$

$$B_{11} = B_{12} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}', \quad D_{21} = D_{22} = 0$$

Analyzing stability of the open loop system ($u(k) = 0$), we find that this system is not quadratically stable (there is no single quadratic Lyapunov fucntion $V(x) = x'Px$, proving stability), nor quadratically stabilizable (one can not compute a control law using quadratic stabilisability conditions). Using Theorem 2, we find that this system is Poly-Quadratically stable. Analyzing the $\mathcal{H}_\infty$ performance level using Theorem 4 we find that the minimum value of γ such that conditions of Theorem 4 are satisfied is $\gamma = 8.39$. A state feedback law can be used to improve this performance level. Using Theorem 5, we get the control law given by

$$u(k) = Kx(k) \quad \text{with} \quad K = \begin{bmatrix} 0.1737 & 0.8226 & 1.6092 & 0.9688 \end{bmatrix}$$

which gives an $\mathcal{H}_\infty$ performance $\gamma = 6.9$.

5.2 *Example 2*

Now consider the system described by (39) with $p = 1$ (one single varying parameter) and:

$$A_0 = \begin{bmatrix} -0.13 & 0.02 & -0.01 & 0.09 \\ 0.22 & -0.41 & -0.48 & 0.53 \\ 0.24 & 0.38 & -0.11 & -0.06 \\ 0 & 0.32 & 0.69 & 0.14 \end{bmatrix}, \quad B_{2[1]} = \begin{bmatrix} 0.05 \\ 0.49 \\ 0.18 \\ 0.21 \end{bmatrix},$$

$$A_{[1]} = \begin{bmatrix} 0.06 & -0.26 & 0.11 & -0.57 \\ -0.13 & -0.09 & -0.16 & -0.01 \\ 0.31 & 0.09 & -0.48 & -0.45 \\ 0.03 & -0.04 & 0.17 & 0.42 \end{bmatrix}, \quad B_{20} = \begin{bmatrix} -0.01 \\ -0.10 \\ -0.14 \\ -0.32 \end{bmatrix},$$

$$\mathcal{B}_1 = B_{10} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}', \quad \mathcal{C}_1 = C_{10} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix},$$
$$\mathcal{D}_1 = D_{10} = \mathcal{D}_2 = D_{20} = 0$$

Note that if $\tilde{\delta} = 1$, we recover the polytopic system with two vertices described in the previous example. First, it is aimed to stabilise the system for the greatest size of uncertainty. Considering quadratic stabilisation, *i. e.* solving problem (36) with $X_1 = X_2 = G$, we find

$$u(k) = Kx(k) \quad \text{with} \quad K = \begin{bmatrix} -0.1883 & -0.2123 & 1.6235 & 2.2568 \end{bmatrix}$$

what leads to $\tilde{\delta}^\star = 0.9426$. This result confirms that the polytopic system previoulsy considered is not quadratically stabilisable.
Solving the Poly-Quadratic stabilisability problem (36) (Theorem 6) yields

$$u(k) = Kx(k) \quad \text{with} \quad K = \begin{bmatrix} -0.0060 & 0.8640 & 1.9786 & 0.9647 \end{bmatrix}$$

that leads to the optimal value $\tilde{\delta}^\star = 1.0788$ which is consistent with the fact that the polytopic system of the previous example is indeed Poly-Quadratically stabilisable.
One can apply the same comparison between quadratic and Poly-Quadratic stabilisability while a $\mathcal{H}_\infty$ performance level is required. This level is $\gamma = 6.9$. Using conditions of Theorem 7, the quadratic approach (*i.e.* $X_1 = X_2 = G$) yields

$$u(k) = Kx(k) \quad \text{with} \quad K = \begin{bmatrix} -0.0628 & 0.1637 & 1.7186 & 1.7422 \end{bmatrix}$$

and the maximal value of $\tilde{\delta}$ is $\tilde{\delta}^\star = 0.8818$. In a Poly-Quadratic context (Theorem 7 with no restriction), we get

$$u(k) = Kx(k) \quad \text{with} \quad K = \begin{bmatrix} 0.1698 & 0.8176 & 1.6125 & 0.9798 \end{bmatrix}$$

and the maximal value of $\tilde{\delta}$ is $\tilde{\delta}^\star = 0.9999$. These results are perfectly coherent with those of the previous paragraph.

6. CONCLUSION

Robust stabilization and robust $\mathcal{H}_\infty$ state-feedback control of discrete-time models against time-varying parametric uncertainty has been handled through the concept of Poly-Quadratic stability. This property was proved to be less pessimistic than the more classical quadratic stability. The proposed results are based on computationally tractable $\mathcal{LMI}$ conditions. The efficiency of the technique has been emphasized on numerical examples. One can extend easily the results proposed in this paper to the well known gain scheduling problem. In this problem, the plant is assumed to switch between different linear models and one is interested by a stabilizing switching control. Under an assumption relying on knowledge of the true model in real time, a stabilizing switching control with an $\mathcal{H}_\infty$ performance level can be derived immediately from the results proposed in this paper.

References

F. Amato, M. Mattei, and A. Pironti. A robust stability problem for discrete-time systems subject to an uncertain parameter. *Automatica*, 34: 521–523, 1998.

F. Blanchini and S. Miani. A new class of universal lyapunov functions for the control of uncertain linear systems. *IEEE Trans. on Aut. Control*, 44:641–647, 1999.

S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory.* SIAM Studies in Applied Mathematics, 1994.

J. Daafouz, O. Bachelier, and J. Bernussou. About reducing conservatism in robust control design of discrete-time systems: The non stationary case. *Internal Report*, 2001.

J. Daafouz and J. Bernussou. Parameter dependent lyapunov functions for discrete time systems with time varying parametric uncertainties. *Systems and Control Letters*, 43:355–359, 2001.

E. Feron, P. Apkarian, and P. Gahinet. Analysis and synthesis of robust control systems via parameter dependant Lyapuonv functions. *IEEE Trans. on Aut. Contr.*, 41:436–442, 1996.

J. C. Geromel, M. C. de Oliveira, and L. Hsu. LMI characterization of structural and robust stability. *Linear Algebra and its Applications*, 285:69–80, 1998.

W.M. Haddad and D.S. Bernstein. Parameter dependent lyapunov functions and the discrete-time popov criterion in robust analysis. *Automatica*, 30:1015–1021, 1994.

M. C. De Oliveira, J. Bernussou, and J. C. Geromel. A new discrete time robust stability condition. *System and Control Letters*, 36, 1999.

D. Peaucelle, D. Arzelier, O. Bachelier, and J. Bernussou. A new robust D-stability condition for real convex polytopic uncertainty. *System and Control Letters*, 40, 2000.

A. Tesi and A. Vicino. Robust stability of state-space models with structured uncertainty. *IEEE Trans. on Aut. Contr.*, 35:191–195, 1990.

A. Trofino and C. E. de Souza. Bi-quadratic stability of uncertain linear systems. *Proc. of the* 38^{th} *Conference on Decision and Control*, pages 3017–3021, December 1999.